Conformal Carroll Scalars

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Main goal: find explicit actions for conformal Carroll theories

- Only few examples of flat space candidate dual field theories
- Conformal Carroll algebra is isomorphic to BMS
- Use small c expansion to construct Carroll equivalent of

$$S = -\frac{1}{2}\int d^d x \sqrt{-g}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{d-2}{4(d-1)}R\phi^2\right)$$

Outline

- Introduce Carroll geometry
  - local tangent space symmetries
  - relation to flat holography
  - connection and curvature
  - from Lorentzian geometry
- Construct conformal Carroll scalar actions
  - timelike
  - spacelike

Carroll symmetries

Are used to `relativistic' Lorentz boosts

$$t \to t + \beta x, \qquad x \to x + \beta t$$

Non-relativistic limit $c \to \infty$ gives Galilean boosts

$$t \to t, \qquad x \to x + vt$$

Instead, taking $c \to 0$ gives Carroll boosts [Levy-Leblond, Sen Gupta]

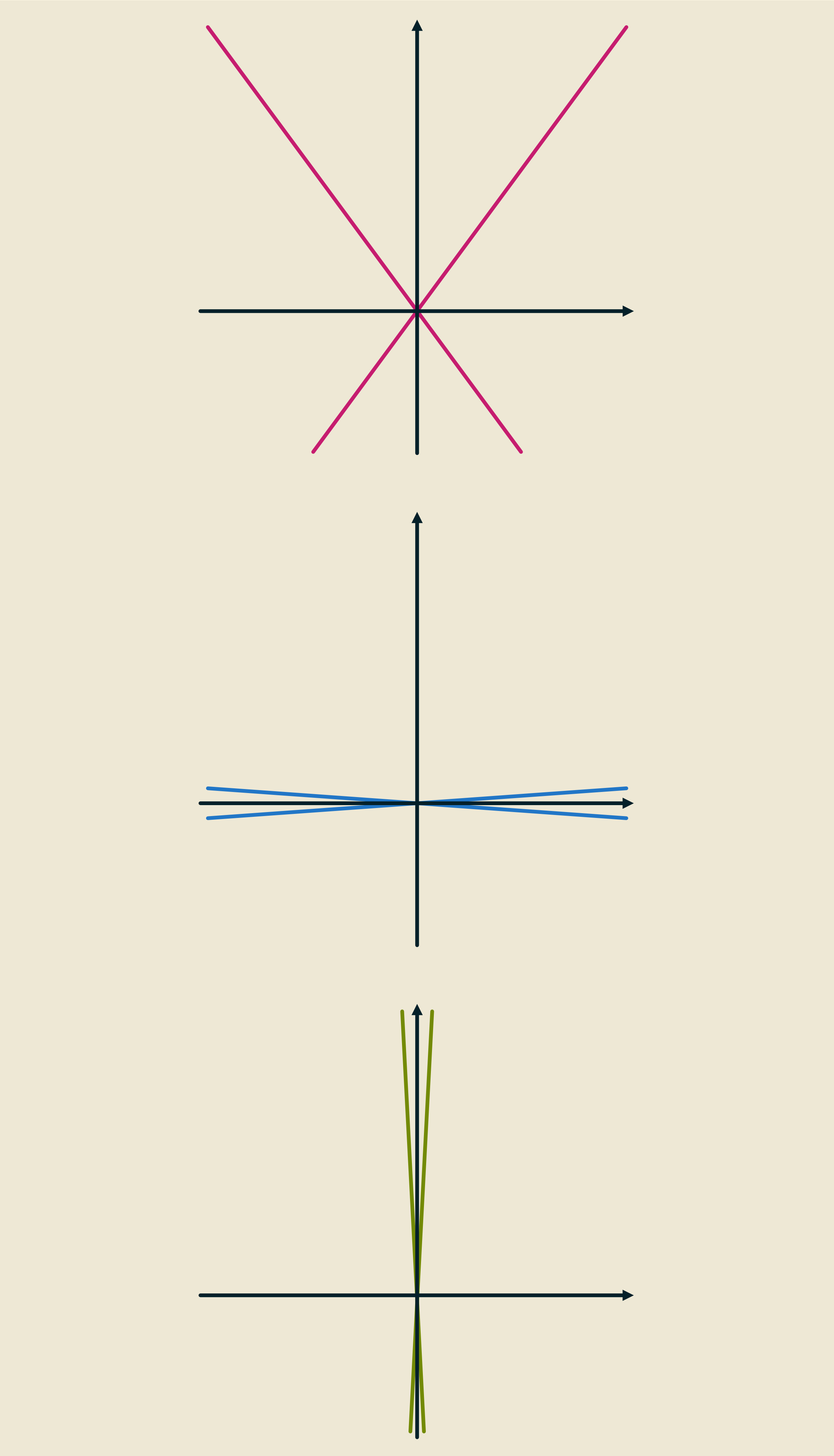
$$t \to t + \lambda x \qquad x \to x$$

Less obviously physical, but

- ultra-local behavior leads to solvable systems
- appears in Lorentzian geometry on null surfaces such as $\mathscr{I}^+$
- BMS asymptotic symmetries are isomorphic to conformal Carroll algebra [Duval, Gibbons, Horvathy, Zhang]

Boost Ward identity implies $\langle \phi(t,x)\phi(0,0)\rangle = \begin{cases} f(t)\delta(x), \\ g(x) \end{cases}$ timelike and spacelike theories

[Henneaux, Salgado-Rebodello] [De Boer, Hartong, Obers, Sybesma, Vandoren]



Carroll symmetries and flat holography

Holographic dual field theory for asymptotically flat spacetimes?

In 3+1 dim: BMS_4 asymptotic symmetries on $\mathscr{I}^+ \simeq \mathbb{R} \times S^2$

supertranslations $u \to u + f(z, \bar{z})$

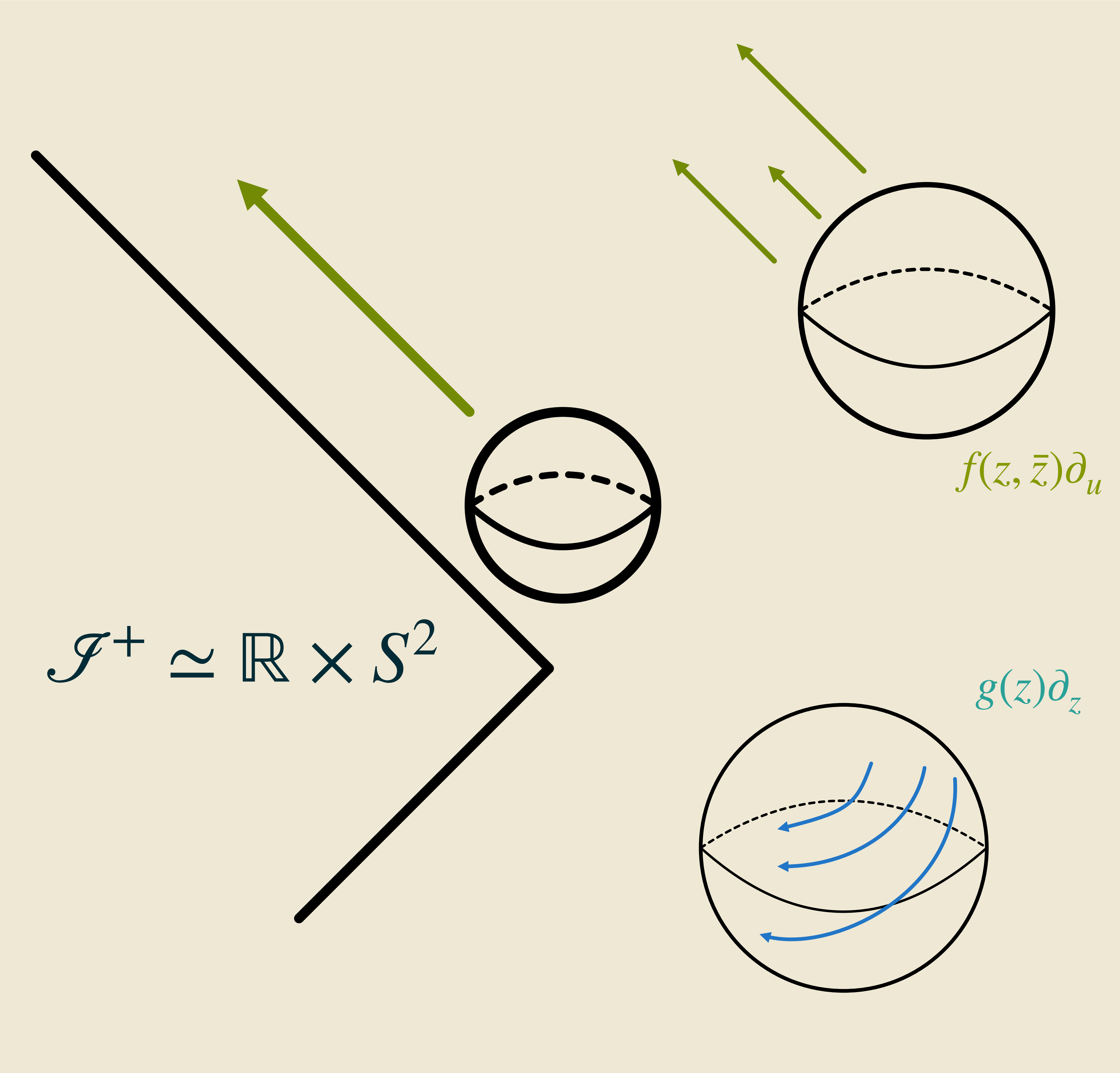
- $\sim$ Carroll boosts at each $(z, \bar{z})$
- suggests 3d Carrollian CFT dual: $\text{BMS}_4 \simeq \text{CCar}_3$

superrotations $z \to g(z), \quad \bar{z} \to \bar{g}(\bar{z})$

- Virasoro symmetries of CFT_2
- suggests 2d celestial CFT dual: CCFT_2

u-direction enters in CCFT_2 as conformal weight $\Delta \in 1 + i\mathbb{R}$
[Pasterski, Shao, Strominger]

Few explicit CCFT_2 theories known,
but *can construct CCar_3 examples from $c \to 0$ limit*



Carroll geometry

Carroll boosts $t \to t + \lambda x$, $x \to x$ and $\partial_t \to \partial_t$, $\partial_x \to \partial_x + \lambda \partial_t$

Geometry defined by time vector field $v^\mu(x^\rho)$ and spatial metric $h_{\mu\nu}(x^\rho)$

[Duval, Gibbons, Horvathy, Zhang] [Hartong] [Ciambelli, Marteau, Petropoulos…] [Hansen, Obers, GO, Søgaard] …

Complemented with $\tau_\mu(x^\rho)$ and $h^{\mu\nu}(x^\rho)$, transform under boosts $\lambda_\mu(x^\rho)$ as

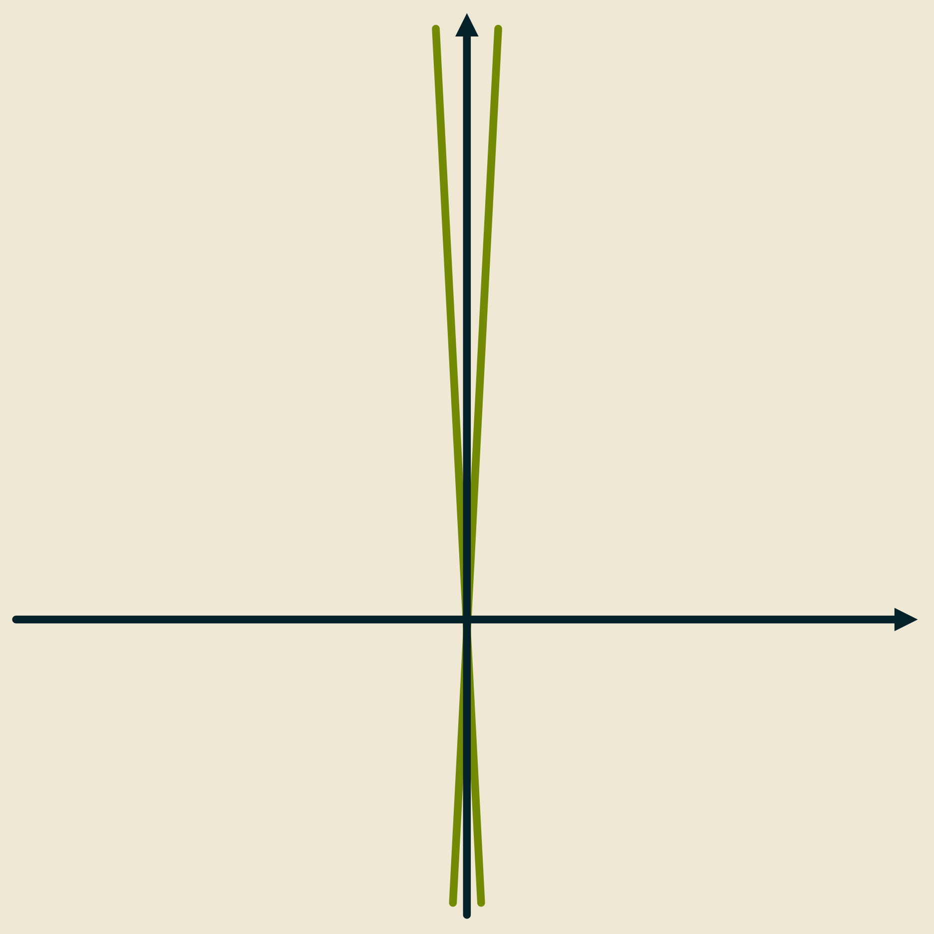
$$\delta_\lambda \tau_\mu = \lambda_\mu \,, \qquad \delta_\lambda h^{\mu\nu} = \lambda^\mu v^\nu + v^\mu \lambda^\nu$$

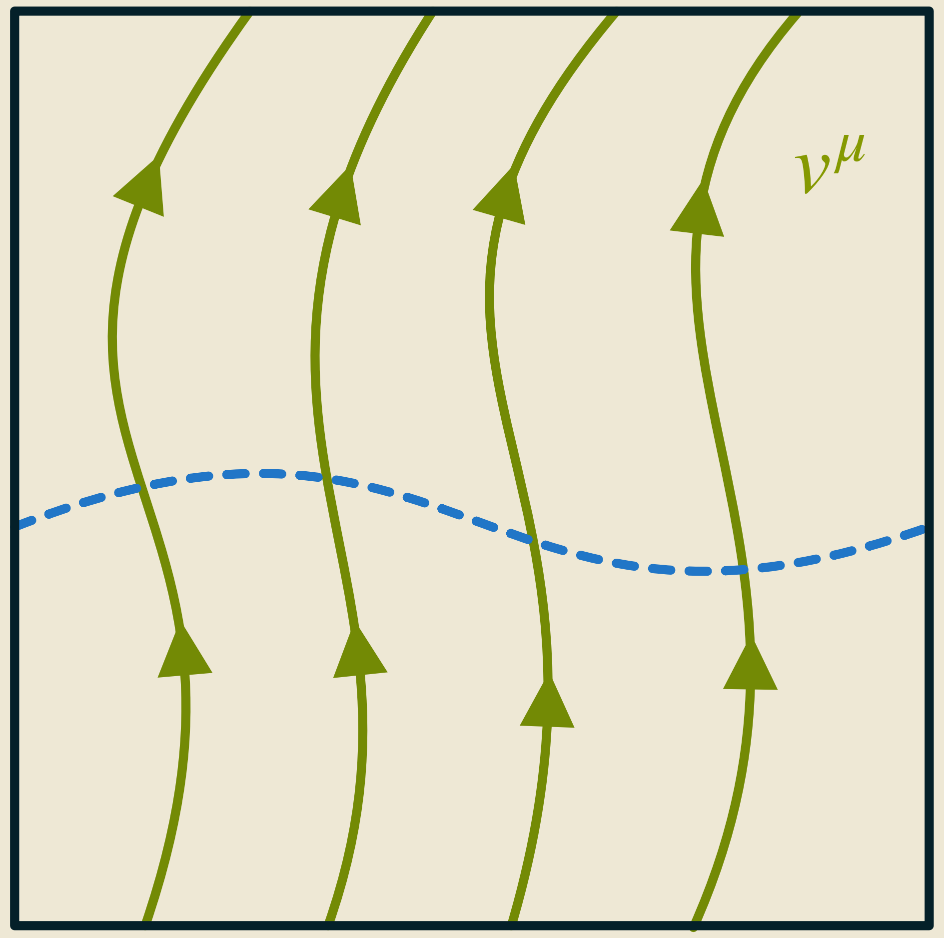
Satisfy $\quad v^\mu h_{\mu\nu} = 0, \quad \tau_\mu h^{\mu\nu} = 0, \quad v^\mu \tau_\mu = -1, \quad \delta^\mu_\nu = -v^\mu \tau_\nu + h^{\mu\rho} h_{\rho\nu}$

Boost parameter is spatial, $v^\mu \lambda_\mu = 0$, raise with $h^{\mu\nu}$

Also need extrinsic curvature $K_{\mu\nu} = -\frac{1}{2} \mathcal{L}_v h_{\mu\nu}$ and acceleration $a_\mu = 2 v^\rho \partial_{[\mu} \tau_{\rho]}$

Both boost-invariant and spatial tensors (so $K^{\mu\nu} = h^{\mu\rho} h^{\nu\sigma} K_{\rho\sigma}$ etc)





Carroll geometry

Want connection that satisfies $\tilde{\nabla}_\mu v^\nu = 0$ and $\tilde{\nabla}_\rho h_{\mu\nu} = 0$,

$$\tilde{\Gamma}^\rho_{\mu\nu} = -v^\rho \partial_{(\mu}\tau_{\nu)} - v^\rho \tau_{(\mu}\mathscr{L}_v \tau_{\nu)} + \frac{1}{2}h^{\rho\sigma}\left(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu}\right) - h^{\rho\sigma}\tau_\nu K_{\mu\sigma}$$

Non-zero torsion $\tilde{T}^\rho{}_{\mu\nu} = 2h^{\rho\sigma}\tau_{[\mu}K_{\nu]\sigma}$ [Bekaert, Morand] [Hansen, Obers, GO, Søgaard]

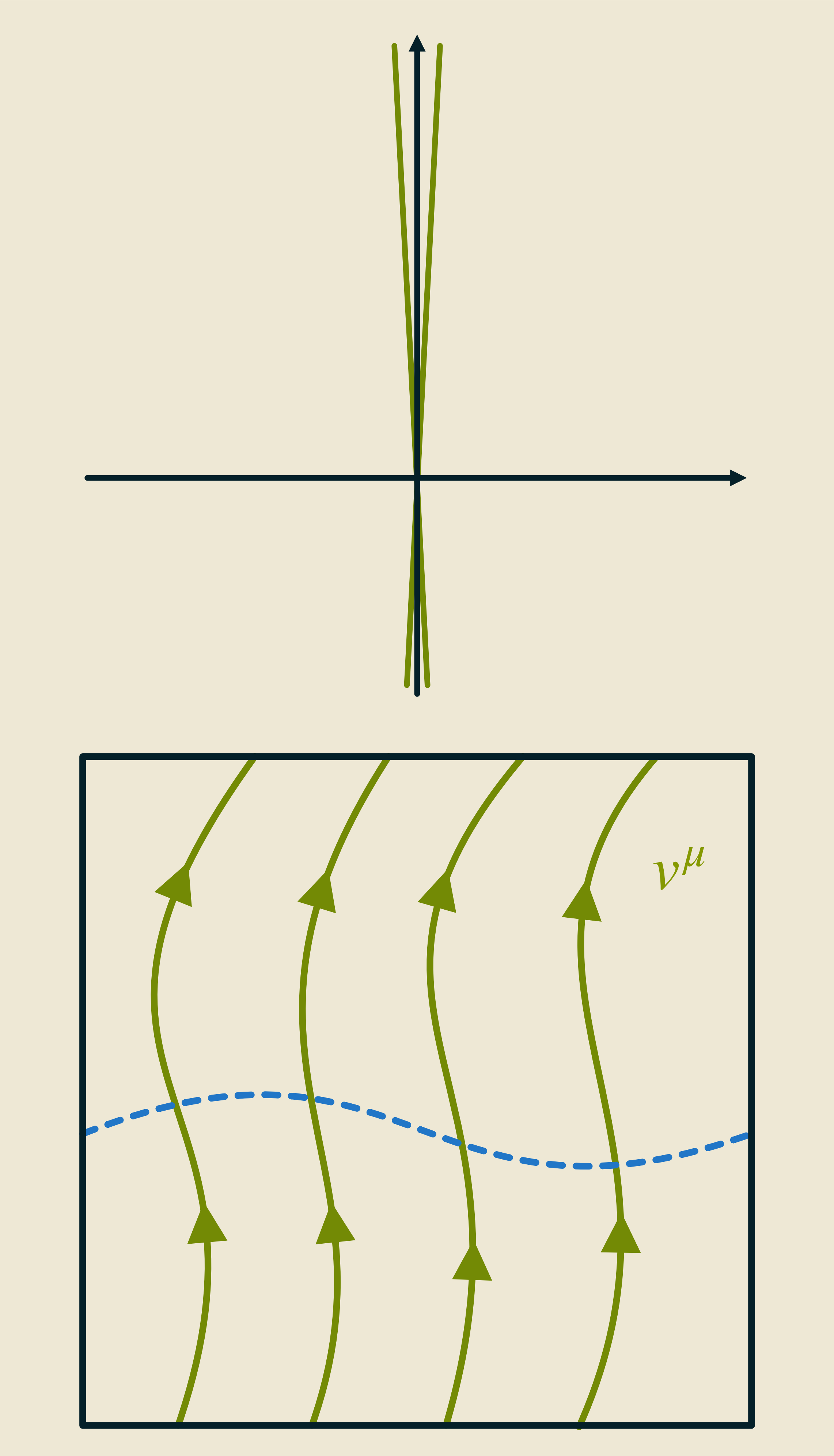
Define energy-momentum tensor using $e = \det(\tau_\mu, h_{\mu\nu})$ and variations

$$T^v_\mu = -\frac{1}{e}\frac{\delta S}{\delta v^\mu}, \qquad T^h_{\mu\nu} = -\frac{2}{e}\frac{\delta S}{\delta h^{\mu\nu}}$$

Boost-invariant combination is

$$T^\mu{}_\nu = -v^\mu T^v_\nu - h^{\mu\rho}T^h_{\rho\nu}$$

Boost Ward identity implies $T^i{}_0 = 0$ in adapted coordinates $x^\mu = (t, x^i)$



Carroll geometry from Lorentzian

From Lorentzian geometry, can get Carroll by expanding in $c \to 0$

Choose time vector V^μ and rewrite

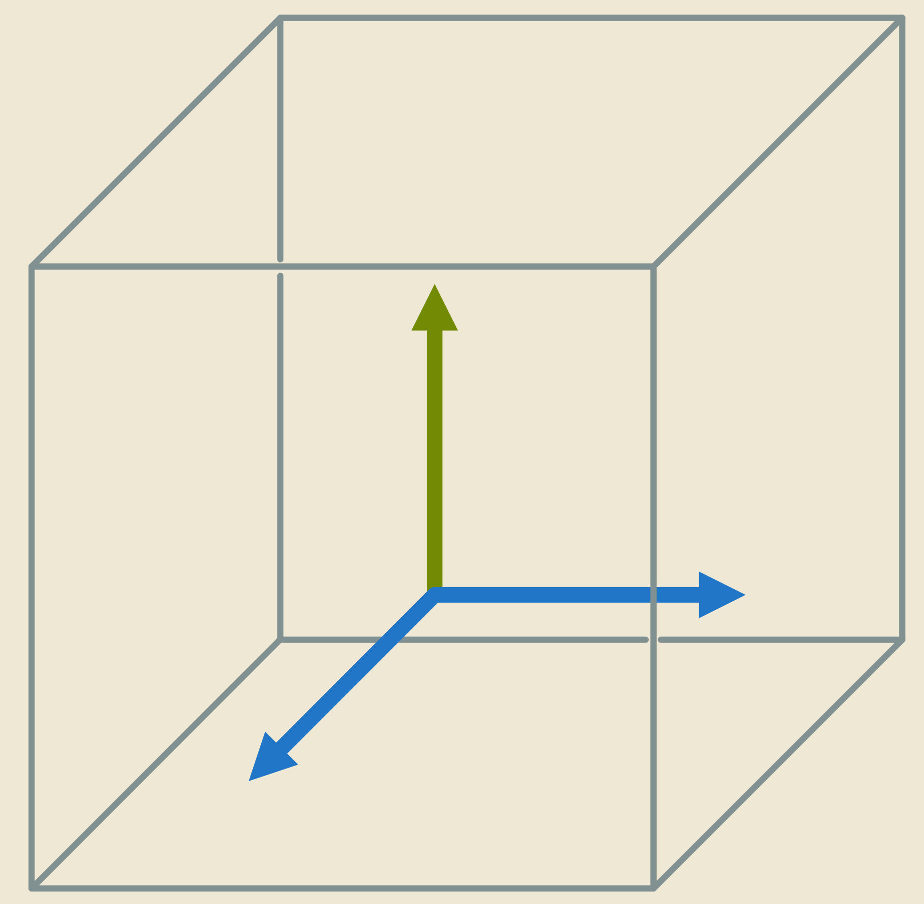
$$g^{\mu\nu} = -\frac{1}{c^2}V^\mu V^\nu + \Pi^{\mu\nu}, \qquad g_{\mu\nu} = -c^2 T_\mu T_\nu + \Pi_{\mu\nu}$$

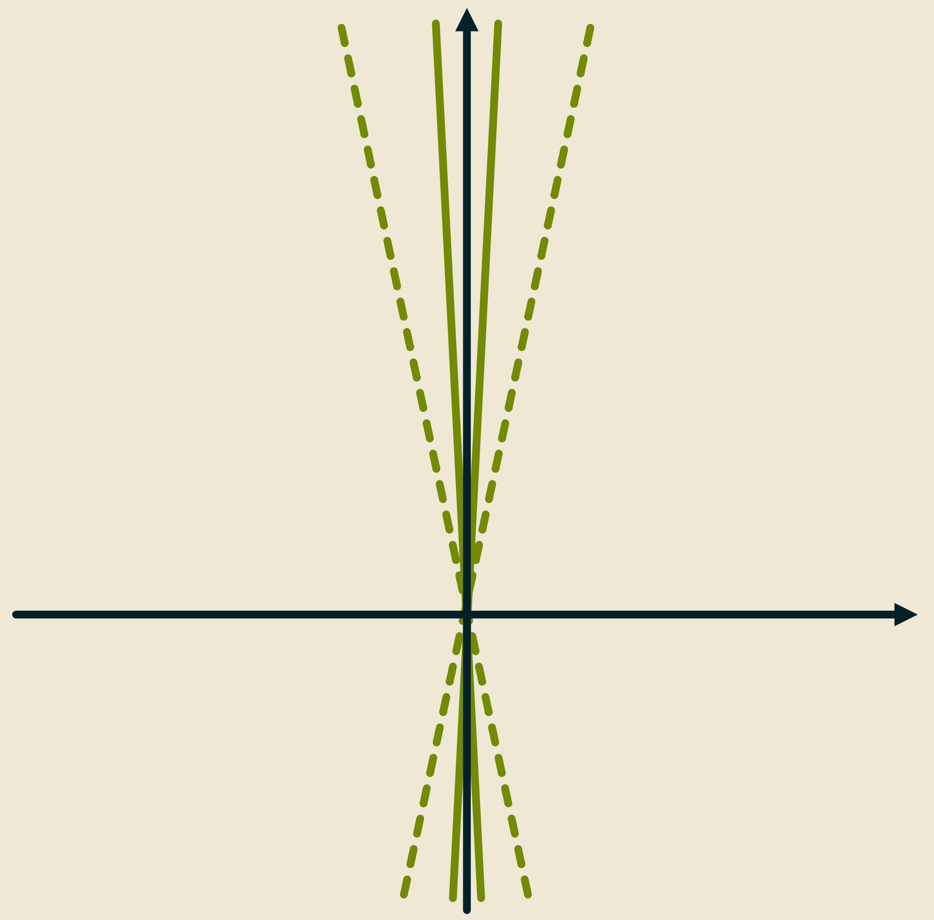
Carroll geometry appears at leading order in expansion,

$$V^\mu = v^\mu + c^2 M^\mu + \cdots, \qquad T_\mu = \tau_\mu + \cdots$$
$$\Pi_{\mu\nu} = h_{\mu\nu} + c^2 \Phi_{\mu\nu} + \cdots, \qquad \Pi^{\mu\nu} = h^{\mu\nu} + \cdots$$

Local Lorentz symmetry $\to$ local Carroll symmetry + corrections

Developed for ultra-local expansion of GR in [Hansen, Obers, GO, Søgaard]
following non-relativistic expansion in [Hansen, Hartong, Obers]





Carroll geometry from Lorentzian

Carroll connection $\tilde{\Gamma}^{\rho}_{\mu\nu}$ can be obtained from Levi-Civita connection,

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{c^2} S_{(-2)}{}^{\rho}{}_{\mu\nu} + \tilde{C}^{\rho}_{\mu\nu} + S_{(0)}{}^{\rho}{}_{\mu\nu} + c^2 S_{(2)}{}^{\rho}{}_{\mu\nu}\,,$$

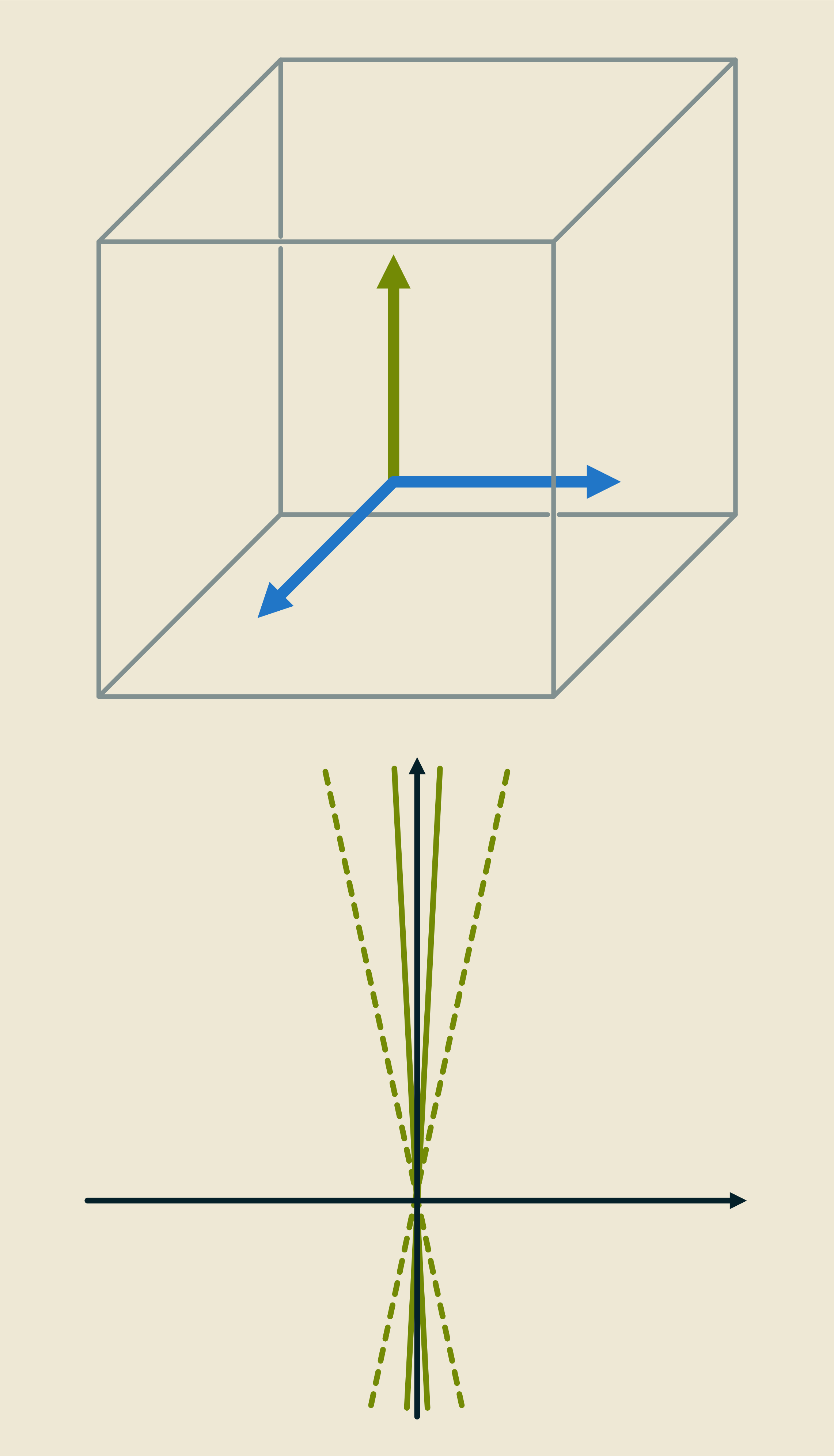
where the $S^{\rho}{}_{\mu\nu}$ are known tensors and $\tilde{C}^{\rho}_{\mu\nu} = \tilde{\Gamma}^{\rho}_{\mu\nu} + \cdots$

Then Levi-Civita Ricci scalar is

$$R = \frac{1}{c^2}\left[\mathscr{K}^{\mu\nu}\mathscr{K}_{\mu\nu} + \mathscr{K}^2 - 2V^{\mu}\partial_{\mu}\mathscr{K}\right] + \Pi^{\mu\nu}\tilde{R}_{\mu\nu} - \tilde{\nabla}_{\mu}A^{\mu} + c^2\Pi^{\mu\rho}\Pi^{\nu\sigma}\partial_{[\mu}T_{\nu]}\partial_{[\rho}T_{\sigma]}$$

where $\mathscr{K}_{\mu\nu} = K_{\mu\nu} + \cdots$ and $A_{\mu} = a_{\mu} + \cdots$

Finally, $\sqrt{-g} = cE = ce + \cdots$ where $E = \det(T_{\mu}, \Pi_{\mu\nu})$ and $e = \det(\tau_{\mu}, h_{\mu\nu})$



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  - timelike
  - spacelike

Conformal scalar actions: timelike

Now can rewrite Lorentzian conformal scalar action,

$$S = -\frac{1}{2}\int d^dx\sqrt{-g}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{d-2}{4(d-1)}R\phi^2\right)$$

$$= -\frac{c}{2}\int d^dx\,E\left[\left(-\frac{1}{c^2}V^\mu V^\nu + \Pi^{\mu\nu}\right)\partial_\mu\phi\partial_\nu\phi\right.$$

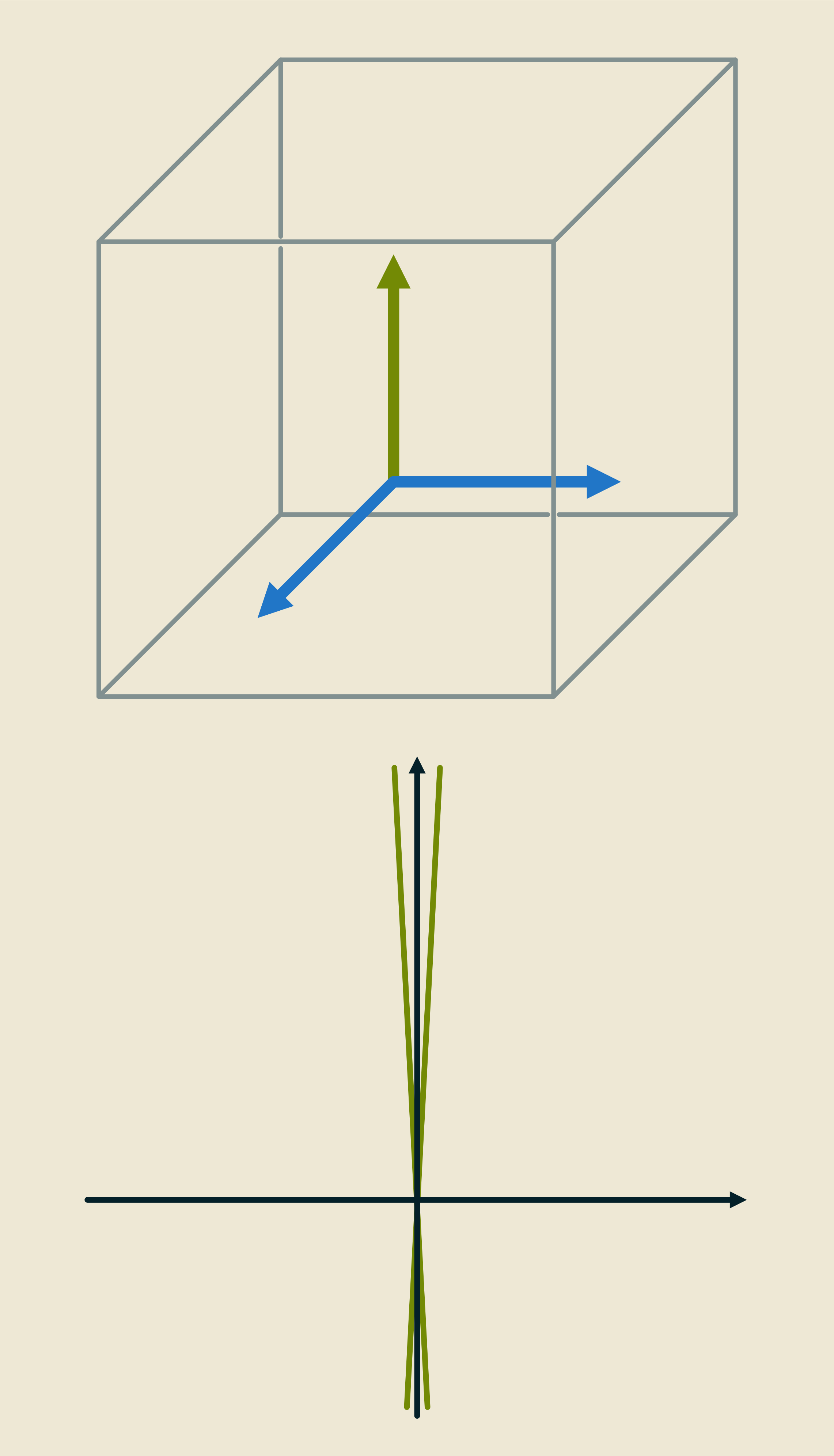
$$\left.+\frac{d-2}{4(d-1)}\left(\frac{1}{c^2}\left[\mathscr{K}^{\mu\nu}\mathscr{K}_{\mu\nu} + \mathscr{K}^2 - 2V^\mu\partial_\mu\mathscr{K}\right] + \Pi^{\mu\nu}\tilde{R}_{\mu\nu} - \tilde{\nabla}_\mu A^\mu + c^2\Pi^{\mu\rho}\Pi^{\nu\sigma}\partial_{[\mu}T_{\nu]}\partial_{[\rho}T_{\sigma]}\right)\phi^2\right]$$

In Carroll limit $c \to 0$, leading-order terms give [Baiguera, GO, Sybesma, Søgaard]

$$S_t = -\frac{1}{2}\int d^dx\,e\left[-(v^\mu\partial_\mu\phi)^2 + \frac{(d-2)}{4(d-1)}\left(K^{\mu\nu}K_{\mu\nu} + K^2 - 2v^\mu\partial_\mu K\right)\phi^2\right]$$

This is timelike conformal Carroll scalar.

Also considered from different perspective in [Gupta, Suryanarayana] [Rivera-Betancour, Vilatte]



Conformal scalar actions: timelike

Obtained timelike conformal Carroll scalar from leading-order terms,

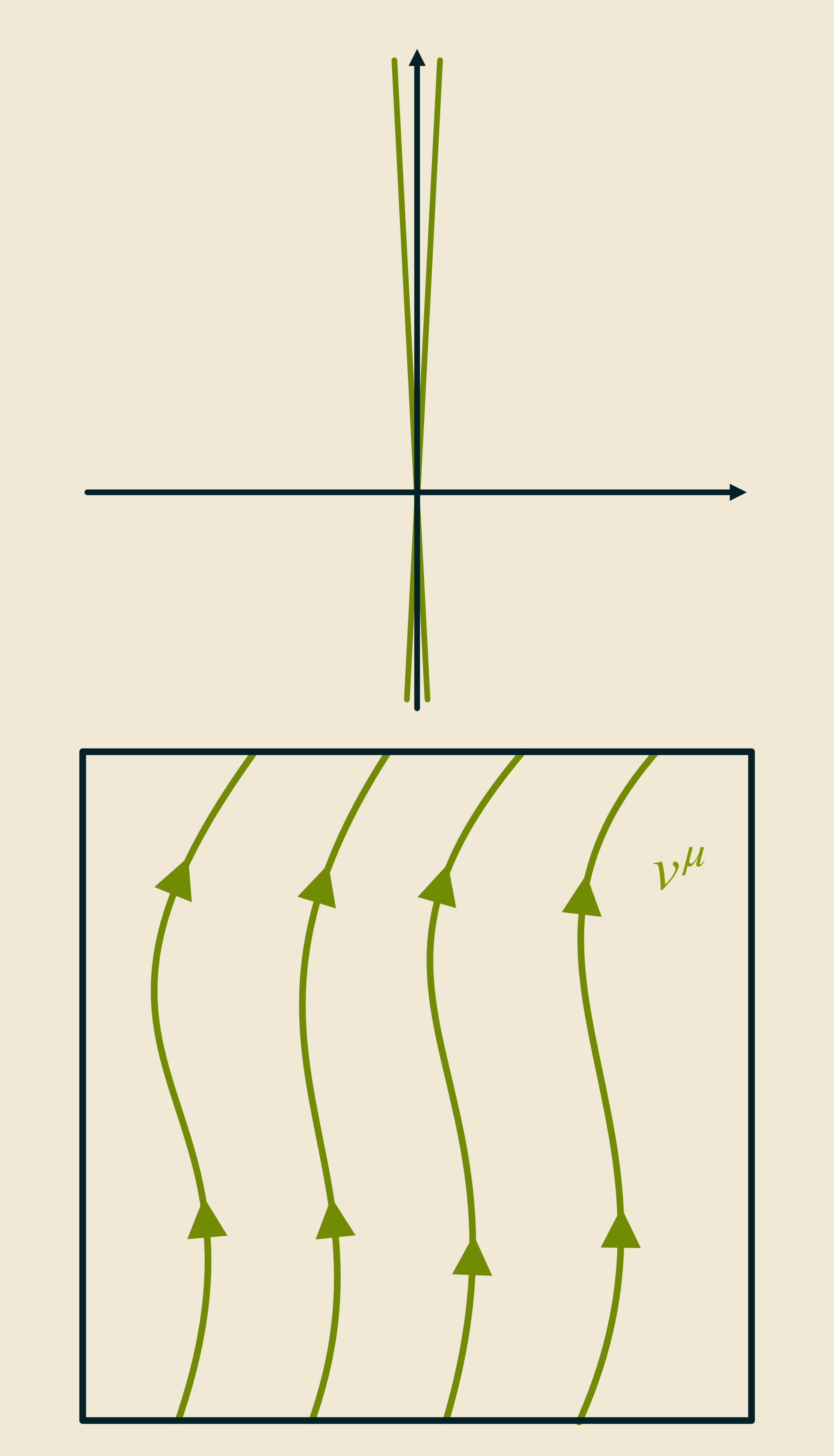
$$S_t = -\frac{1}{2}\int d^d x\, e \left[-(v^\mu \partial_\mu \phi)^2 + \frac{(d-2)}{4(d-1)} \left(K^{\mu\nu} K_{\mu\nu} + K^2 - 2 v^\mu \partial_\mu K \right) \phi^2 \right]$$

- invariant under local Carroll boosts and Weyl transformations
- energy-momentum tensor satisfies $T^i{}_0 = 0$ and $T^\mu{}_\mu = 0$

Flat space propagator $\sim t\,\delta(x)$ of timelike form

Reduction along Carroll time results in CCFT_2-like behavior

- for correlators [Bagchi, Banerjee, Basu, Dutta]
- for energy-momentum tensor?
- build on related work on BMS currents [Donnay, Fiorucci, Herfray, Ruzziconi]



Conformal scalar actions: spacelike

From rewritten Lorentzian conformal scalar action,

$$S = -\frac{1}{2}\int d^d x \sqrt{-g}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{d-2}{4(d-1)}R\phi^2\right)$$

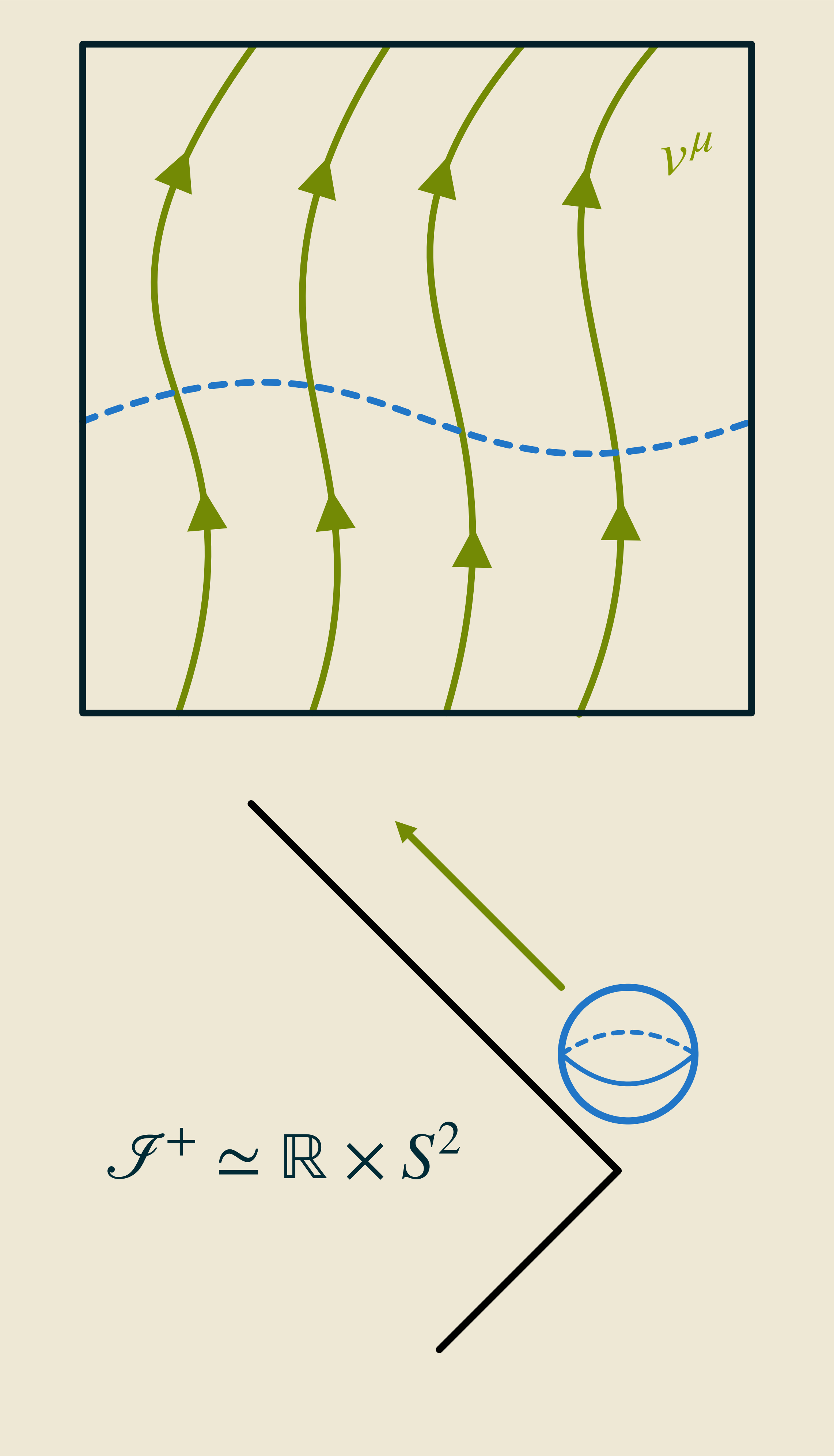
$$= -\frac{c}{2}\int d^d x\, E\left[\left(-\frac{1}{c^2}V^\mu V^\nu + \Pi^{\mu\nu}\right)\partial_\mu\phi\partial_\nu\phi + \frac{d-2}{4(d-1)}\left(\frac{1}{c^2}\left[\mathscr{K}^{\mu\nu}\mathscr{K}_{\mu\nu} + \mathscr{K}^2 - 2V^\mu\partial_\mu\mathscr{K}\right] + \Pi^{\mu\nu}\tilde{R}_{\mu\nu} - \tilde{\nabla}_\mu A^\mu + c^2\Pi^{\mu\rho}\Pi^{\nu\sigma}\partial_{[\mu}T_{\nu]}\partial_{[\rho}T_{\sigma]}\right)\phi^2\right]$$

can take alternative Carroll limit $c \to 0$ using Lagrange multipliers,

$$S_s = -\frac{1}{2}\int d^d x\, e\left[h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{(d-2)}{4(d-1)}\left(h^{\mu\nu}\tilde{R}_{\mu\nu} - \tilde{\nabla}_\mu a^\mu\right) + \chi\left(v^\mu\partial_\mu\phi + \frac{(d-2)}{4(d-1)}K\right) + \chi^{\mu\nu}\breve{K}_{\mu\nu}\phi\right]$$

This is spacelike conformal Carroll scalar. [Baiguera, GO, Sybesma, Søgaard]

- invariant under boosts and Weyl transformations
- extrinsic curvature must be pure trace $K_{\mu\nu} = \frac{h_{\mu\nu}}{d-1}K$
- time-dependence $v^\mu\partial_\mu\phi$ is fixed, so only spacelike dynamics



Conformal scalar actions: spacelike

Spacelike conformal Carroll scalar from next-to-leading-order terms,

$$S_s = -\frac{1}{2}\int d^d x\, e \left[h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{(d-2)}{4(d-1)}\left(h^{\mu\nu}\tilde{R}_{\mu\nu} - \tilde{\nabla}_\mu a^\mu\right)\phi^2 + \chi\left(v^\mu\partial_\mu\phi + \frac{(d-2)}{4(d-1)}K\right) + \chi^{\mu\nu}\check{K}_{\mu\nu}\phi\right]$$

- invariant under local Carroll boosts and Weyl transformations
- energy-momentum tensor satisfies ${T^i}_0 = 0$ and ${T^\mu}_\mu = 0$

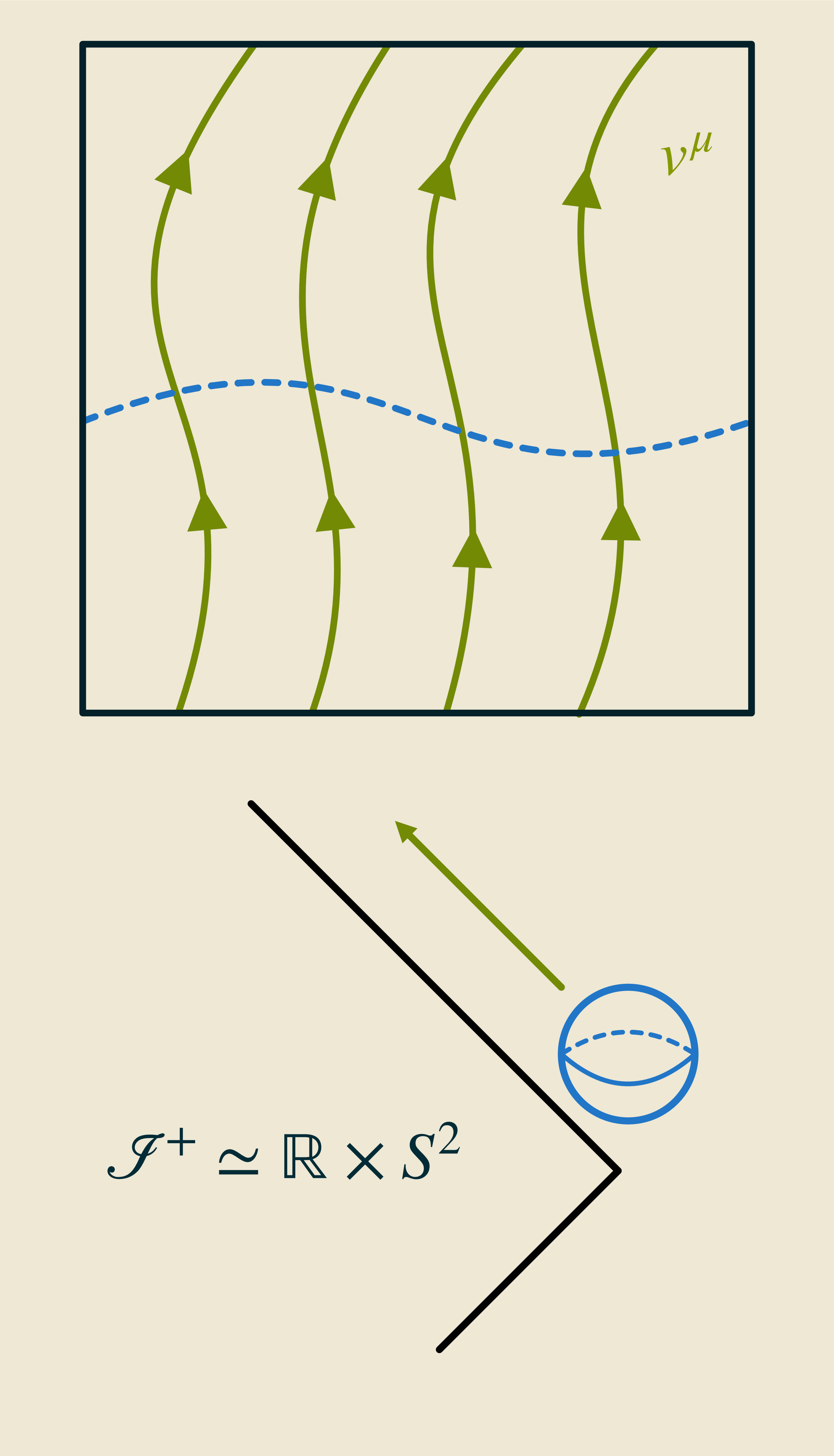
Flat space propagator $\sim \log(x)^2$ of spacelike Euclidean free boson

Remarkably, can dimensionally reduce the action explicitly using constraints,

$$S_s = -\frac{1}{2}\int d^{d-1}x\sqrt{h}\left(h^{ij}\partial_i\hat{\phi}\partial_j\hat{\phi} + \frac{(d-3)}{4(d-2)}\hat{R}\hat{\phi}^2 + \frac{1}{4(d-1)(d-2)}A^{-2}\hat{R}_{A^{-2}h_{ij}}\right)$$

where $\hat{\phi} = A^{1/2}\phi$ and the background field $A = \int_v \tau$ encodes former 'Carroll time'

but otherwise this is $(d-1)$-dimensional Euclidean conformal scalar!



Conformal scalar actions: spacelike

Dimensional reduction to Euclidean theory

$$S_s = -\frac{1}{2}\int d^d x\, e \left[h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{(d-2)}{4(d-1)}\left(h^{\mu\nu}\tilde{R}_{\mu\nu} - \tilde{\nabla}_\mu a^\mu\right)\phi^2 + \chi\left(v^\mu\partial_\mu\phi + \frac{(d-2)}{4(d-1)}K\right) + \chi^{\mu\nu}\breve{K}_{\mu\nu}\phi\right]$$
$$= -\frac{1}{2}\int d^{d-1}x\sqrt{h}\,\left(h^{\mu\nu}\partial_\mu\hat{\phi}\partial_\nu\hat{\phi} + \frac{(d-3)}{4(d-2)}\hat{R}\hat{\phi}^2 + \frac{1}{4(d-1)(d-2)}A^{-2}\hat{R}_{A^{-2}h_{ij}}\right)$$

Reminiscent of embedding space formalism!

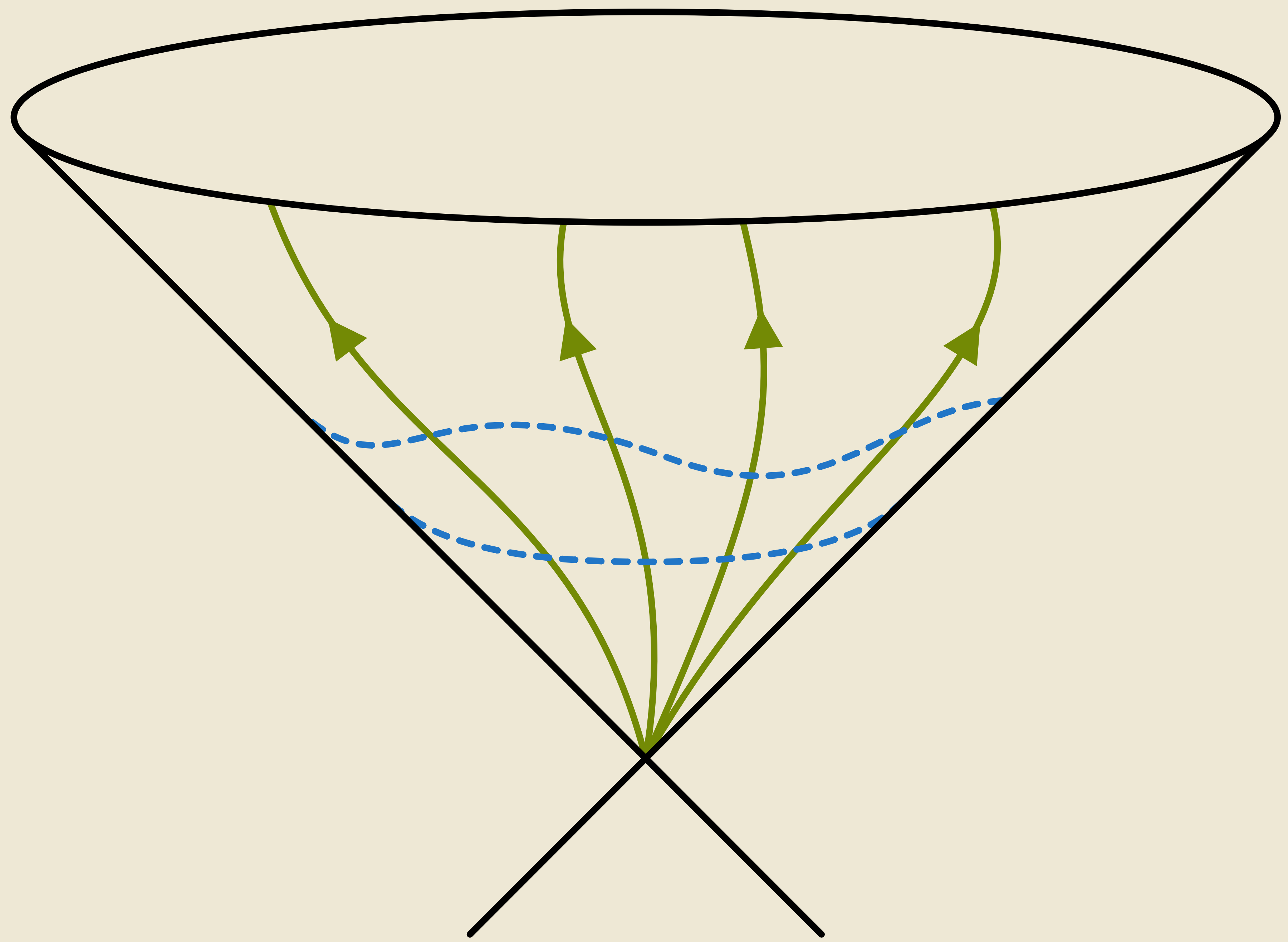
Get $(d-1)$-dim conformal $SO(d,1)$ representations from $(d+1)$-dim Lorentz representations in $\mathbb{R}^{1,d}$

Restriction to light cone

$\implies$ Carrollian spacelike theory

$\implies$ Euclidean theory

Similar procedure for other spacelike Carroll theories?



Conformal Carroll anomalies

Can geometrically classify all possible Weyl anomalies

In Lorentzian case, find $\langle T^\mu{}_\mu \rangle = \begin{cases} -\frac{c}{24\pi} R & d = 2, \\ aE_4 - cW^2 & d = 4, \\ & \vdots \end{cases}$

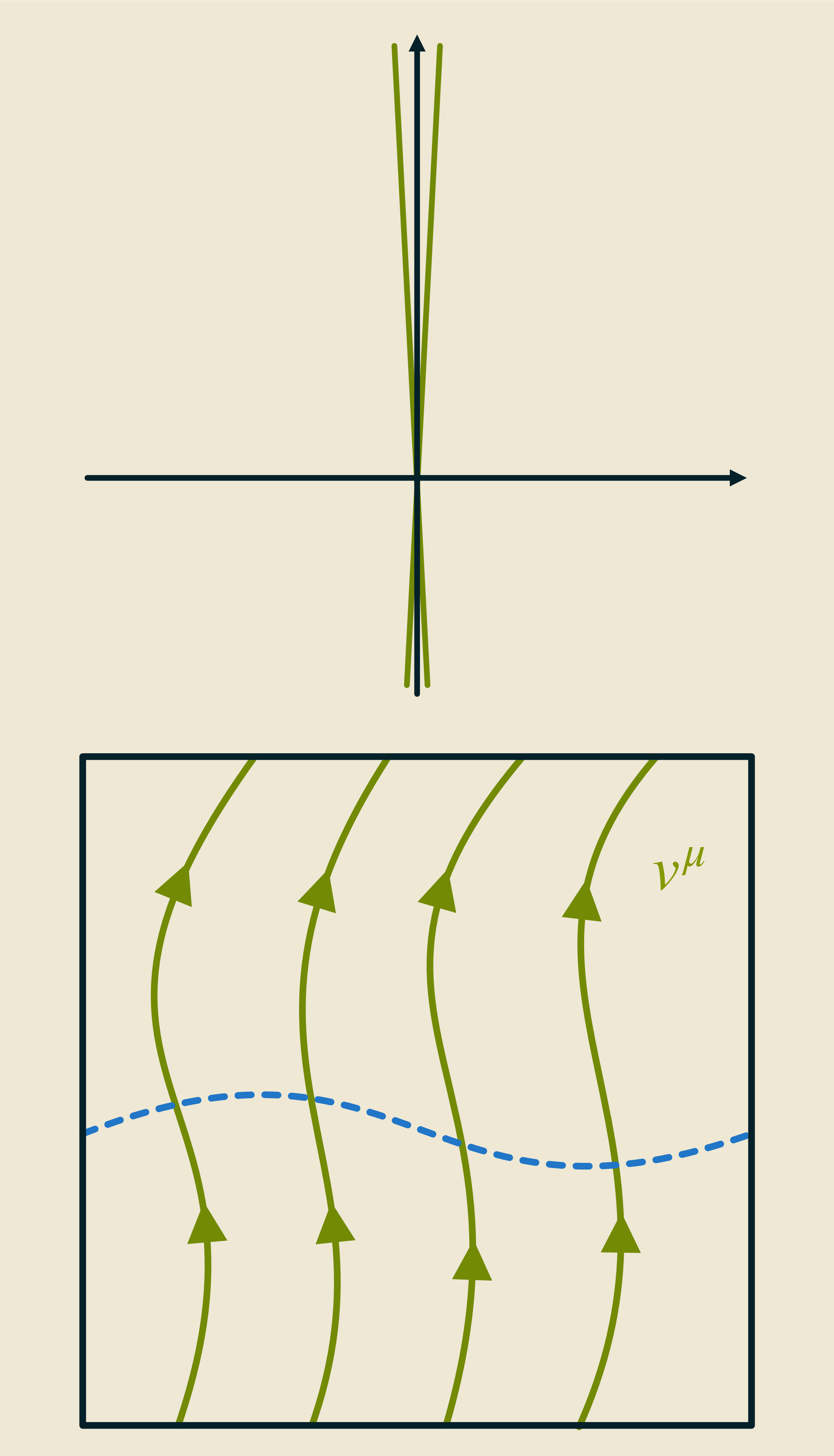
In Carrollian case, have different connection and curvature, so other invariants

Timelike anomalies: using only v^μ , $h_{\mu\nu}$, $\mathscr{L}_v$ and $K_{\mu\nu}$ get

$$\langle T^\mu{}_\mu \rangle = \begin{cases} \varnothing & d = 2 \\ \varnothing & d = 3 \\ b_1 \left(-7\mathrm{Tr}(K^4) + \frac{1}{3} K \mathrm{Tr}(K^3) + \mathrm{Tr}(K^2)(\mathscr{L}_v K) + (\mathscr{L}_v K)^2 \right) + b_2(\cdots) & d = 4, \\ & \vdots \end{cases}$$

[Baiguera, GO, Sybesma, Søgaard] [Arav, Chapman, Oz]

No 3d Carroll anomalies $\implies$ no 2d CCFT anomalies $\sim$ celestial $T_{\mu\nu}$ not renormalized?



Summary and outlook

Constructed timelike and spacelike conformal Carroll scalar actions

Enables direct computations using only basic QFT techniques

Ongoing and future work:

- direct computation of scalar anomalies
- complete general anomaly classification
- build conformal Carroll $\Longleftrightarrow$ CCFT dictionary
  - map energy-momentum tensor
  - reduce (quantum) timelike theory?

Top-down flat holography from $c \to 0$ limit of AdS/CFT?