Linear Theory and perturbations Growth

The Universe is not homogeneous on small scales. We want to study how seed perturbations (like the ones we see in the Cosmic Microwave Background) evolve in an expanding univserse.

The expanding Universe is a key ingredient: expansion si the only "force" gravity has to fight against. If gravity would be the only force each perturbation will grow and collapse (sooner or later), untill all the fluctuations will end up in one large object.

Jeans theory

The Jeans theory describes perturbation growth in the linear regime $\delta\rho/\rho < 1$. Let's consider a fluid with pressure p (non relativistic) , density ρ and velocity $\vec{u}$ (let's say the hubble expantio). The fluid evolution obbeis to three equations: Continuity, Euler and Poisson.

$$\left(\frac{\partial \rho}{\partial t}\right)_{\vec{r}} + \vec{\nabla} \cdot (\rho \vec{u}) = 0 \tag{1}$$

$$\frac{d\vec{u}}{dt} = \left(\frac{\partial \vec{u}}{\partial t}\right)_{\vec{r}} + \left(\vec{u} \cdot \vec{\nabla}_{\vec{r}}\right) \vec{u} = -\vec{\nabla}_{\vec{r}}\Phi - \frac{1}{\rho}\vec{\nabla}_{\vec{r}}\, p \tag{2}$$

$$\vec{\nabla}^2_{\vec{r}} = 4\pi G\rho \tag{3}$$

Let's now introduce a small parturbation in our field, a small density bump:

$$\rho(\vec{r}, t) = \rho_0(t)[1 + \delta(\vec{r}, t)] \tag{4}$$

where $\rho_0(t)$ is the average densitu at the time t. Because of the density bump we will also induce peculiar velocities:

$$\vec{u} = \frac{d\vec{r}}{dt} = \frac{d(a\vec{x})}{dt} = \dot{a}\vec{x} + \vec{v}(\vec{x}, t). \tag{5}$$

The first term ($\dot{a}\vec{x}$) is the Hubble expantion while the second one $\vec{v}(\vec{x}, t)$ it is the peculiar velocity. In an anologuos way there will be also a peculiar potential ϕ, defined as $\Phi = \Phi_0 + \phi$.
The perturbated quantities can be used instead of the corresponding "smooth" unperturbed ones in equations 1, 2 and 3.

Let's start with rewriting the term $\frac{1}{\rho}\vec{\nabla}_{\vec{r}}\, p$

$$\frac{1}{\rho}\vec{\nabla}_{\vec{r}}\, p = \frac{1}{a\rho_0(1+\delta)}\frac{\partial p}{\partial \rho}\nabla\rho_0(1+\delta) = \frac{1}{a(1+\delta)}c_s^2\vec{\nabla}\delta \tag{6}$$

where we introduce the sound speed $c_s^2 = \partial p/\partial\rho$. We will now convert the perturbed equations from physical (or proper) coordinates to comoving ones in order to make explicit the expantion factor a. Remember that:

$$\nabla_{\vec{x}} = a\nabla_{\vec{r}} \tag{7}$$

$$\left(\frac{\partial f}{\partial t}\right)_{\vec{r}} + \left(\frac{\dot{a}}{a}\right)\vec{r} \cdot \vec{\nabla}_{\vec{r}} f = \left(\frac{\partial f}{\partial t}\right)_{\vec{x}} \tag{8}$$

Then to get the new equation of motions we subtract unperturbed equations from perturbed ones (and we will omit the subscrit $\vec{x}$ from now on). What we are left with is a new system of equations in the perturbed quantities δ, v and ϕ.

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\vec{\nabla} \cdot [(1+\delta)\vec{v}] = 0 \tag{9}$$

$$\frac{\partial \vec{v}}{\partial t} + \frac{\dot{a}}{a}\vec{v} + \frac{1}{a}(\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{1}{a}\vec{\nabla}\phi - \frac{1}{a(1+\delta)}c_s^2\vec{\nabla}\delta \tag{10}$$

$$\nabla^2\phi = 4\pi G\rho_0 a^2\delta \tag{11}$$

Now we can linearize these equations by removing second order terms and assuming $(1+\delta)\approx 2$

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\vec{\nabla}\cdot\delta = 0 \tag{12}$$

$$\frac{\partial \vec{v}}{\partial t} + \frac{\dot{a}}{a}\vec{v} + \frac{1}{a}\nabla\phi + \frac{1}{a}c_s^2\vec{\nabla}\delta = 0 \tag{13}$$

Finally peculiar velocities can be eliminated with few more algebrical passages, using that $\frac{1}{a}\frac{\partial}{t}$ (12) $= 0$ and $\nabla(13) = 0$ and the poisson equation for the perturbe potential ϕ: $\nabla\phi = 4\pi H\rho_0 a^2\delta$ Negletting again second order terms in δ we arrive at the final equation describing the evolution of a perturbation in the linear regime:

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\rho_0\delta - \frac{c_s^2}{a}\nabla^2\delta = 0 \tag{14}$$

If delta is small ($\delta \ll 1$) we can decompose it in Fourier space and treat each k mode as independent:

$$\ddot{\delta}_k + 2H\dot{\delta}_k + \left[\frac{c_s^2k^2}{a^2} - 4\pi G\rho_0\right]\delta_k = 0 \tag{15}$$

Equation 15 can have two different types of solutions depending on the sign of the last term, the one in square brackets.

- If the sign is positive, then solutions for δ_k are of sinusoidal type, which says that at any given place, or point in time, the density excess, δ_k oscillates. The sign of the square-brackets term is positive when the pressure term, $c_s^2k^2/a^2$, dominates. The physical interpretation is that the pressure forces are strong enough to effectively resist gravity. Hence the oscillations.
- If the sign of the term is negative, then δ_k will have solutions that are not oscillating, but monotonic in time, for example solutions like hyperbolic sine or cosine are possible. These non-oscillatory solutions lead to a monotonic increase or decrease in the density excess. The physical interpretation is that gravity takes over and collapses the density perturbation before pressure forces can dissipate it. Thus, monotonic increase in δ_k can result in a bound structure at some later time.

The dividing line between these two types of behavior is the Jeans scale, when the square-brackets terms is zero:

$$\frac{c_s^2k^2}{a^2} = 4\pi G\rho_0 \tag{16}$$

This defines a scale, usually called the Jeans scale, $k_J = \sqrt{4\pi G\rho_0}ac_s^{-1}$ that sets the future evolution of the perturbation. The corrisponding Jeans length is $\lambda_J = 2\pi/k_J$ and the Jeans mass is $M_J = \rho_0\lambda_J^3$.

Growth of DM perturbations in an Einstein-de Sitter Universe

In the next few sections we will be considering growth of density fluctuations in the **dark matter alone**. Because dark matter does not have any pressure of its own, and it does not couple to photons, the pressure term in the Jeans equation, $c_s^2 k^2/a^2$, will be ignored. As usual, if we assume E-dS model, i.e. $\Omega_m = 1$ things simplify a lot. Using $\rho_0 = \rho_{crit} = 3H_0^2/8\pi G$ and $t = 2/3H$, eq. 15 and its exact solution become:

$$\ddot{\delta}_k + \frac{4}{3t}\dot{\delta}_k = \frac{2}{2t^2}\delta_k; \qquad \delta_k = At^{2/3} + Bt^{-1}. \tag{17}$$

where A and B are constants.
For the growing mode (the first of two terms in the solution) $\delta_k \propto a$ that is, the amplitude of fluctuations grows proportionately to the scale factor of the universe. This is an important result. Notice that the growth rate is independent of the value of k. This means that the spatial regions of the Universe whose $\delta(\vec{x})$ is composed of a range of k-modes, each with its own amplitude δ_k also grow at the rate $\propto a$ (from now on we will omit the subscript k in δ). The growth in the linear regime (in the spatial regions where $\delta < 1$) continues forever in an E-dS Universe. The spatial extent of structures within which average $\delta\rho/\rho \sim 1$ continues to grow, while the density deep inside of these structures continues to increase well into the $\delta \gg 1$ regime. The growth of overdense regions occurs at the expense of the underdense regions, which, in the linear regime experience negative growth (Bt^{-1}).

Growth of DM perturbations in an open, $\Omega_\Lambda = 0, \Omega_K \neq 0$ Universe

Friedmann equation takes the form,

$$H^2 = H_0^2 \left[\Omega_m a^{-3} + (1 - \Omega_k)a^{-2}\right] \tag{18}$$

We can assume that during the epoch when curvature dominates, the term $\Omega_m a^{-3}$ will be negligible compared to $(1 - \Omega_k)a^{-2}$ so the matter density contribution to the expansion dynamics is tiny, and can be neglected. The Jeans equation now becomes,

$$\ddot{\delta} + 2H\dot{\delta} = 0 \tag{19}$$

From the Friedmann equations we see that when curvature dominates $H \propto a^{-1}$, therefore $t \propto a$, so that $Ht = 1$. With this, the Jeans equation has two solutions,

$$\delta \propto A \times const + Bt^{-1}. \tag{20}$$

The "growing mode" solution in this case is the least rapidly decaying one, $\delta \propto$ const, and implies that the amplitude of the density fluctuations in the linear regime is not going to change once curvature comes to dominate the global dynamics of the Universe. In an open Universe no growth of the density fluctuations in the linear regime took place after the curvature terms prevales on the mass one. This effect is sometimes called the "freezing of the fluctuations".

Growth of DM perturbations in a flat, $\Omega_\Lambda \neq 0, \Omega_K = 0$ Universe

Friedmann equation takes the form,

$$H^2 = H_0^2 \left[\Omega_m a^{-3} + \Omega_\Lambda\right] \tag{21}$$

In this model, just like in the open Universe, growth of density perturbations will virtually stop when Λ begins to dominate, i.e. when $\Omega_m a^{-3}/\Omega_\Lambda < 1$. The equation then becomes the same as eq.

19, but the solutions are different because H and t now have a different relation connecting them: from Friedmann $H^2 = H_0^2\Omega_\Lambda$, so Hubble parameter is constant, and the solutions are:

$$\delta \propto A \times const + B \exp^{-2Ht} . \qquad (22)$$

WARNING: It is important to note that here we are talking about growth shutting off on linear length scales. At the present epoch in our own Universe these are scales larger than about 20-40 Mpc, i.e. scales of superclusters, voids, and larger. The shutting off of growth on these scales does not mean that the dynamical evolution within non-linear structures $\delta \gg 1$ that are already assembled has stopped: these objects continue to evolve until they virialize. Also, interactions between objects, like galaxy collisions, etc, can stimulate additional dynamical evolution in mildly non-linear regions.

Growth of sub-horizon DM perturbations during radiation dominated epoch

During radiation domination the largest contribution to the energy density is from photons, which, being relativistic, do not cluster. That means that the last term in eq: 15 is 0 ($\delta_{tot} \sim \delta_{phot} \sim 0$). The Jeans equation reduces to:

$$\ddot{\delta} + 2H\dot{\delta} = \ddot{\delta} + \frac{1}{t}\dot{\delta} = 0 \qquad (23)$$

The second step follows because $a \propto t^{-1/2}$ during this epoch. This equation has two solutions:

$$\delta \propto A \times const + B \ln t. \qquad (24)$$

Perturbations in particles not coupled to photons (like CDM) grow at best logarithmically during this epoch, which is not very fast at all. Remember that baryons during this epoch are tightly coupled to the photons and so their density perturbations are well represented by an oscillating solution to eq. 15.

Growth of superhorizon DM perturbations

In general an horizon is surface you cannot see beyond. Let's define a sphere drawn around an observer such that light from sources on the sphere, if emitted at $t = 0$ will just reach the observer now at $t = t_0$. In formulas

$$d_H = \int_0^{t_0} \frac{dt}{a(t)} = \int_0^{r_0} \frac{dr}{(1 - r^2/k)} \qquad (25)$$

The particle horizon is defined as $D_H = d_H a(t)$, which is the proper radius corresponding to d_H. If $a(t) \propto t^{2/[3(1+w)]}$ the particle horizon is finite an equal to

$$D_H = [3(1 + w)/(1 + 3w)]t \qquad (26)$$

For matter and radiation dominated era this translates into $D_H = 3t$ and $D_H = 2t$, respectively. This means that the region encompassed by D_H grow with time, and, more important, grows *faster* than the rate at which two observers are carried away from each other due to cosmic expantion. More and more region will be in causal contact as time goes by.

If a fluctuation has a scale larger than the horizon at that time (i.e. it is *outside* the horizon), it does not care about microphysical process like fluid pressure, but it simply evolves only according to General Relativity (Friedman equations).

Let's assume for semplicity $\Omega_{tot} = 1$, then we have:

$$H^2 = \frac{8\pi G}{3}\bar{\rho} \qquad (27)$$

where $\bar{\rho}$ is the background density. An overdensity $\delta\rho$ will then induce a positive curvature in the local space ($k > 0$).

$$H^2 = \frac{8\pi G}{3}(\bar{\rho} + \delta\rho) - \frac{k}{a^2} \quad (28)$$

then:

$$\frac{\delta\rho}{\rho} = \frac{\rho - \bar{\rho}}{\bar{\rho}} = \frac{(k/a^2)}{8\pi G\bar{\rho}/3} \propto (a^2\bar{\rho})^{-1} \quad (29)$$

This implies that super horizon fluctuations will be always growing like $\delta \propto a^2$ and $\delta \propto a$ if radiation or matter dominates, respectively.

Power Spectrum and Transfer Function

One simple way to describe the perturbation of a smooth medium is thorugh its Power Spectrum. If the quantity $\delta\rho(\vec{r})$ is reasonable continuous we can decompose it into the Fourier space.

$$\delta\rho(\vec{r}) = \int_{-\inf}^{\inf} \delta_{\vec{k}} \exp(-i\vec{k} \cdot \vec{r}) \mathrm{d}^3\mathrm{k} \quad (30)$$

Isotropy tells us that the Universe is the same in all directions so we can simply assume that $\delta_{\vec{k}}$ is just a function of $|k|$. Then the power spectrum is defined as:

$$P(k) \equiv |\delta_k|^2 \quad (31)$$

A foundamental quantity in cosmology is the root mean square (r.m.s.) of the fluctuations on a given scale R:

$$\sigma_R^2 \equiv \langle|\frac{\delta\rho}{\rho}|^2\rangle_R \propto \int_{-\inf}^{\inf} |\delta_k|^2 k^2 |W_k|^2 \mathrm{dk} \quad (32)$$

where W_k is the fourier transform of a spatial filter (window function: i.e. top-hat or gaussian) which weights regions at distances larger than R less than the central ones.
Associated with the density variance on a scale R there is a mass variance σ_M^2, where $M = 4/3\pi\bar{\rho}R^3$.

Usually the primordial power spectrum is assumed to be a power law of the form $P(k) = ak^n$. Power law shape is what is expected for a scale-free force like Gravity. Moreover inflation predicts such a spectrum with $n = 1$.

The constant A can be determined through observations, by measuring σ_R from galaxy distribution. For historical reasons, the variance is measured on a scale of 8 Mpch^{-1}, the associated parameter is called σ_8 and fixes the normalization of the power spectrum.
For $P(k) \propto k^n$ we have that:

$$\langle|\frac{\delta\rho}{\rho}|^2\rangle_R \propto k^{n+3} \quad (33)$$

Since $k \propto \lambda^{-1}$ and $M \propto \lambda^3$, $\sigma_M = M^{-\alpha}$, where $\alpha = n/6 + 1/2$. Since σ_8 is computed on a large scales, that today is still linear, its evolution is the same as a linear perturbation. Therefore $\sigma_8 \propto a$ in the matter dominated era.

The Transfer Function

As time goes by the universe Horizon grows faster than the Hubble expantion. This means that larger and larger scales will enter the Horizon. Because of the different rates of growth of perturbations inside and outside the Horizon, the power spectrum, changes shape compared to its original primordial Harrison-Zeldovich ($n = 1$) spectrum, $P(k) \propto k$

These changes are described by the so-called transfer function which relates the primordial spectrum to the processed one. If we consider DM only, there exists a special epoch relevant to the evolution of fluctuations in DM. That epoch is matter-radiation equality (MRE, or just equality), and there will be a special length scale as well, corresponding to the size of the horizon at z_{eq}.

Lengthscales less than $R_{hor,eq}$ would have entered the horizon before z_{eq} , and so would have undergone a period of slow (logarithmic) growth, whereas scales larger than $R_{hor,eq}$ would have not experienced such a phase. Hence the processed power spectrum has a (smooth) break at $R_{hor,eq}$, and is no longer scale-free. The original Harrison-Zeldovich shape $\propto k$ is retained on the longest scales only.

To sum up transfer function in a dark matter dominated universe: long wavelength modes always grow well, because when they make the transition from being super-horizon to sub-horizon the Universe is already matter dominated, and so they never have to go through the growth suppressing period of being sub-horizon during radiation-domination. Unlike smaller length scales.

The more time a given scale has to spend in the sub-horizon-radiation-dominated growth purgatory the smaller its amplitude is going to be. So the smallest scales are most severely suppressed.
The processed spectrum $P(k, z)$ is usually parametrized in the following way:

$$P(k, z) = Ak^n T^2(k, z) \tag{34}$$

where $T(k, z)$ is the transfer function, which depends on the cosmological parameters: $\Omega_m, \Omega_\Lambda, \Omega_r, \Omega_b$ ad well as on the equation of state of Dark Energy, $w(t)$.

The Zel'dovich Approximation

In the linear regime the growth of a perturbation can be written as:

$$\delta(x, a) = D(a)\delta_i(x) \tag{35}$$

where $\delta_i(x)$ is the density perturbation at some initial time t_i, and $D(a)$ is the so called "linear growth factor" and it is normalized such that $D(a_i) = 1$. For example for an E-dS universe $D(a) = a$.
If we substitute eq 35 into the poisson equation we obtain:

$$\Phi(x, a) = D(a)\Phi_i(x) \qquad \text{where} \qquad \mathrm{\nabla^2\Phi_i = 4\pi G\rho_0 a^2 \delta_i(x)} \tag{36}$$

We can substitute this result in the linearized Euler equaiton (eq: 13 $\dot{v} + (\dot{a}/a)v = \nabla\Phi/a$) which give us:

$$v = -\frac{\nabla\Phi_i}{a}\int\frac{da}{dt}. \tag{37}$$

Because by definition $D(a)$ satisfies the fluctuation growth equation, $\ddot{\delta} + (2\dot{a}/a)\dot{\delta} = 4\pi G\rho_0\delta$ so that $\int(D/a)dt = \dot{D}/4\pi G\rho_0 a$, equations 37 can be written as:

$$v = -\frac{\dot{D}}{4\pi G\rho_0 a^2}\nabla\Phi_i(x), \tag{38}$$

which shows that the peculiar velocity is proportional to the current gravitational accelleration. Since $v = a\dot{x}$, integrating the above equation once again and to the first order of pertubations, so that $\nabla\Phi_i(x)$ can be replaced by $\nabla\Phi_i(x_i)$ (with x_i the initial position of the mass element at x), we obtain

$$x = x_i - \frac{D(a)}{4\pi G\rho_0 a^3}\nabla\Phi_i(x_i). \tag{39}$$

This formulation of linear perturbation theory, which is applicable to a pressurless fluid, is due to Zeldovich (1970). It is a Lagrangian description that specifies the growth of structure by giving the displacement $x - x_i$ and the peculiar velocity v of each mass element in terms of the initial position x_i.

The Zeldovich approximation is extremely useful to set up the inital conditions of numerical simulations. What we want is to start with our mass elements (hereafter defined as particles) at rest on a regular grid and then compute their peculiar velocity and displacement according to the power spectrum at that redshift.

According to Zeldovich approximation, (which is valid in the mild linear regime, $\delta < 1$) the displacement of a particle can be written as:

$$x = x_i - \frac{D(a)}{4\pi G\rho_0 a^3} \sum_k S_k(x_i) \tag{40}$$

and its peculiar velocity as:

$$v = -\frac{\dot{D}}{4\pi G\rho_0 a^2} \sum_k S_k(x_i) \tag{41}$$

where the displacement vector S is related to the potential Φ and the power spectrum of the fluctuations $P(k)$:

$$S_k(x_i) = \nabla_x \Phi_k(x_i), \qquad \Phi_k = \sum_k a_k \cos(kx_i) + b_k \sin(kx_i) \tag{42}$$

where a and b are gaussian random numbers with the mean zero and dispersion $\sigma^2 = P(k)/k^4$:

$$a_k = \sqrt{P(k)}\frac{Gauss(0,1)}{k^2}, \qquad b_k = \sqrt{P(k)}\frac{Gauss(0,1)}{k^2}. \tag{43}$$

The normalization of the Power Spectrum (e.g σ_8) defines the normalization of the fluctuations.

In order to set the initial conditions, we choose the size of the computational box L and the number of particles N^3. The phase space is divided into small equal cubes of size $2\pi/L$. Each cube is centered on a harmonic $k = 2\pi/L \times i, j, k$, where i, j, k are integer numbers with limits from zero to N/2. We make a realization of the spectrum of perturbations a_k and b_k, and find displacement and momenta of particles with $x_i = L/N \times i, j, k$ using the equations 40 and 41. Here $i, j, k = 1, N$.