The case for non-Lorentzian strings

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NBI mini workshop: What is new in gravity?

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Outline

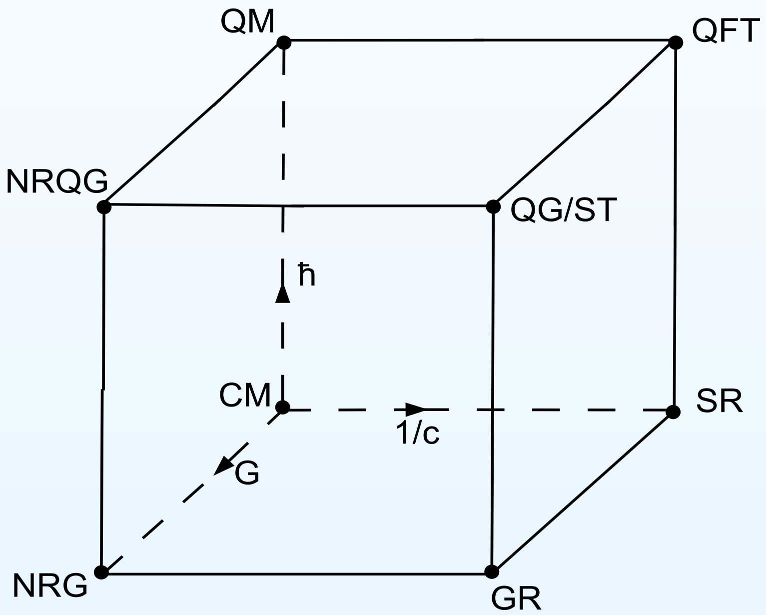
- Introduction and motivation
- Non-Lorentzian geometries
- Non-Lorentzian Strings
- Outlook

Introduction

- Gravity and quantum mechanics clash.
- Usual approach: study e.g. black holes and graviton scattering
- Question: from a low-energy bottom-up perspective, when do we start seeing a clash and why?

Introduction

- Bronstein cube



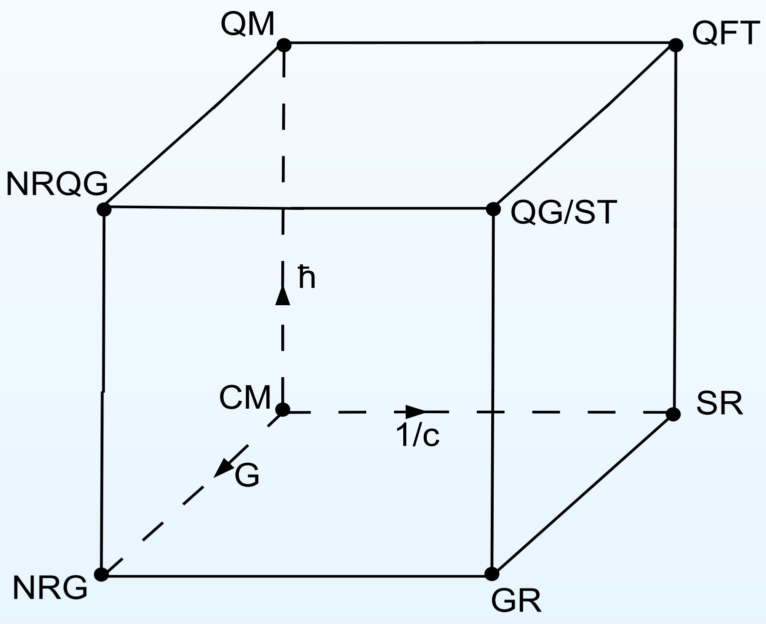
- NRG = non-relativistic gravity ⊃ Newtonian gravity

- NRQG = non-relativistic quantum gravity

- NRQG is a bit of a misnomer. There is no dynamical gravity to quantise. Think of quantum matter (described by quantum mechanics) backreacting with a background that reacts instantaneously.

Introduction

- Is there a well-defined non-relativistic limit of quantum gravity/string theory?



- In string theory there exist dualities between QG and specific QFTs (holography, AdS/CFT, ...).
- Do such dualities exist in the NR domain, i.e. can NR strings be dual to some NR field theory?

- Limits of AdS/CFT called Spin Matrix Theory give rise to NR strings dual to quantum mechanical limits of AdS/CFT. See e.g. [Harmark, Kristjansson, Orselli, 2007/8], [Harmark, JH, Obers, 2017].

Introduction

- Many of the recent developments in NR gravity, and NR string theory rely on an improved understanding of NR geometry.
- The most common example of a NR geometry is Newton–Cartan geometry which is the arena of every day life.
- Many other NR geometries have been found: type II Newton–Cartan, string Newton–Cartan, Aristotelian, Carrollian geometries, ...
- NR geometries have found applications in non-AdS holography (boundary geometries), fluid dynamics, condensed matter physics, Hořava–Lifshitz gravity, 2D/3D gravity, ...

Non-Lorentzian geometries in a nutshell

- Nowhere vanishing 1-forms (vielbeine) τ^A, e^a that transform linearly under some group $G \supset SO(1,p) \times SO(d-p)$. ($A = 0, 1, \ldots, p$ and $a = p+1, \ldots, d$)

- G is typically a subgroup or a contraction of the Lorentz group $SO(1,d)$.

- Typically the geometry comes equipped with additional gauge fields that also transform under G.

- Newton–Cartan geometry: τ, e^a ($a = 1, \ldots, d$), m with $G = \mathbb{R}^d \rtimes SO(d)$ acting as

$$\tau' = \tau\,, \qquad e'^a = R^a{}_b e^b + \Lambda^a \tau\,, \qquad m' = m + \delta_{ab}\Lambda^a e^b + \frac{1}{2}\delta_{ab}\Lambda^a\Lambda^b\tau$$

- m is also a gauge field transforming as $m' = m + d\sigma$.

Newton–Cartan gravity

- Using a suitable G-invariant affine connection $\Gamma^{\rho}_{\mu\nu}$, with Ricci tensor $R_{\mu\nu}$, Newtonian gravity can be written covariantly as

$$R_{\mu\nu} = 8\pi G \frac{d-2}{d-1} \rho \tau_\mu \tau_\nu \,, \qquad d\tau = 0$$

- This follows from expanding Einstein's equations sourced by a point particle with mass density ρ in $1/c$ using

$$g_{\mu\nu} = -c^2 \tau_\mu \tau_\nu + \delta_{ab} e^a_\mu e^b_\nu - m_\mu \tau_\nu - m_\nu \tau_\mu + O(c^{-2})$$

[Dautcourt, 1996], [Van den Bleeken, 2017], [Hansen, JH, Obers, 2018-20]

- An action principle for Newtonian gravity was found in [Hansen, JH, Obers, 2018]

A very incomplete history of non-Lorentzian strings

- Limits of closed bosonic strings with a flat target spacetime: [Gomis, Ooguri, 2000] and 2x [Danielsson, Güijosa, Kruczenski, 2000]
- String Newton–Cartan (SNC) geometry: [Andringa, Bergshoeff, Gomis, de Roo, 2012]
- Null reduction: [Harmark, JH, Obers, 2017]
- Strings in SNC backgrounds from limit: [Bergshoeff, Gomis, Yan, 2018]
- Limits and null reductions are equivalent: [Harmark, JH, Menculini, Obers, Oling, 2019]
- Strings with NR worldsheets & spin matrix limit of $\mathcal{N} = 4$ SYM: [Harmark, JH, Obers, 2017]; [Harmark, Menculini, JH, Obers, Oling, Yan, 2018]
- For a nice and recent review article see [Oling, Yan, 2022]

The $1/c$ expansion of closed bosonic strings

[JH, Have, 2021 and to appear]

- The $1/c$ expanded closed bosonic string theory follows from

$$
\begin{aligned}
\mathcal{L}_{\text{NG}} &= -Tc\left(\sqrt{-\det g_{\alpha\beta}} + \frac{1}{2}\epsilon^{\alpha\beta}B_{\alpha\beta}\right) \\
g_{\mu\nu} &= c^2\left(-T_\mu^0 T_\nu^0 + T_\mu^1 T_\nu^1\right) + \delta_{ab}E_\mu^a E_\nu^b\,, \qquad a = 2,\ldots,d \\
T_\mu^A &= \tau_\mu^A + \frac{1}{c^2}m_\mu^A + \mathcal{O}(c^{-4})\,, \qquad E_\mu^a = e_\mu^a + \mathcal{O}(c^{-2}) \\
B_{\mu\nu} &= c^2 B_{(-2)\mu\nu} + B_{(0)\mu\nu} + \mathcal{O}(c^{-2}) \\
X^\mu &= x^\mu + \frac{1}{c^2}y^\mu + \mathcal{O}(c^{-4})
\end{aligned}
$$

- Expanding the NG Lagrangian leads to a cascade of theories

$$
\mathcal{L}_{\text{NG}} = c^2\mathcal{L}_{\text{NG-LO}} + \mathcal{L}_{\text{NG-NLO}} + \mathcal{O}(c^{-2})
$$

- @LO and @NLO we find (with $T_{\text{eff}} = cT$)

$$\mathcal{L}_{\text{NG-LO}} = -T_{\text{eff}}\left[\sqrt{-\tau} + \frac{1}{2}\epsilon^{\alpha\beta}B_{(-2)\alpha\beta}\right]$$

$$\mathcal{L}_{\text{NG-NLO}} = -\frac{T_{\text{eff}}}{2}\left[\sqrt{-\tau}\tau^{\alpha\beta}H_{\alpha\beta} + \epsilon^{\alpha\beta}B_{(0)\alpha\beta}\right] + y^\mu \frac{\delta\mathcal{L}_{\text{NG-LO}}}{\delta x^\mu}$$

- $\tau_{\alpha\beta}$ and $H_{\alpha\beta}$ are pullbacks (using x^μ) of

$$\tau_{\mu\nu} = \eta_{AB}\tau_\mu^A\tau_\nu^B = g_{\mu\nu}|_{\mathcal{O}(c^2)}$$

$$H_{\mu\nu} = \delta_{ab}e_\mu^a e_\nu^b + 2\eta_{AB}\tau^A_{(\mu}m^B_{\nu)} = g_{\mu\nu}|_{\mathcal{O}(c^0)}$$

- The Polyakov action can be obtained by expanding $\mathcal{L}_{\text{P}} = -\frac{cT}{2}\left(\sqrt{-\gamma}\gamma^{\alpha\beta}g_{\alpha\beta} + \epsilon^{\alpha\beta}B_{\alpha\beta}\right)$ where

$$\gamma_{\alpha\beta} = \gamma_{(0)\alpha\beta} + c^{-2}\gamma_{(2)\alpha\beta} + \mathcal{O}(c^{-4})$$

- $\gamma_{(0)\alpha\beta}$ is proportional to $\tau_{\alpha\beta}$ which is Lorentzian (by assumption).

- The Polyakov Lagrangian acquires the following expansion

$$\mathcal{L}_{\text{P}} = c^2 \mathcal{L}_{\text{P-LO}} + \mathcal{L}_{\text{P-NLO}} + \mathcal{O}(c^{-2})\,,$$

where

$$\begin{aligned}
\mathcal{L}_{\text{P-LO}} &= -\frac{T_{\text{eff}}}{2}\left[\sqrt{-\gamma_{(0)}}\,\gamma_{(0)}^{\alpha\beta}\tau_{\alpha\beta} + \epsilon^{\alpha\beta}B_{(-2)\alpha\beta}\right] \\
\mathcal{L}_{\text{P-NLO}} &= -\frac{T_{\text{eff}}}{2}\left[\sqrt{-\gamma_{(0)}}\left(\gamma_{(0)}^{\alpha\beta}H_{\alpha\beta} - \frac{1}{2}G_{(0)}^{\alpha\beta\gamma\delta}\tau_{\alpha\beta}\gamma_{(2)\gamma\delta}\right) + \epsilon^{\alpha\beta}B_{(0)\alpha\beta}\right] \\
&\quad + y^M \frac{\delta\mathcal{L}_{\text{P-LO}}}{\delta x^M}
\end{aligned}$$

with $G_{(0)}^{\alpha\beta\gamma\delta} = \gamma_{(0)}^{\alpha\gamma}\gamma_{(0)}^{\delta\beta} + \gamma_{(0)}^{\alpha\delta}\gamma_{(0)}^{\gamma\beta} - \gamma_{(0)}^{\alpha\beta}\gamma_{(0)}^{\gamma\delta}$.

- A not entirely straightforward rewriting shows that

$$\begin{aligned}\mathcal{L}_{\text{P-NLO}} &= -\frac{T_{\text{eff}}}{2}\left[\sqrt{-\gamma_{(0)}}\gamma_{(0)}^{\alpha\beta}H_{\alpha\beta}+\epsilon^{\alpha\beta}B_{(0)\alpha\beta}\right] \\ &\quad -\frac{T_{\text{eff}}}{2}\epsilon^{\alpha\beta}\left[\lambda_{++}e_\alpha^+\tau_\beta^+ + \lambda_{--}e_\alpha^-\tau_\beta^-\right]+y^M\frac{\delta\mathcal{L}_{\text{NG-LO}}}{\delta x^M}\end{aligned}$$

where $\tau_\alpha^\pm = \tau_\alpha^0 \pm \tau_\beta^1$ and $\gamma_{(0)\alpha\beta} = -(e_\alpha^+ e_\beta^- + e_\alpha^- e_\beta^+)/2$.

- λ_{++} and λ_{--} are Lagrange multipliers that are built out of $\gamma_{(2)\alpha\beta}$ and y^M-dependent terms.

- This is equivalent to the string-Newton-Cartan string, i.e. the Gomis–Ooguri string on a general curved target spacetime, [Harmark, JH, Menculini, Obers, Oling, 2019] iff $\frac{\delta\mathcal{L}_{\text{NG-LO}}}{\delta x^M}=0$, i.e.

$$\epsilon^{\alpha\beta}\partial_\alpha x^K\partial_\beta x^L\left(d\left[\frac{1}{2}\epsilon_{AB}\tau^A\wedge\tau^B+B_{(-2)}\right]\right)_{MKL}=0$$

- Take a flat spacetime $S^1 \times \mathbb{R}^{1,24}$ with a circle of radius cR_{eff} parametrised by v: $\tau^0 = dt$, $\tau^1 = dv$, $H_{\mu\nu}dx^\mu dx^\nu = dx^i dx^i$
- The string winds $w > 0$ times around S^1.
- For a flat metric $\gamma_{(0)\alpha\beta} = \eta_{\alpha\beta}$, the Virasoro constraints are

LO: $$\partial_+ x^+ \partial_+ x^- = 0\,, \qquad \partial_- x^+ \partial_- x^- = 0$$

NLO: $$\partial_+ y^- = \frac{1}{wR_{\text{eff}}}\partial_+ x^i \partial_+ x^i\,, \qquad \partial_- y^+ = \frac{1}{wR_{\text{eff}}}\partial_- x^i \partial_- x^i$$

where $x^\pm = x^t \pm x^v$ and $y^\pm = y^t \pm y^v$.

- After fixing residual gauge transformations we find

$$\begin{aligned}
x^\pm &= x_0^\pm + wR_{\text{eff}}\left(\sigma^0 \pm \sigma^1\right) \\
x^i &= x_0^i + \frac{1}{2\pi T_{\text{eff}}}p_i\sigma^0 + \frac{1}{\sqrt{4\pi T_{\text{eff}}}}\sum_{k\neq 0}\frac{i}{k}\left[\alpha_k^i e^{-ik\sigma^-} + \tilde{\alpha}_k^i e^{-ik\sigma^+}\right] \\
y^\pm &= y_0^\pm + \frac{1}{2\pi T_{\text{eff}}}\left(E_{NLO} \pm p_v\right)\sigma^0 + \frac{1}{\sqrt{4\pi T_{\text{eff}}}}\sum_{k\neq 0}\frac{i}{k}\beta_k^\pm(\alpha)e^{-ik\sigma^\mp}
\end{aligned}$$

- For a flat target space $S^1 \times \mathbb{R}^{1,24}$ with radius cR_{eff} and constant B field $B = c^2\lambda dt \wedge dv$ where $v \sim v + 2\pi R_{\text{eff}}$, the energy of the string with tension $T_{\text{eff}} = cT$ and nonzero winding w is [JH, Have, 2021]

$$
\begin{aligned}
E &= c^2 E_{\text{LO}} + E_{\text{NLO}} + \mathcal{O}(c^{-2}) \\
&= -c^2 \oint d\sigma^1 \frac{\partial \mathcal{L}_{\text{P-LO}}}{\partial \dot{x}^t} - \oint d\sigma^1 \frac{\partial \mathcal{L}_{\text{P-NLO}}}{\partial \dot{x}^t} + \mathcal{O}(c^{-2}) \\
&= (1-\lambda)c^2 \frac{wR_{\text{eff}}}{\alpha'_{\text{eff}}} + \frac{\alpha'_{\text{eff}}}{2wR_{\text{eff}}} p_i p_i + \frac{N + \bar{N} - 2\hbar}{wR_{\text{eff}}} + \mathcal{O}(c^{-2})
\end{aligned}
$$

where $\alpha'_{\text{eff}} = \frac{1}{2\pi T_{\text{eff}}}$ and where $p_i = \oint d\sigma^1 \frac{\partial \mathcal{L}_{\text{P-NLO}}}{\partial \dot{x}^i}$.

- Using $\frac{\partial \mathcal{L}_{\text{P-LO}}}{\partial \dot{x}^t} = \frac{\partial \mathcal{L}_{\text{P-NLO}}}{\partial \dot{y}^t}$ we can compute the energy up to $\mathcal{O}(c^{-2})$ entirely from $\mathcal{L}_{\text{P-NLO}}$.
- Level matching condition: $p_v = \oint d\sigma^1 \frac{\partial \mathcal{L}_{\text{P-NLO}}}{\partial \dot{x}^v} = \frac{\hbar n}{R_{\text{eff}}} = \frac{N - \bar{N}}{wR_{\text{eff}}}$

- Global symmetries of $\mathcal{L}_{\text{P-NLO}}$

$$\mathcal{L}_{\text{P-NLO}} = -\frac{T_{\text{eff}}}{2}\eta^{\alpha\beta}\partial_\alpha x^i \partial_\beta x^i - T_{\text{eff}}\eta_{AB}\eta^{\alpha\beta}\partial_\alpha x^A \partial_\beta y^B$$

where we used the gauge $\gamma_{(0)\alpha\beta} = \eta_{\alpha\beta}$ and $\gamma_{(2)\alpha\beta} = 0$, are

$$\begin{aligned}
\delta x^A &= \lambda^A_{(0)B} x^B + a^A_{(0)} \\
\delta x^i &= \lambda^i_{(0)j} x^j + \lambda^i_{(0)A} x^A + a^i_{(0)} \\
\delta y^A &= \lambda^A_{(0)B} y^B + \lambda^A_{(2)B} x^B + \lambda^A_{(0)i} x^i + a^A_{(2)}
\end{aligned}$$

- They form the string Bargmann algebra [Bergshoeff, Gomis, Rosseel, Şimşek, Yan, 2019].

- Canonical quantisation and NNLO theory [JH, Have, to appear].

- To do: Spectrum and irreps of the string Bargmann algebra, critical dimension via quantum string Bargmann symmetry.

- The beta functions have been computed and lead to non-relativistic theories of gravity coupled to NSNS-type fields [Yan, Yu, 2019] and [Gallegos, Gürsoy, Zinnato, 2019], but no action is known. Is it the $1/c$ expansion of NSNS gravity?

- Spectrum of quantised modes consists of massive fields (nonzero winding).

- In string scattering we have to include zero winding states as off shell fields (these lead to instantaneous forces mediated by string Newton–Cartan background fields) [Danielsson, Güijosa, Kruczenski, 2001].

- Currently we are looking into $1/c$ expansions of non-relativistic open strings, T-duality and NR D-branes. These have NR gauge theories on the worldvolume [Gomis, Yan, Yu, 2020].

Strings with non-relativistic worldsheets

- The $1/c$ expansion for a fine-tuned B-field (such that the LO Lagrangian cancels) leads to the torsional string Newton–Cartan string given by [Bidussi, Harmark, JH, Obers, Oling, 2021]

$$\mathcal{L} = -\frac{T_{\text{eff}}}{2}\left[\sqrt{-\tau}\tau^{\alpha\beta}h_{\alpha\beta} + \epsilon^{\alpha\beta}m_{\alpha\beta}\right]$$

  where $h_{\alpha\beta}$ is the pullback of $\delta_{ab}e^a_\mu e^b_\nu$ and $m_{\alpha\beta}$ is the pullback of a combination of $B_{(0)\mu\nu}$ and m^A_μ.

- Consider the scaling limit for $c \to \infty$

$$T_{\text{eff}} = \tilde{T}/c\,, \quad \tau^0_M = c^2\tilde{\tau}^0_M\,, \quad \tau^1_M = c\,\tilde{\tau}^1_M\,, \quad h_{MN} = \tilde{h}_{MN}\,, \quad m_{MN} = c\,\tilde{m}_{MN}$$

  which (dropping tildes) leads to

$$\mathcal{L} = -\frac{T}{2}\left[\sqrt{-\tau}\tau^\alpha_1\tau^\beta_1 h_{\alpha\beta} + \epsilon^{\alpha\beta}m_{\alpha\beta}\right]$$

$\tau^\tau_1 \neq 0$: wrong sign kinetic terms

[Bidussi, Harmark, JH, Obers, Oling, to appear]

- Impose the gauge condition $\tau_\sigma^0 = 0$ with a Lagrange multiplier.
- Typical target space: $\mathbb{R}_{\text{time}} \times S^1 \times \mathcal{M}^{2n}$ with coordinates t, v, x^i and geometric data

$$\tau^0 = dt\,, \qquad \tau^1 = dv\,, \qquad h_{\mu\nu}dx^\mu dx^\nu = h_{ij}dx^i dx^j\,, \qquad m = a_i dx^i \wedge dv$$

- $\mathcal{M}^{2n}$ is symplectic (2-form da) and Riemannian (metric h_{ij}).

$$\mathcal{L} = -T a_i \left(X'^v \dot{X}^i - \dot{X}^v X'^i \right) - \frac{T}{2} \frac{\dot{X}^t}{X'^v} h_{ij} X'^i X'^j - \frac{T}{2} \omega X'^t$$

This is a NR sigma model with only first order time-derivatives.

- Agrees with the spin matrix limit of strings on AdS$_5 \times S^5$.
- There also exists a Carrollian analogue obtained by taking a different scaling limit.

Outlook

	standard string theory	Gomis–Ooguri string theory	particle-string	Carrollian string
target spacetime geometry	Lorentzian	string Newton–Cartan	limit of string Newton–Cartan	limit of string Newton–Cartan
worldsheet	Lorentzian	Lorentzian	Galilean	Carrollian
worldsheet algebra of first class constraints	Virasoro$\oplus$Virasoro	Virasoro$\oplus$Virasoro	Virasoro$\oplus U(1)$	BMS

- How rich is the landscape of non-Lorentzian string theories? Are there holographic dualities?
- Open strings with NR worldsheets?
- Interpretation of the Carrollian strings (new limit of strings on AdS$_5 \times S^5$).