Beyond the Linear Regime

Our next step is to look at regions with $\delta \gg 1$. As one can imagine the dynamics of these regions can be complicated, so we will make simplifying assumptions. Let's consider an overdense spherically symmetric piece of the Universe, and let the background cosmology be E-dS ($\Omega_m = 1.0$). Imagine dividing up the region into spherically concentric shells. The density increases inward, so that inner most and densest regions will collapse first, then outer regions, etc. The equation of motion for a particle at radius r is:

$$\ddot{r} = -\frac{GM}{r^2} \tag{1}$$

Where $M = M(< r)$ is the mass within r. A general parametric solution to this equation is

$$r = A(1 - \cos\theta) \quad \text{and} \quad \mathrm{t} = \mathrm{B}(\theta - \sin\theta) \tag{2}$$

The angle θ is sometimes called the evolution angle; it is a surrogate for time; constants A and B are to be determined. When $\theta = \pi$ the particle reaches its maximum distance from the center, r_{max} , therefore:

$$A = r_{max}/2 \tag{3}$$

This radius is the **turnaround**, and it occurs at later cosmic times for shells further out. In other words, innermost shells are the first ones to turnaround and start collapsing. A relation between A and B can be established if we plug in r from the previous equation into the equation of motion for r:$\ddot{r} = -\frac{GM}{r^2}$, this relation leads to the value of B:

$$A^3 = GMB^2 \leftarrow B = \left(\frac{r^3_{max}}{8GM}\right)^{1/2} \tag{4}$$

Let us calculate average density within a given shell of the sphere, $\rho(t) = 3M/4\pi r^3$. If we use the results of eq: 2 we obtain:

$$\rho(t) = \frac{3}{4\pi G t^2}\frac{(\theta - \sin\theta)^2}{(1 - \cos\theta)^3} = \frac{9}{2}\rho_0\frac{(\theta - \sin\theta)^2}{(1 - \cos\theta)^3}. \tag{5}$$

One can show that for small values of θ, i.e. early on, the density scales as t^{-2} , which is the same as for the average density in a E-dS universe.

Density at turnaround

At turnaround, $\theta = \pi$, hence we have that:

$$\frac{\rho}{\rho_0} = \frac{9\pi^2}{16} \approx 5.5, \qquad \text{or} \qquad \delta_{\mathrm{TA}} \approx 4.5 \tag{6}$$

This is an important result. So an initially small perturbation grows, first linearly, then faster. If you follow any one particle its velocity with respect to the center of the bump will steadily decrease compared to the Hubble flow. At some point it stops - that's turnaround - and starts to collapse. The turn-around marks the time when the material at that radius decouples from the Hubble flow; from then onwards that material proceeds with its own dynamical evolution. The turn-around radius (the radius within which the average overdensity is 5.5) increases with time, so collapse of objects proceed from inside out.
After turnaround our approximation of no-shell crossing breaks down, and the θparametric model we have used so far is no longer valid.

Density at virialization

Let us not pay attention to the ensuing messy process of halo virialization; suppose next time we look at the galaxy all the matter has settled, and the galaxy attained a state of (relative) stability: the potential has stopped evolving. Now, what is the density inside the portion of the sphere that is virialized?
The virial theorem tells us that for virialized objects:

$$K_E = -\frac{1}{2} P_E \tag{7}$$

where K_E and P_E are the total kinetic and potential energy of the dark matter halo, respectively. At turnaround all the energy is in the form of potential energy:

$$E_{tot} = P_E(r_{max}) = P_E(r_{TA}) \tag{8}$$

while at virialization (using the virial theorem) we have

$$E_{tot} = \frac{1}{2} P_E(r_{vir}). \tag{9}$$

From equations 8 and 9 we can derive that $r_{vir} = 0.5 r_{TA}$. Which implies that $\rho_{vir} = 8\rho_{r_{TA}}$. But this is not the whole story. What we want to compute is the overdensity of a perturbation at virialization, what we want is $\delta(R_{vir})$. To compute this quantity we have to consider that also the background density evolved from t_{TA} and t_{vir}. Roughly, it will take as much time for the structure to collapse as it did to attain turnaround. In E-dS universe $r \propto t^{2/3}$ and $\rho \propto t^{-2}$. Since the time doubled since turnaround, the background density dropped by a factor of 4. This implies that:

$$\frac{\rho_{vir}}{\rho_b} = \Delta_{vir} = 8 \times 4 \times \frac{\rho_{TA}}{\rho_0} = 32 \times 5.5 = 176 \approx 200 \tag{10}$$

This (over)density threshold is usually referred as the virial density, and it is model and redshift dependent. A useful fitting formula to determi Δ_{vir} as been porposed by Brian & Norman (1998). For a flat universe, ($\Omega_K = 0$):

$$\Delta_{vir} = 18\pi^2 + 82x - 39x^2 \tag{11}$$

while for a Universe without cosmological constant ($\Omega_\Lambda = 0$, and arbitrary geometry):

$$\Delta_{vir} = 18\pi^2 + 60x - 32x^2 \tag{12}$$

where $x = \Omega(z) - 1$ and $\Omega(z) = \Omega_m(1+z)^3/E(z)^2$. The function $E(z)$ describes the evolution of the Hubble parameter and is defined as:

$$E(z)^2 = (100hE(z))^2 = \Omega_m(1+z)^3 + \Omega_\Lambda + \Omega_K(1+z)^2 \tag{13}$$

So an ob ject whose average density is Δ_{vir} is considered to be virialized. As time goes on larger and larger regions become virialized, ans the boundary enclosing an average overdensity of Δ_{vir} grows larger.

Linear density at virialization

It is also possible to compute the linearized density for virialization. This density can be described as the density that a perturnation δ would have if it will always stay in the linear regime.

Equation 2 tells us that the minimum radius is achieved for $\theta = 2\pi$. Using the relation between *linear* density and time ($\rho(t) = 1/(6\pi G t^2)$) and the second part of eq: 2 we have:

$$\delta_c = \frac{\rho_L}{\rho_b} - 1 = \frac{3}{5}\left(\frac{3}{4}\right)^{2/3}(\theta - \sin\theta)^{2/3} = [\theta = 2\pi] = 1.686 \tag{14}$$

This density threshold will be useful for the so called Press & Schecter approach, which we will describe next.

out of the Hubble flow, and via gravitational instability. Stars, planets, giant molecular clouds, etc. are not included in this category because they did not form out of the Hubble flow—they formed inside of galaxies, in regions of space that were dynamically dominated by self-gravity of the parent galaxy, and completely oblivious to the Hubble expansion. The following analysis applies to dark matter halos exclusively. However, in our Universe it appears that dark matter halos of all masses are occupied by some sort of light emitting matter: DM halos with $M \lesssim 10^{12} M_\odot$ are occupied by individual galaxies (in general, one galaxy per halo), $M \sim 10^{13} M_\odot$ DM halos are occupied by groups of galaxies, and $M \gtrsim 10^{14} M_\odot$ DM halos are occupied by clusters of galaxies (again, usually one cluster per halo). So the analysis in effect, applies to galaxies, groups and clusters.

5.2 Mass function of collapsed objects

On average, the distribution of mass is assumed to be homogeneous and isotropic. Let the rms dispersion in mass (or density) in spheres that, *on average*, contain $\bar{M}$, and have radius $R^3 = \bar{M}/[(4/3)\pi\bar{\rho}]$, ($\bar{\rho}$ is the average mass density at that epoch) be

$$\sigma_M = \frac{\langle (M - \bar{M})^2 \rangle}{\bar{M}^2} \propto \bar{M}^{-\alpha}, \tag{29}$$

α is related to the power spectrum index, $\alpha = n/6 + 1/2$, and $P(k) \propto k^n$. Since α is positive rms dispersion in mass decreases with increasing scale. This makes sense because on small scales the Universe is lumpy, but gets smoother on large scales. Since the amplitude of δ grows with time, so does the amplitude of σ_M: $\sigma_M \propto t^{2/3}$ in an Einstein-de Sitter cosmology.

At any given time there will be regions in space that are overdense compared to others; these will accrete surrounding particles to form denser objects, which proceed to collapse after they have reached critical overdensity, $\delta \sim 4.5$. As time goes on more and more particles get caught in collapsed objects. In fact, there will be collapsed objects that become 'ingested' into larger objects that are also collapsed. For example, already formed galaxies become 'particles' incorporated into collapsed clusters of galaxies. In other words, we have a hierarchy of objects. In spite of the fact that an ever increasing amount of mass in the Universe becomes trapped in collapsed objects there are still areas of linear growth in the Universe, but typically you have to go to larger and larger spatial scales to find these. Linear regimes, i.e. where fractional overdensity δ does not exceed ~ 1, often surround objects that are just decoupling from the Hubble flow and are beginning to collapse. So linear formalism approximately applies to spatial scales where collapsed objects are just forming.

Consider many randomly placed volumes of size $V \propto R^3 \propto \bar{M}$. Each one of these is characterized by an overdensity δ, and the volumes are large enough so that typical δ's are not too large. Press and Schechter argued that in such a case δ's are Gaussian distributed:

$$p(\delta, V) = \frac{1}{(2\pi\sigma_M^2)^{1/2}} e^{-\delta^2/2\sigma_M^2} = \frac{1}{(2\pi\sigma_M^2)^{1/2}} e^{-x^2}, \qquad x = \frac{\delta}{\sqrt{2}\sigma_M} \tag{30}$$

At a given time perturbations in the high overdensity tail of the Gaussian are just collapsing or have already collapsed. The fraction of collapsed volumes of all masses is

$$P(\delta, V) = \int_{\delta_c}^{\infty} p(\delta, V) d\delta = \frac{1}{\pi^{1/2}} \int_{x_c}^{\infty} e^{-x^2} dx, \tag{31}$$

The threshold $\delta_c = 1.69$ is the *linearized* fractional overdensity[4], independent of the epoch. Linearized means that it is the overdensity collapsed objects would have had had they not gone

[4]The corresponding non-linear overdensity is 200. This is the actual overdensity.

non-linear. The reason we have to use linearized δ here is because we start with the linear theory and eq. 30 is valid when density fluctuations are small. For objects just collapsing the exponent in eq. 31 becomes:

$$x_c{}^2 = \frac{\delta_c{}^2}{2AM^{-2\alpha}} = \frac{2AM_\star{}^{-2\alpha}}{2AM^{-2\alpha}} = \Big(\frac{M}{M_\star}\Big)^{2\alpha}, \tag{32}$$

where we have replaced δ_c with a new parameter, $M_\star$.

The fraction of mass volumes that are just becoming bound, i.e. are just forming at that epoch is $P(M) - P(M + dM)$, it is the fraction of all objects that have formed by that epoch, minus the fraction of objects that have formed during previous epochs:

$$P(M) - P(M + dM) = \frac{dP}{dM}\,dM = \frac{dP}{dx}\frac{dx}{dM}\,dM = \frac{1}{\pi^{1/2}}\,e^{-x_c^2}\,\alpha\,\frac{M^{\alpha-1}}{M_\star{}^{\alpha}}\,dM \tag{33}$$

To convert this into the *fraction of mass* contained in these objects per unit volume of space we have to multiply by $\bar{\rho}$, and to get *number density* of objects per unit volume we divide by $\bar{M}$. This gives us the mass function,

$$n(M)\,dM = \frac{dP}{dM}\,dM\,\frac{\bar{\rho}}{M} = \frac{\alpha\bar{\rho}}{\pi^{1/2}}\,\frac{1}{M^2}\Big(\frac{M}{M_\star}\Big)^{\alpha} e^{-x_c^2}\,dM \tag{34}$$

The final form of the mass function is

$$n(M)\,dM = \frac{\alpha\bar{\rho}}{\pi^{1/2}}\,\frac{1}{M_\star{}^2}\Big(\frac{M}{M_\star}\Big)^{\alpha-2} e^{-(M/M_\star)^{2\alpha}}\,dM \tag{35}$$

This is the mass function of objects just reaching virialization at time t. The units on both sides of the "=" sign are: per unit volume. The PS mass function is a power law in mass with an exponential cut-off at masses above $M_\star$. The mass function tells us that massive objects are rearer than less massive objects. $M_\star$ is time dependent because of σ_M. As time progresses larger and larger mass scales reach overdensities necessarily for collapse, and so more massive objects collapse later than less massive objects: galaxies form before clusters, etc.

The derivation of $n(M)\,dM$ made many assumptions, and many relevant effects were not taken into account at all.[5] So it is surprising that the derived mass function agrees very well with the results of high-resolution computer simulations.

6 Clustering of dark matter halos on large scales

In the previous sections we dealt with the mass function of dark matter halos, spanning the mass range from low mass galaxies, to groups and clusters of galaxies. That is, distribution of dark matter halos in mass. We now turn to the distribution of galaxies in space (clustering), and the motion of galaxies on scales $\gtrsim$ few Mpc (dynamics). Both clustering and dynamics can tell us about the global properties of the Universe: Ω's and the shape and amplitude of the matter power spectrum. The hope is that both types of methods give the same results for the parameters, and in fact they do, at least in the general sense. (The details sometimes do not agree; measument errors are a major cause of this.) In other words, we are pretty sure we have the big picture of how structure evolved in the Universe, and all the aspects of the observed large scale structure (LSS) amount to a self-consistent picture.

[5] One of the main weaknesses of Press-Schechter formalism is that it does not treat objects collapsing in underdense regions, i.e. regions with $\delta < 1$. Such regions, voids, do form dark matter halos, but PS ignores them completely. One must correct for this 'by hand'. Since about half of the total mass of the Universe is contained in underdense regions, eq. 35 must be multiplied by a factor of 2. This is a handwaving argument, but one can also show that using more convincing lines of reasoning.