Wormholes and Late Time Complexity Growth
of Black Holes

Vyshnav Mohan

University of Iceland

November 15, 2022

In collaboration with Lárus Thorlacius and Fridrik Freyr Gautason

Complexity and the Volume of Maximal Surfaces

Complexity=Volume Proposal

- The computational complexity of a black hole can be thought of as the minimal number of gates required for a quantum circuit to prepare the corresponding time evolved state in the dual CFT.
- [Susskind 2014] proposed that the complexity of a black hole equals the volume of the extremal codimension-1 surface anchored at the boundary.
- Complexity of a chaotic system is expected to grow linearly at late times and saturate at extremely late times ($t \sim e^S$).
- The maximal volume surfaces display the expected linear growth at late times. However, this volume keeps growing and never saturates.

Complexity=Volume Proposal

- The computational complexity of a black hole can be thought of as the minimal number of gates required for a quantum circuit to prepare the corresponding time evolved state in the dual CFT.
- [Susskind 2014] proposed that the complexity of a black hole equals the volume of the extremal codimension-1 surface anchored at the boundary.
- Complexity of a chaotic system is expected to grow linearly at late times and saturate at extremely late times ($t \sim e^S$).
- The maximal volume surfaces display the expected linear growth at late times. However, this volume keeps growing and never saturates.

Complexity=Volume Proposal

- The computational complexity of a black hole can be thought of as the minimal number of gates required for a quantum circuit to prepare the corresponding time evolved state in the dual CFT.
- [Susskind 2014] proposed that the complexity of a black hole equals the volume of the extremal codimension-1 surface anchored at the boundary.
- Complexity of a chaotic system is expected to grow linearly at late times and saturate at extremely late times ($t \sim e^S$).
- The maximal volume surfaces display the expected linear growth at late times. However, this volume keeps growing and never saturates.

Complexity=Volume Proposal

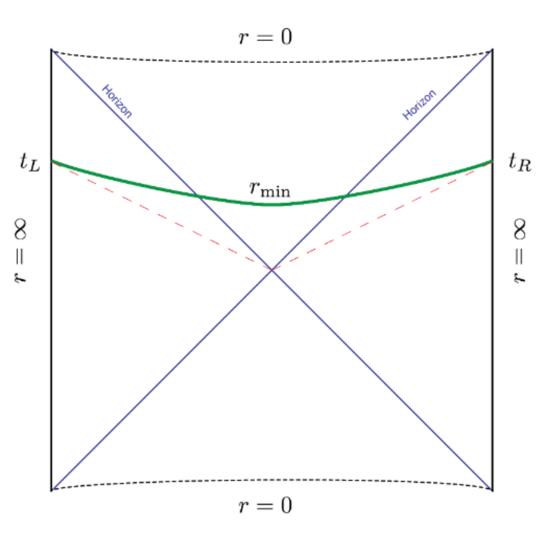
- The computational complexity of a black hole can be thought of as the minimal number of gates required for a quantum circuit to prepare the corresponding time evolved state in the dual CFT.
- [Susskind 2014] proposed that the complexity of a black hole equals the volume of the extremal codimension-1 surface anchored at the boundary.
- Complexity of a chaotic system is expected to grow linearly at late times and saturate at extremely late times ($t \sim e^S$).
- The maximal volume surfaces display the expected linear growth at late times. However, this volume keeps growing and never saturates.

- In JT gravity, the plateau can be obtained by adding wormhole corrections to the volume computations.
- JT gravity can be realised explicitly as a random matrix theory. Therefore, the wormhole corrections can be added systematically.
- Higher dimensional gravity computations require further investigation to make sense of such wormhole inclusions.

- In JT gravity, the plateau can be obtained by adding wormhole corrections to the volume computations.
- JT gravity can be realised explicitly as a random matrix theory. Therefore, the wormhole corrections can be added systematically.
- Higher dimensional gravity computations require further investigation to make sense of such wormhole inclusions.

- In JT gravity, the plateau can be obtained by adding wormhole corrections to the volume computations.
- JT gravity can be realised explicitly as a random matrix theory. Therefore, the wormhole corrections can be added systematically.
- Higher dimensional gravity computations require further investigation to make sense of such wormhole inclusions.

Maximal Surfaces in Eternal Black Hole



The metric in the infalling Eddington-Finkelstein coordinates is given by

$$v = t + r^*(r); \quad ds^2 = -f(r)dv^2 + 2dvdr + r^2 d\Sigma_{k,d-1}^2 \tag{1}$$

where

$$f(r) = r^2 + 1 - \frac{\omega^{d-2}}{r^{d-2}} \tag{2}$$

The volume of a spherically symmetric spacelike codimension-1 surface is given by

$$\mathcal{V} = \Omega_{d-1} \int d\lambda r^{d-1} \sqrt{-f(r)\dot{v}^2 + 2\dot{v}\dot{r}} \equiv \Omega_{d-1} \int d\lambda \mathcal{L}(\dot{v}, r, \dot{r}) \tag{3}$$

The integrand has a conserved quantity E:

$$E = -\frac{\partial \mathcal{L}}{\partial \dot{v}} = \frac{r^{d-1}(f\dot{v} - \dot{r})}{\sqrt{-f\dot{v}^2 + 2\dot{v}\dot{r}}} \tag{4}$$

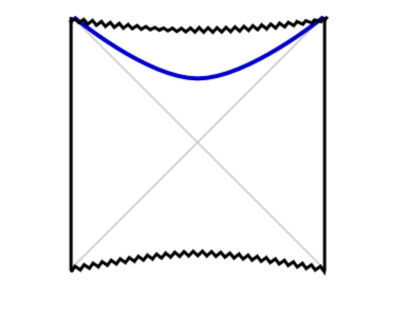
Extremizing the action, we get the EOM

$$E = r^{2(d-1)}(f(r)\dot{v} - \dot{r})$$
$$\dot{r}^2 + r^{-4(d-1)}V_{\text{eff}} = 0 \quad \text{where} \quad V_{\text{eff}} = r^{2(d-1)}f(r) + E^2 \tag{5}$$

The turning point r_{min} can be obtained by solving

$$\boxed{V_{\text{eff}}(r_{min}) = 0} \tag{6}$$

Late time behavior



The location of the constant radial surface $\tilde{r}_{min}$ is given by

$$\boxed{\mathcal{V}'_{\text{eff}}(\tilde{r}_{min}) = 0} \tag{7}$$

The End

$$\mathcal{V} = \Omega_{d-1} \int d\lambda r^{d-1} \sqrt{-f(r)\dot{v}^2 + 2\dot{v}\dot{r}}$$

Length of Geodesics

$$\ell = \int d\lambda \sqrt{-f(r)\dot{v}^2 + 2\dot{v}\dot{r}} \tag{8}$$

Extremizing the action, we get the EOM

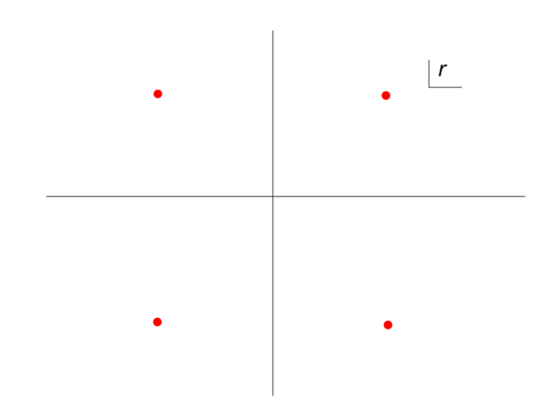
$$\begin{aligned} E &= f(r)\dot{v} - \dot{r} \\ \dot{r}^2 + \tilde{V}_{\text{eff}} &= 0 \quad \text{where} \quad \tilde{V}_{\text{eff}} = f(r) + E^2 \end{aligned} \tag{9}$$

The turning point r_{min} can be obtained by solving

$$\boxed{\tilde{V}_{\text{eff}}(r_{min}) = 0} \tag{10}$$

Critical Points of $\tilde{V}_{\text{eff}}$

In $d = 4$, we have



$$\tilde{V}'_{\text{eff}}(\tilde{r}_{min}) = 0$$

Picking the right contour

Using the equations of motion, we can see that the volume and the boundary time can be written as integrals over the complex contour.

$$\ell = \int_\gamma dr \frac{1}{\sqrt{f(r)+E^2}} \tag{11}$$

and

$$t_R = \int_\gamma dr \left[\frac{E}{f(r)\sqrt{f(r)+E^2}}\right] \tag{12}$$

We want to impose the **boundary condition**:

$$\boxed{\mathrm{Im}(t_R) = 0} \tag{13}$$

Growth of the Geodesics

- Imposing the boundary conditions, we find that there are two contours, or equivalently, two complex geodesics that satisfy $\mathrm{Im}(t_R) = 0$.
- The length of the geodesic along the two contours turn out to be complex conjugates of each other.
- At late times, we can explicitly show that the real part of the length grows linearly in time!

Relation to Boundary Correlation Functions

[Fidkowsk 2003] argued that, in the geodesic approximation, the boundary correlation function is related to the length of the complex geodesic through the relation

$$\langle \chi_L(t)\chi_R(t)\rangle = e^{-\Delta\ell} + e^{-\Delta\bar{\ell}} \tag{14}$$

where $\chi_{L,R}$ are probe operators acting on the heavy states of the boundary theory. Therefore, we have

$$2\,\mathrm{Re}\{(\ell(t))\} = -\lim_{\Delta\to 0} \frac{\partial \langle \chi_L(t)\chi_R(t)\rangle}{\partial \Delta} \tag{15}$$

Late-Time Behavior of Correlation Functions

We expect the probe operators $\chi_{L,R}$ acting on the heavy states of the boundary theory to behave as random matrices. The density of states of a RMT can be argued to have a universal correction that is non-perturbative in e^S:

$$\rho(E)\rho(E') = \rho(E)\rho(E') - \frac{\sin^2\left[e^S\left(E-E'\right)\right]}{\left[\pi e^S\left(E-E'\right)\right]^2} \tag{16}$$

Using

- Eigenstate Thermalization Hypothesis (ETH)
- Non-perturbative correction

we can explicitly see that the correlation function saturates at $t > e^S$. Therefore, (real part of) the length of complex geodesics also saturate!

Outlook

- We saw that the length of complex geodesics have a linear growth at *early* late times and then it saturates when $t > e^S$.
- Therefore, complexity defined thorough the length of these complex geodesics display the exact behavior one would expect complexity to demonstrate.
- New bulk dual for complexity?

- The length saturated only when we added the non-perturbative correction to the calculation.
- The non-perturbative correction admits a formal topological expansion and the gravitational dual of these corrections can be thought of as wormhole corrections to the length computation.
- We have already seen similar wormhole corrections show up in the Page curve calculations.
- These wormholes help us go beyond what one would expect semiclassical gravity to capture.

Thank You

Extra Slides

Late time behavior of correlation functions

We have

$$\langle \chi_L(t)\chi_R(t)\rangle = \int dE_m dE_n \rho(E_n)\rho(E_m) e^{-\frac{\beta}{2}(E_m+E_n)} e^{-it(E_n-E_m)} \left|\langle E_n|\chi|E_m\rangle\right|^2 \quad (17)$$

We expect the probe operators $\chi_{L,R}$ acting on the heavy states of the boundary theory to behave as random matrices.

$$\rho(E)\rho(E') = \rho(E)\rho(E') - \frac{\sin^2\left[e^S\left(E-E'\right)\right]}{\left[\pi e^S\left(E-E'\right)\right]^2} \quad (18)$$

Using ETH

$$\langle E_n|\chi|E_m\rangle = \chi(\bar{E})\delta_{mn} + e^{-S/2} f(\bar{E},\omega) R_{mn} \quad (19)$$

we can show that the correlation functions plateau at $t > e^S$!