Cosmic Strings and Celestial Entanglement

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Asymptotic symmetries of flat space

Asymptotic symmetries of flat space: supertranslations and superrotations

$$\mathrm{d}s^2 = -\mathrm{d}u^2 - 2\mathrm{d}u\,\mathrm{d}r + \frac{4r^2}{(1+|z|^2)^2}\mathrm{d}z\,\mathrm{d}\bar{z} + \dots$$

Superrotations:

$$z \to w(z) + \dots\ , \qquad u \to \frac{1+|z|^2}{1+|w|^2}|w'|u + \dots\ , \qquad r \to \frac{1+|w|^2}{1+|z|^2}\frac{1}{|w'|}r + \dots\ .$$

$=$ Lorentz transformations when $w(z) = \frac{az+b}{cz+d}$

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4D flat space scattering amplitudes $=$ 2D CFT correlators

Celestial holography

Cosmic strings & superrotations

Superrotation $w(z) = \left(\frac{z-z_1}{z-z_2}\right)^{1-4G_\mathrm{N}\mu}$ [Strominger, Zhiboedov, 1610.00639]

Inserts cosmic string: [Penrose]

$$\mathrm{d}s^2 = -\mathrm{d}\tilde{u}^2 - 2\mathrm{d}\tilde{u}\,\mathrm{d}\tilde{r} + \tilde{r}^2\left[\mathrm{d}\tilde{\theta}^2 + (1-4G_\mathrm{N}\mu)\,\mathrm{d}\tilde{\phi}^2\right], \qquad \tilde{\phi} \sim \tilde{\phi} + 2\pi.$$

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Uniformisation map

Entanglement entropy in QFT

Quantum system, $\mathcal{H} = \mathcal{A} \otimes \mathcal{B}$.

Density matrix $\rho \quad \Rightarrow \quad$ reduced density matrix $\rho_{\mathcal{A}} = \operatorname{tr}_{\mathcal{B}} \rho$.

Entanglement entropy:

$$S = -\operatorname{tr}_{\mathcal{A}} \left(\rho_{\mathcal{A}} \log \rho_{\mathcal{A}}\right)$$

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Rényi entropy

$$S_n = \frac{1}{1-n} \ln \mathrm{tr}_{\mathcal{A}}\, \rho_{\mathcal{A}}^n.$$

Entanglement entropy

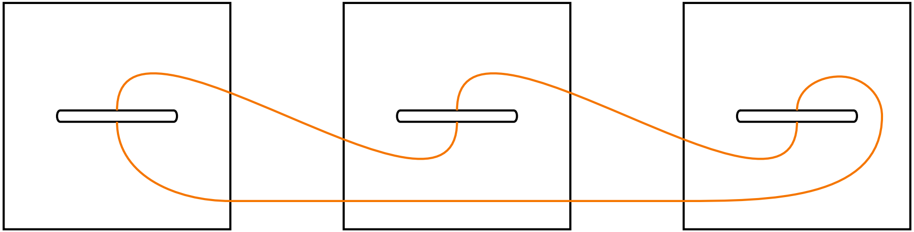
$$S = \lim_{n \to 1} S_n.$$

Replica trick

Rényi entropy

$$S_n = \frac{1}{1-n} \ln \operatorname{tr}_{\mathcal{A}} \rho_{\mathcal{A}}^n .$$

For integer n, $\operatorname{tr} \rho_{\mathcal{A}}^n = Z_n / Z_1^n$ is partition function on multiple sheets:



Replica trick in 2D CFT

[Calabrese, Cardy, hep-th/0405152]

Entanglement entropy of interval $z = \bar{z} \in [z_1, z_2]$

Uniformisation map

$$w(z) = \left(\frac{z - z_1}{z - z_2} \right)^{1/n}$$

n-sheeted replica manifold $\rightarrow$ complex plane

Same as cosmic string superrotation

$$\mu = \frac{n-1}{4nG_{\mathrm{N}}}.$$

Replica trick in 2D CFT

Use conformal symmetry:

$$S_n = \frac{c}{6}\left(1+\frac{1}{n}\right)\ln\left(\frac{|z_2 - z_1|}{\varepsilon}\right) + a_n.$$

$$S = \frac{c}{3}\ln\left(\frac{|z_2 - z_1|}{\varepsilon}\right) + a_1.$$

On unit sphere

$$S_n = \frac{c}{6}\left(1+\frac{1}{n}\right)\ln\left[\frac{2}{\varepsilon}\sin\left(\frac{\ell}{2}\right)\right] + a_n.$$

$\ell =$ great circle distance

Entanglement entropy from cosmic strings?

$$S_n = \frac{1}{1-n} \ln \mathrm{tr}_{\mathcal{A}}\, \rho_{\mathcal{A}}^n = \frac{1}{1-n} \left(\ln Z_n - n \ln Z_1\right).$$

Replica symmetry

$$\ln Z_n = n \ln \bar{Z}_n$$

Partition function on n-sheeted manifold

Partition function on single sheet

$$S_n = \frac{n}{1-n} \left(\ln \bar{Z}_n - \ln Z_1\right).$$

Entanglement entropy from cosmic strings?

Working assumption: CCFT partition function = bulk partition function.

$$\bar{Z}_{\mathrm{n,CCFT}} = \bar{Z}_{\mathrm{n,grav.}} = e^{iI_n}$$

I_n = on-shell gravity action in presence of cosmic string, tension

$$\mu = \frac{n-1}{4nG_{\mathrm{N}}}$$

Rényi entropies

$$S_n = \frac{in}{1-n}\left[I_n - I_1\right]$$

$$\partial_n\left[\left(1-\frac{1}{n}\right)S_n\right] = -i\partial_n I_n$$

AdS case: [Dong, 1601.06788]

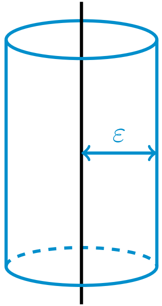
Computing on-shell action

Conical singularity at location of string – cut out small tube.

Einstein gravity (+ minimally coupled matter):

$$\partial_n I_n = -\frac{1}{16\pi G_{\mathrm{N}}} \int_{\mathrm{tube}} \mathrm{d}^3x \, \sqrt{-\gamma} \, (\nabla^\mu \partial_n g_{\mu\nu} - g^{\nu\rho} \nabla_m \partial_n g_{\nu\rho})$$

[Lewkowycz, Maldacena, 1304.4926; Dong, 1601.06788]



Key assumption: boundary terms vanish at infinity.

Single interval Rényi entropy

Cosmic string spacetime

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}r^2 + \frac{r^2}{n^2}\mathrm{d}\phi^2$$

Then

$$\partial_n I_n = -\frac{1}{16\pi G_\mathrm{N}} \int_\mathrm{tube} \mathrm{d}^3x \sqrt{-\gamma}\,(\nabla^\mu \partial_n g_{\mu\nu} - g^{\nu\rho}\nabla_m \partial_n g_{\nu\rho})$$

$$= -\frac{1}{4G_\mathrm{N} n^2} \int_\mathrm{tube} \mathrm{d}t\,\mathrm{d}x$$

Tempting: $\partial_n I_n = -\frac{\mathrm{Area[string]}}{4G_\mathrm{N} n^2}$ **— leads to n-independent Rényi entropy.**

Regulating integral - hyperbolic slicing

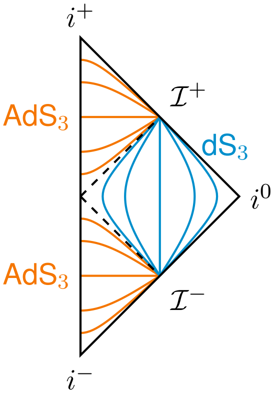
$$t = \tau \frac{1+\rho^2}{2\rho}, \qquad x = \tau \frac{1-\rho^2}{2\rho} \cos\theta, \qquad r = \tau \frac{1-\rho^2}{2\rho} \sin\theta.$$

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + \tau^2 \underbrace{\left[\frac{\mathrm{d}\rho^2}{\rho^2} + \frac{(1-\rho^2)^2}{4\rho^2} \left(\mathrm{d}\theta^2 + \frac{\sin^2\theta}{n^2} \mathrm{d}\phi^2 \right) \right]}_{\mathrm{AdS}_3}$$

[de Boer, Solodukhin, hep-th/0303006]

IR cutoff $\tau < L$

Fefferman-Graham cutoff $\rho > \rho_c(\theta, \phi)$



Choosing regulator surface

Uniformisation

$$w(z) = \left(\frac{z - z_1}{z - z_2}\right)^{1/n}$$

Singular conformal factor

$$\frac{2}{(1+|z|^2)^2}\mathrm{d}z\,\mathrm{d}\bar{z} = 2\Bigg(\underbrace{\frac{n|z_2 - z_1||w|^{n-1}}{|w^n - 1|^2 + |z_2 w^n - z_1|^2}}_{\Omega}\Bigg)^2 \mathrm{d}w\,\mathrm{d}\bar{w}$$

Need singular conformal factor on celestial sphere ($\mathfrak{t} = \tan(\theta/2)$, $\delta \ll 1$)

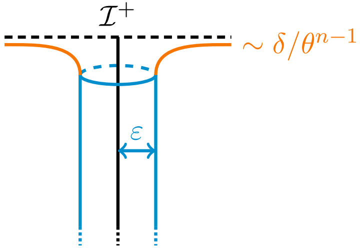
$$\begin{aligned}\rho_c &= \Omega^{-1}\delta \\ &= \delta\frac{\mathfrak{t}}{n|z_2 - z_1|}\left[(1+z_1^2)\mathfrak{t}^{-n} + (1+z_2^2)\mathfrak{t}^n - 2(1+z_1 z_2)\cos\phi\right]\end{aligned}$$

c.f. AdS case [Hung, Smolkin, Myers, Yale, 1110.1084]

Single interval Rényi entropy

Near-string and near-$\mathcal{I}^+$ limits don't commute.

[Adjei, Donelly, Py, Speranza, 1910.05435]



$$S_n = \frac{iL^2}{8G_{\mathrm{N}}}\left(1+\frac{1}{n}\right)\ln\left[\frac{2}{\varepsilon}\sin\left(\frac{\ell}{2}\right)\right]$$

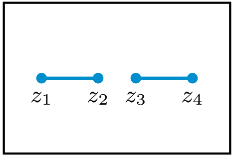
$$S_n = \frac{c}{6}\left(1+\frac{1}{n}\right)\ln\left[\frac{2}{\varepsilon}\sin\left(\frac{\ell}{2}\right)\right] \quad ✓$$

$$c = \frac{3iL^2}{4G_{\mathrm{N}}} \quad ✓ \quad \longleftarrow$$

[Cheung et al., 1609.00732]
[Pasterski, Verlinde, 2201.01630]
[Ogawa et al., 2207.06735]

Outlook

- Entanglement of multiple intervals
  — entanglement inequalities?
- Higher dimensions
- Meaning of central charge $\propto$ IR cutoff?
- Relation to Carollian approach?



$$S = \frac{c}{3} \ln \left[\frac{|z_{12} z_{34} z_{14} z_{23}|}{\varepsilon^2 |z_{13} z_{24}|} \mathcal{F}\left(\frac{z_{12} z_{34}}{z_{13} z_{24}} \right) \right]$$

[Calabrese, Cardy, Tonni, 0905.2069]

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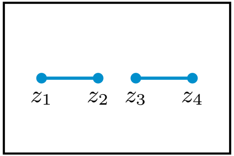

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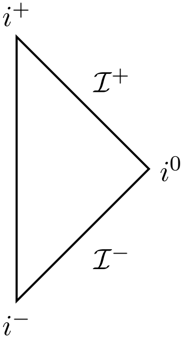
Thanks!

Backup slides

Asymptotically flat spacetimes

$$\mathrm{d}s^2 = g_{uu}\,\mathrm{d}u^2 + 2g_{ur}\,\mathrm{d}u\,\mathrm{d}r + r^2 g_{ab}\,\mathrm{d}\theta^a\,\mathrm{d}\theta^b + 2g_{ua}\,\mathrm{d}u\,\mathrm{d}\theta^a, \qquad (u = t - r).$$



$$\partial_r \det\left(r^{-2} g_{ab}\right) = 0$$

At large r:

$$g_{uu} = -1 + \mathcal{O}(r^{-1}), \qquad g_{ur} = -1 + \mathcal{O}(r^{-2}),$$
$$g_{ab} = r^2\gamma_{ab} + \mathcal{O}(r), \qquad g_{ua} = \mathcal{O}(1).$$

[Bondi, van der Burg, Metzner 1962; Sachs 1962]

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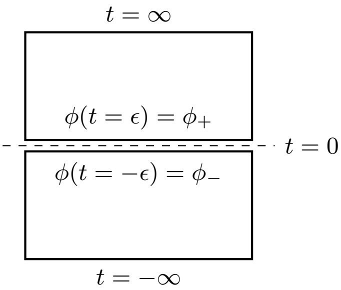

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