Expanding Universe and dynamics

The Universe is assumed to be homogeneous and isotropic. No preferential locations and no preferential directions.

We have to define a metric for such a universe, especially for its spatial part (dl^2):

$$ds^2 = c^2dt^2 - dl^2 \tag{1}$$

Given our assumptions, a spherical coordinate system is an appropriate one:

$$dl^2 = \left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi\right)\right] \tag{2}$$

where k is a scalar (the curvature) and we will discuss it later.
We also know that our Universe is expanding, so we want to introduce a stretching factor in front of the metric, which will be time dependent:

$$dl^2 = a^2(t)\left[...\right]. \tag{3}$$

The expansion factor

Let's now try to understand what is the meaning of the **expansion factor** $a(t)$. Let's supposed a signal is emitted at t_e, r_e and received by an observer at t_o, r_o, we can always define our system such that $\theta = 0$ and $\phi = 0$. Our signal propagates on a geodesic, this implies $ds^2 = 0$, which gives us:

$$\int_{t_e}^{t_o}\frac{dt}{a(t)} = \int_{r_e}^{r_o}\frac{dr}{\sqrt{(1-r^2k)}} \tag{4}$$

A second signal is emitted at $t_e + \delta t_e$ and received at $t_o + \delta t_o$. Since our observer did not move, and we assume that the scale factor stayed constant between t_e and $t_e + \delta t_e$ we have that:

$$\int_{t_e+\delta t_e}^{t_o+\delta t_o}\frac{dt}{a(t)} = \int_{r_e}^{r_o}\frac{dr}{\sqrt{(1-r^2k)}} \tag{5}$$

which implies:

$$\int_{t_e+\delta t_e}^{t_o+\delta t_o}\frac{dt}{a(t)} = \int_{t_e}^{t_o}\frac{dt}{a(t)} \tag{6}$$

Therefore,

$$\frac{\delta t_o}{a(t_o)} = \frac{\delta t_e}{a(t_e)}, \qquad \text{or,} \qquad \frac{\lambda_\mathrm{o}}{\mathrm{a(t_o)}} = \frac{\lambda_\mathrm{e}}{\mathrm{a(t_e)}}. \tag{7}$$

This result tell us that the difference in wavelength depends only on the ratio of scale factors at the time of emission and absorption (detection). Put in more astronomical familiar form and using the definition of redshift z:

$$z = \frac{\lambda_o - \lambda_e}{\lambda_e} \rightarrow a = (1+z)^{-1}. \tag{8}$$

Spectra of QSO allows us to measure some element line emission up to $z \approx 7-8$. In this case z tells us about the difference in the value of the expansion factor between emission and detection. It is **not** related to the receding velocity of the object (no Hubble like redshift!).

The Hubble parameter

For nearby Universe I can expand $a(t)$ in a Taylor series:

$$a(t) = a_0 + \dot{a}(t - t_0) + \frac{1}{2}\ddot{a}(t - t_0)^2 ... \tag{9}$$

that we can rewrite as:

$$\frac{a(t)}{a_0} = \left[1 + \frac{\dot{a}}{a_0}(t - t_0) + \frac{1}{2}\frac{\ddot{a}}{a_0}(t - t_0)^2 ...\right] \tag{10}$$

Since we know that $1 + z = a(t)/a_0$ we can rewrite the previous equation, and for small Δt just stop at the first order:

$$z = \left(\frac{\dot{a}}{a}\right)_{t=t_0} \Delta t = \frac{\dot{a}}{a}\frac{\text{Dist}}{\text{c}}. \tag{11}$$

This is another incarnation of the Hubble law: $vel = H \times Dist$, if this time we assume that the redshift is due to Doppler effect and can be translated in a velocity. This implies that:

$$H_0 = \left(\frac{\dot{a}}{a}\right)_{t=t_0}. \tag{12}$$

Let us notice that the Hubble *constant* it is not a constant, since by definition it is defined at a precise time $t = t_0$. On the other hand it is *at least* constant in space.

Friedman Equations

We will not derive the Friedman equations, a formal derivation can be found in any General Relativity book. The ingredients needed are the following:

- The metric we defined before: $ds^2 = c^2dt^2 - a^2(t)dl^2$
- The Einstein equations: $R_{\mu\nu} + \frac{1}{2}Rg_{\mu\nu} + (-\Lambda g_{\mu\nu}) = 8\pi T_{\mu\nu}$. Where the term $(-\Lambda g_{\mu\nu})$ takes into account a possible cosmological constant contribution.
- The Stress-Energy tensor $T_{\mu\nu}$, which is defined as a diagonal matrix $[\rho, p, p, p]$ (density and pressure), for analogy with a perfect fluid.
- Some mathematical manipulations

The 00, i.e. the time-time component of the field equation gives:

$$3\frac{\dot{a}^2}{a^2} + 3\frac{k}{a^2} - \Lambda = 8\pi G\rho \tag{13}$$

The three spatial components (all identical by symmetry, i.e the cosmological principle: homogeneity and isotropy) give:

$$\frac{\ddot{a}}{a} = -\frac{4}{3}\pi G(\rho + 3p) \tag{14}$$

Subtracting equations 13 from 14, and dividing the resulting equation by (-2), we get

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}, \tag{15}$$

which is an equation of motion and looks very similar to its Newtonian analog. Also the continuity equation $\dot{\rho} + 3(\dot{a}/a)(\rho + p) = 0$ is *hidden* in equations 13 and 14.

Newtonian dynamics

Let us considering an expanding sphere with total energy density ρ and volume V. The total energy within the sphere is $U = \rho \times V$. During the expansion the first law of thermodynamics tells us that:

$$\mathrm{dU} + \mathrm{pdV} = 0 \tag{16}$$

since there are not heat sources (homogeneity) and not net inflow-outflow (isotropy). Using the definition of U:

$$\mathrm{d}(\rho \mathrm{V}) + \mathrm{pdV} = 0 \rightarrow \mathrm{pdV} + \rho\mathrm{dV} + \mathrm{V}\mathrm{d}\rho = 0 \tag{17}$$

Assuming a short period dt for the change ($\mathrm{d} \rightarrow \dot{}$) we have:

$$(\dot{\rho V}) + p\dot{V} = 0 \rightarrow p\dot{V} + \rho\dot{V} + V\dot{\rho} = 0 \tag{18}$$

The volume V grows proportionally to a^3, and so $\mathrm{dV/V} = 3\mathrm{da/a}$, which leads us to the following equation for the energy density:

$$\dot{\rho} = -3(\rho + p)\frac{\dot{a}}{a} = -3H(\rho + p) \tag{19}$$

This equation describes the evolution of the (average) energy density in an expanding universe. Let's analyse now some specific cases. If we assume that $p = w\rho$, where w is a constant, we can rewrite equation 19 as:

$$\frac{\dot{\rho}}{\rho} = -3\frac{\dot{a}}{a}(1 + w) \tag{20}$$

which has the following general solution:$\rho = \rho_0 a^{-3(1+w)}$. The evolution of ρ will then depend on which "substance" rules the evolution of the Universe.

- **Matter domination:** Collisional matter (e.g. Dark Matter) has no pressure, hence $w = 0$, which implies $\rho_m(t) = \rho_m(0)a^{-3}$. Which tells us that the density will simply scale due to the increase Volume.

- **Radiation domination:** For radiation $w = 1/3$, therefore $\rho_r = \rho_r(0)a^{-4}$. This result can be understood in the following way: a term a^{-3} is due to the increased volume and an extra term a^{-1} is due to the redshift effect.

- **Cosmological constant:** By definition $\rho_\Lambda = const.$, which results in $w = -1$, and $\rho = -p$. That's why sometimes the cosmological constant is also (wrongly) referred as "anti-gravity".

- **Curvature k:** from the first Friedman equation (eq: 13): $\rho \propto 3k/a^2$, that gives $w = -1/3$.

A simple conclusion of this analysis is that DE will always dominate the expansion of the universe at later times, if there is no cosmological constant, than for non flat universes the curvature will dominate, otherwise the non-collisional matter component.

The Ω parameter

A common way to describe the contribution of a given component to the total energy density of the Universe is through its density paramer:

$$\Omega_i = \frac{\rho_i}{\rho_{cr}} \tag{21}$$

where ρ_{cr} is the critical density of the universe and it is defined as the density the Universe has in order to have a flat geometry ($K = 0$). By definition:

$$\Omega_{tot} = 1 - \Omega_k = \Omega_m + \Omega_r + \Omega_\Lambda(+\Omega_\nu) \tag{22}$$

where the last term Ω_ν is the density of massive neutrinos.
The exact value of ρ_{cr} can be easily obtained from the first Friedman equation (eq. 13):

$$3\frac{\dot{a}^2}{a^2} = 8\pi G\rho + \Lambda = 8\pi G(\rho_{m+r} + \rho_\Lambda) = 8\pi G\rho_{cr} \tag{23}$$

where we defined $\rho_\Lambda = \Lambda/4\pi G$. From the previous equation we can compute the present value of the crital density using the Hubble constant:

$$\rho_{cr}(z=0) = \frac{3}{8\pi G}\left(\frac{\dot{a}}{a}\right)^2_{z=0} = \frac{3H_0}{8G\pi} = 2.775 \times 10^{11} \quad h^2 \quad \mathrm{M}_\odot \quad \mathrm{Mpc}^{-3} \tag{24}$$

the symbol h is called the Hubble parameter and it express the value of H_0 in units of 100 km s^{-1} Mpc^{-1}. Today observations (CMB + Lensing + Hubble telescope) suggest $h = 0.72 \pm 0.03$.
Finally we can rewrite the first Friedmann equation in the following way:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_{cr}(0)\left[\Omega_m(0)a^{-3} + \Omega_r(0)a^{-4} + \Omega_k a^{-2} + \Omega_\Lambda\right]. \tag{25}$$