Queen Mary
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Hyperboloidal Numerical Algorithm in the Time Domain for Scalar and Gravitational Self-Force Applications

Towards a numerical recipe: From BHPT through Regge-Wheeler-Zerilli Formalism towards GSF computation

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30/06/2023

Lidia Gomes Da Silva

Outline

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Part I – Overview

- The machinery necessary for modelling scalar and gravitational perturbations by a point-particle on a circular geodesic on a Schwarzschild background
- Current state of TD methods and difficulties hindering progress
- Motivation

Part II – Hyperboloidal discontinuous time-symmetric algorithm with high order jumps in a nutshell

- How our algorithm addresses said difficulties

Part III – Numerical Results for both SF and GSF applications

- Test 1 – Measurement of the radiation fluxes at "the infinities"
- Test 2 – Evaluating the field and its derivatives where necessary
- How can these results be used for a full SSF/GSF computation

Part IV – Summary & Outlook

This talk closely follows our paper: https://arxiv.org/abs/2306.13153

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Scalar & Gravitational Perturbations in a Schwarzschild BH – In a nutshell….

- Expanding the Einstein tensor to linear order:

$$G_{\alpha\beta} = \bar{G}_{\alpha\beta} + \delta G_{\alpha\beta} \quad \rightarrow \quad \delta G_{\alpha\beta} = \frac{8\pi G}{c^4}\boldsymbol{T_{\alpha\beta}} \qquad\qquad \boldsymbol{T_{\alpha\beta}} = \mu \int d\tau (-g)^{1/2} u_\alpha u_\beta \delta^4\left(x^\mu - z^\mu(\tau)\right)$$

- Split **4D** spacetime $x^\alpha = (x^a, \theta^A)$: **2D** Lorentzian manifold $\boldsymbol{x^a} = (\boldsymbol{t}, \boldsymbol{r})$ **+** **2D** Sphere $\boldsymbol{\theta^A} = (\boldsymbol{\theta}, \boldsymbol{\phi})$

- Apply scalar **OR** tensor spherical harmonic decomposition such that

Scalar

$$\Phi(t, r, \theta, \phi) = \sum_{l=0}^{\infty} \phi(t, r) Y_{lm}(\theta, \phi)$$

- Apply Klein Gordon eq. for a scalar field

$$\blacksquare\Phi = \nabla_\mu \nabla^\nu \Phi = \frac{1}{\sqrt{-\boldsymbol{g}}}\partial_\mu[\sqrt{-\boldsymbol{g}}\, g^{\mu\nu}\partial_\nu \Phi] = S_\Phi$$

where,

$$S_\Phi = -4\pi q \int_{-\infty}^{\infty} \frac{1}{\sqrt{-g}} \delta^4\left(x^\mu - z_p^\mu(\tau)\right) d\tau$$

- Simplify to "amicable" form s.t

Gravitational

$$h_{\alpha\beta}(x) = \sum_a \sum_{lm} h_{lm}^a(t, r)(\boldsymbol{t}_{lm}^a)_{\alpha\beta}(\theta, \phi)$$

$$a = \{tt, Rt, Et, Bt, L0, T0, E1, B1, E2, B2\}$$

- Rewrite the perturbations **in the RW gauge** and vary $\delta G_{\alpha\beta}$

- Should find
  - **Odd/Axial** → 3 eqs.
  - **Even/Polar** → 7 eqs.

- Solve as a system of algebraic equations

- Simplify to "amicable" form s.t

Transform in space to the tortoise coordinate, $\boldsymbol{x}$

Towards the SSF & GSF Computations

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$$[-\partial_t^2 + \partial_x^2 - V^{S/RW/Z}(r)]\phi^{S/RW/Z}(t,r) = S^{S/RW/Z}(t,r) = G(t,r)\delta\left(r - \xi_p(t)\right) + F(t,r)\delta'\left(r - \xi_p(t)\right)$$

Example

$$\boldsymbol{F}(t, r_p(t)) = e_l(r) Y^{lm}(t)$$

$$e_l(r) = \frac{8\,\pi m_p}{(1+n_l)} \frac{f^3(r)}{r\,\Lambda_l(r)} \frac{1}{E_p}\left(1 + \frac{L_p^2}{r^2}\right)$$

Computational Strategy:

$$\mathcal{F}^{self} = \mathcal{F}_{dissipative} + \mathcal{F}_{conservative}$$

■ = ● + ■

▲ Flux balance Laws - ●

▲ Mode-Sum - ● ■

- Effective Source - ● ■
- Worldline Convolution- ● ■

Scalar

Phase 1: Solve the Scalar equation in the FD or TD

Phase 2: Calculate the SSF

▲ ●

▲ ● ■

https://arxiv.org/pdf/gr-qc/0502028.pdf
https://arxiv.org/pdf/gr-qc/0311017.pdf
https://arxiv.org/abs/1811.04432

Gravitational

Phase 1: Solve the RWZ in the FD or TD

Phase 2: ▲ ● or ▲ ● ■

▲ Reconstruct the metric perturbations

We have 6 perturbations

Odd → 2 Eqs

Even → 4 Eqs

Phase 3: Use reconstructed terms to calculate the GSF

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Motivation - State of GSF calculations in the time domain (TD)

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$$\left[-\partial_t^2 + \partial_x^2 - V^{S/RW/Z}(r)\right]\phi^{S/RW/Z}(t,r) = S^{S/RW/Z}(t,r) = G(t,r)\delta\left(r - \xi_p(t)\right) + F(t,r)\delta'\left(r - \xi_p(t)\right)$$

GSF results in RWZ in the TD:

- Computing the GSF on a CO plunging into a Schwarzschild BH – *Barack, Lousto* https://arxiv.org/pdf/gr-qc/0205043.pdf

Frequency Domain:

- A Consequence of GSF for circular orbits of the Schwarzschild geometry – *Detweiler* https://arxiv.org/abs/0804.3529
- GSF Regularization in the RW and Easy Gauges – *Thompson et al* https://arxiv.org/abs/1811.04432

Main Difficulties in the TD:

- D1 - Handling the source
- D2 - The outer radiation boundary problem - what boundary conditions to pick ?
  1+1D S/RW/Z wave like equation is in an infinite domain
- D3 - Time integrator schemes must allow for long term EMRI orbital evolutions which may take from months to a few years

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D2 - Conformal methods for BHPT

$g_{\mu\nu} = \Omega^2 \widetilde{g_{\mu\nu}}$ [1]

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Radiation BCs – Not ideal at all for our case

- We want to extract GWs at $\mathcal{I}^+$
- May have a dissipative effect as time grows, long time evolution needed, would it be clear when we reached the horizon and $\mathcal{I}^+$??

$$[-\partial_t^2 + \partial_x^2 - V^{S/RW/Z}(r)]\phi^{S/RW/Z}(t,r) = S^{S/RW/Z}(t,r) = G(t,r)\delta\left(r - \xi_p(t)\right) + F(t,r)\delta'\left(r - \xi_p(t)\right)$$

$$[\blacksquare - V^{S/RW/Z}(\sigma)]\phi^{S/RW/Z}(\sigma,\tau) = S^{S/RW/Z}(\sigma,\tau) = G(\tau,\sigma)\delta\left(\sigma - \sigma_p(\tau)\right) + F(\tau,\sigma)\delta'\left(\sigma - \sigma_p(\tau)\right)$$

Use **hyperboloidal theory**

$$\begin{cases} t = \tau - h(\sigma) \\ x = g(\sigma) \end{cases}$$

- Foliate spacetime with surfaces that intersect $\mathcal{I}^+$ but remain spacelike everywhere
- Spatial Compactification

Automatically has outflow at boundaries

For those who want to learn more about hyperboloidal methods as useful tools to help solving EFEs, consider attending,
https://hyperboloid.al/event/copenhagen-23/
(next week following the Capra meeting)

For more see:
https://arxiv.org/pdf/1604.02261.pdf
https://arxiv.org/abs/2004.06434

[1] Penrose, 1962, "Asymptotic properties of fields and space-times"

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Numerical recipe towards computing the GSF

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We aim to solve numerically the RW/Z equations given as:

$$\left[\blacksquare - V^{S/RW/Z}(\sigma)\right]\phi^{S/RW/Z}(\sigma,\tau) = S^{S/RW/Z}(\sigma,\tau) = G(\tau,\sigma)\delta\left(\sigma-\sigma_p(\tau)\right) + F(\tau,\sigma)\delta'\left(\sigma-\sigma_p(\tau)\right)$$

Our strategy:

- Decompose in spin weighted spherical harmonics, obtain 1+1D S/RWZ equation
- Use hyperboloidal grid covering this 2D Lorentzian manifold from the horizon to $\mathcal{J}^+$ ✔
- Evolve in time using the **Method of Lines** (MoLs) **recipe**: ✔

Reduce it to a first order system

$$\frac{d\boldsymbol{U}}{d\tau} = \boldsymbol{L}\,\boldsymbol{U} + \boldsymbol{Source}, \qquad \boldsymbol{U} = \begin{pmatrix}\phi \\ \dot{\phi}\end{pmatrix}$$

- **Difficulty 1 -** Discontinuous collocation methods for <u>spatial discretization</u> (Discretize U, L in space using finite difference, pseudospectral or Fourier collocation nodes).

- **Difficulty 1, 3 -** Discontinuous symmetric methods for <u>time integration</u> (Integrate in time using time symmetric methods as a coupled ODE system, example Hermite integration).

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D1 - Discontinuous Spatial Discretisation

$$[\blacksquare - V^{S/RW/Z}(\sigma)]\phi^{S/RW/Z}(\sigma,\tau) = S^{S/RW/Z}(\sigma,\tau)$$

Jumps in solution (and its derivatives) across the particle known a priori

$$\boldsymbol{\phi^{(k)}(\sigma_p{}^{+}) - \phi^{(k)}(\sigma_p{}^{-}) = J_k, \qquad k = 0, 1, 2 \ldots}$$

Problem admits a weak-form solution:

$$\boldsymbol{\phi(\tau,\sigma) = \phi^{S}(\tau,\sigma) + \phi^{J}(\tau,\sigma)}$$

Ansatz for non-smooth part of our master function

$$\boldsymbol{\phi^{J}(\tau,\sigma) = \sum_{n=0}^{\infty} J_n(\tau)\, \psi_n(\sigma;\sigma_p)}$$

Collocation methods (FD or Chebyshev) methods are based on Lagrange interpolation:

$$\boldsymbol{p(\sigma) = \sum_{j=0}^{N} \phi_j\, \pi_j(\sigma),}$$

Construct discontinuous generalisation to LI that uses J_k as input.

$$\boldsymbol{\phi(\sigma) = \sum_{j=0}^{N}[\phi_j + \Delta(\sigma_j - \sigma_p; \sigma - \sigma_p)]\text{ , e.g } \quad \partial_\sigma^n\, \phi(\tau,\sigma)|_{\sigma=\sigma_i} \approx \sum_{j=0}^{N} D_{ij=0}^{(n)}[\phi_j + \Delta(\sigma_j - \sigma_p; \sigma_i - \sigma_p)]}$$

Where higher order jumps are given by

$$\boldsymbol{J_{k+2} = \sum_{k=0}^{\infty}\sum_{r=0}}\text{(all from the LHS that does not match the deltas on the RHS)}$$

For more details see, https://arxiv.org/abs/2306.13153

D3 - Time Integration – The case for GI Hermite Methods

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Runge-Kutta (RK) methods:

- Violate energy conservation and symplectic structure
- CFL restricts our time step, not ideal when we want to run a long-term EMRI simulation

Abandon RK methods, adopt Hermite integration methods

Apply the fundamental theorem of calculus and discretize in time

$$U(t_{n+1}) = U(t_n) + \int_{t_n}^{t_{n+1}} LU(t)dt$$

2-point Taylor expansion

$$U^{n+1} = U^n + \frac{\Delta t}{2} L.(U_n + U_{n+1}) + \frac{\Delta t^2}{12} L.\left(\dot{U}_n - \dot{U}_{n+1}\right) + \mathcal{O}(\Delta t^5)$$

$$U^{n+1} = \left(I - \frac{\Delta t}{2} L + \frac{\Delta t^2}{12} L^2 + \dots\right)^{-1} . \left(I + \frac{\Delta t}{2} L + \frac{\Delta t^2}{12} L^2 + \dots\right) U^n$$

For more details see:
Hyperboloidal discontinuous time-symmetric numerical algorithm with higher order jumps for gravitational self-force computations in the time domain: https://arxiv.org/abs/2306.13153
Conservative Evolution of Black Hole Perturbations with Time-Symmetric Numerical Methods – https://arxiv.org/abs/2210.02550
Lidia's Thesis Appendix C: Symmetry, Symplecticity and Stability of ICN Schemes – Lidia J. Gomes Da Silva et al

Generic hyperboloidal symmetric algorithm with higher order jumps

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We want to solve

$$[\blacksquare - V^{S/RW/Z}(\sigma)]\phi^{S/RW/Z}(\sigma,\tau) = S^{S/RW/Z}(\sigma,\tau) = G(\tau,\sigma)\delta\left(\sigma - \sigma_p(\tau)\right) + F(\tau,\sigma)\delta'\left(\sigma - \sigma_p(\tau)\right)$$

Discontinuous Time Integration via:

$$[\![\boldsymbol{u}]\!] = \int_{\tau_n}^{\tau_{n+1}} F(\tau,\sigma_p)\,\delta(\sigma-\sigma_p)\;d\tau = \frac{1}{|d\sigma_p/d\tau|}\,F(\tau_i,\sigma_p)\theta(\tau_{n+1} - \tau_i)\,\theta(\tau_i - \tau_n)$$

Discontinuous numerical evolution 2nd order algorithm

$$\boldsymbol{U}^{n+1} = \boldsymbol{U}^n + \boldsymbol{M}.\left(\boldsymbol{L}.(\Delta t\,\boldsymbol{U}^n) - (\Gamma) + \frac{\Delta t}{2}(S^n + S^{n+1}) + [\![\boldsymbol{u}]\!]\right)$$

where: S/ Γ highlight the discontinuous spatial discretization and time integration, respectively.
$[\![\boldsymbol{u}]\!]$ only turns on when the particle worldline $\sigma_p(\tau)$ crosses σ_i at a time $\tau_i \in [t_n, t_{n+1}]$.

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Is this **ACTUALLY** a numerical algorithm for SSF & GSF applications ???

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Plan is to solve: Phase 1 > Phase 2 > Phase 3;

Q1 – How is accuracy gauged from Phase 1?

- Test 1 - Compute the radiation fluxes at the boundaries
  - Is it accurate ? (compare to both previous TD and FD works)
    - – If yes – can compute dissipative GSF
- Test 2 – Can it evaluate the fields and it's derivatives at the particle and particle limits ?
  - Is it accurate ? (compare to both previous TD and FD works)
    - – If yes – can compute conservative (and dissipative) GSF through mode-sum

Q2 – How will we calibrate our numerical algorithm ?

With 4 numerical optimisation factors

i. Number of nodes
ii. Number of jumps
iii. Time discretisation step
iv. Minimal time for *steady-state* evolution

Q3 – Is our Phase 1 numerics enough for optimal calibration ?

https://arxiv.org/pdf/2306.13153.pdf

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Phase 1: Results - Particle on a circular orbit

https://arxiv.org/pdf/2306.13153.pdf

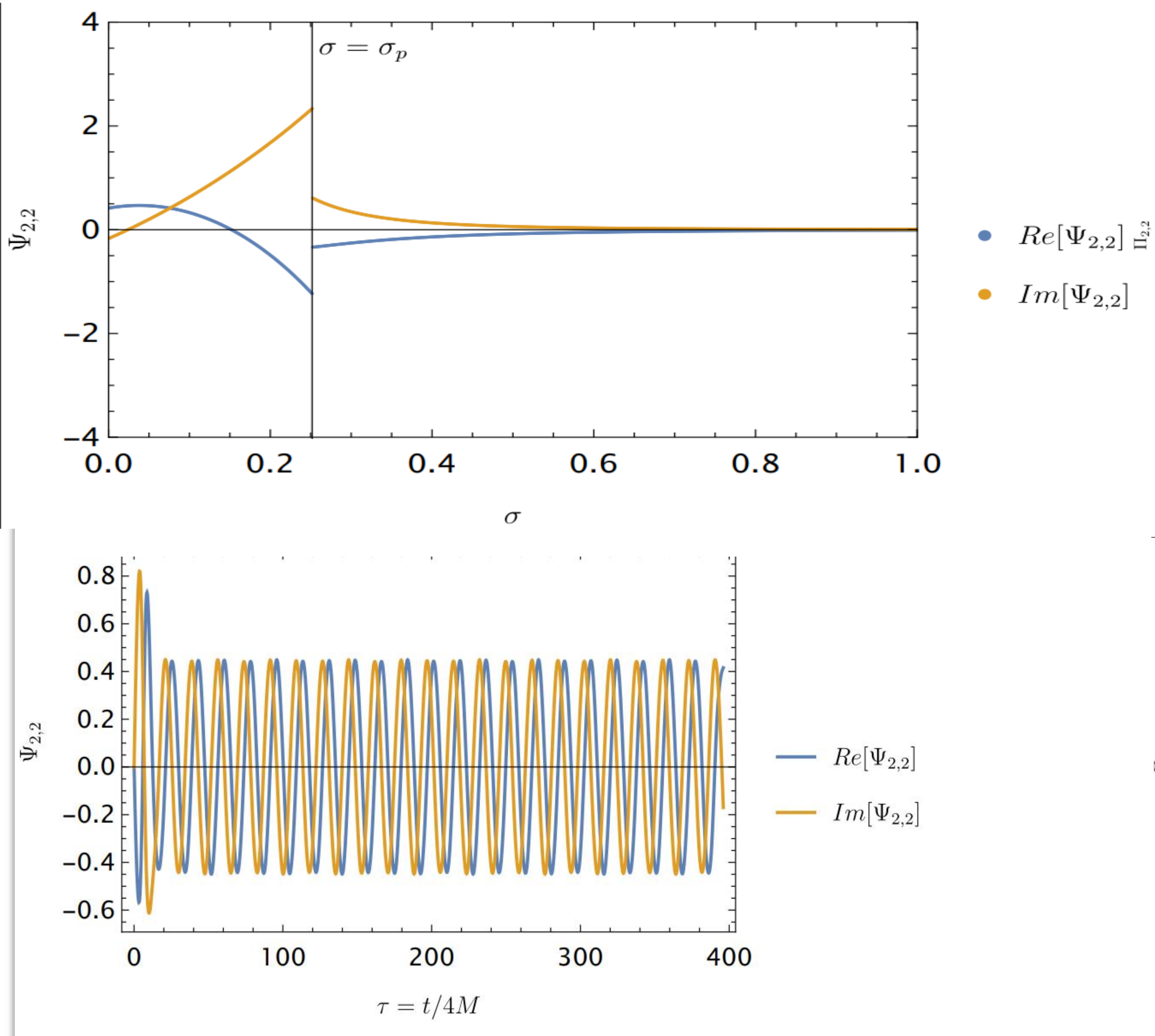


Fig 1 – Top: Retarded hyperboloidal field for a point particle on a circular geodesic where r_p= 7.9456 M and τ =100.

Bottom: Gravitational waveform for the axial component $(l, m) = (2,2)$ of the Z master function

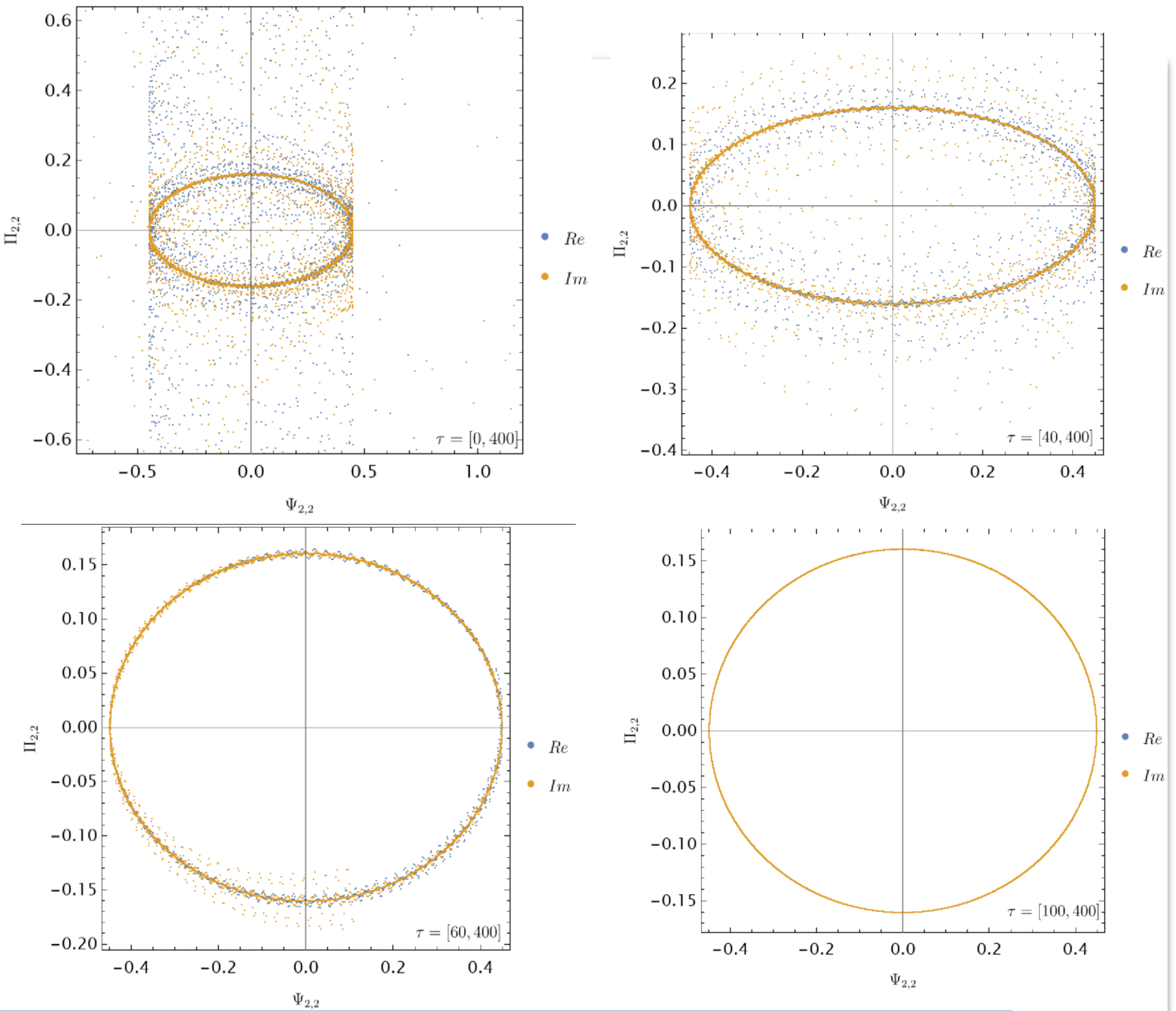
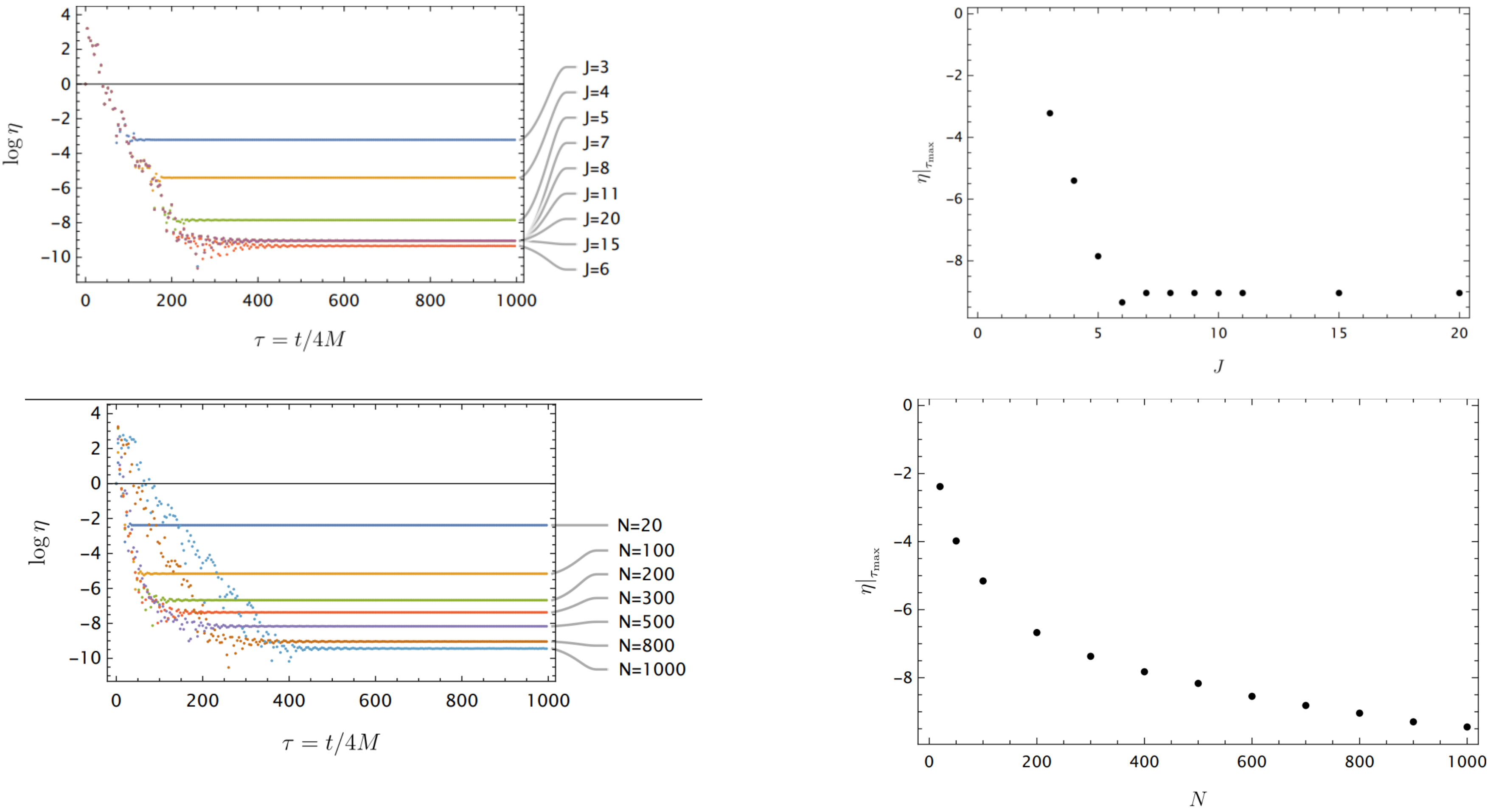


Fig 2 - Phase portrait of our numerical evolution for the $\Psi_{2,2}(\tau, \sigma)$. From the top left to the bottom right we include the evolution from an initial time of τ = {0, 40, 60, 100}.

Phase 1: Numerical optimisation factors for test 1

https://arxiv.org/pdf/2306.13153.pdf



Numerics:

Collocation nodes: Equidistant
Integration scheme: Hermite 4

Numerical Optimisation factors:

i. N= 800
ii. J > 6, we chose J=11
iii. Delta t = 0.02
iv. T extraction = 400

Fig 3 – Convergence studies to determine the optimisation factors Top: Convergence study determining the optimal number of jumps required for accurate evolution and convergence rate showing the numerical error associated with the choice of jumps at the final evolution time τ_{max} . **Bottom:** Convergence study determining the optimal number of nodes required for accurate evolution and convergence rate showing the numerical error associated with the choice of nodes at the final evolution time τ_{max}

Phase 1 – Test 1 – Radiation fluxes at the "infinities"

https://arxiv.org/pdf/2306.13153.pdf

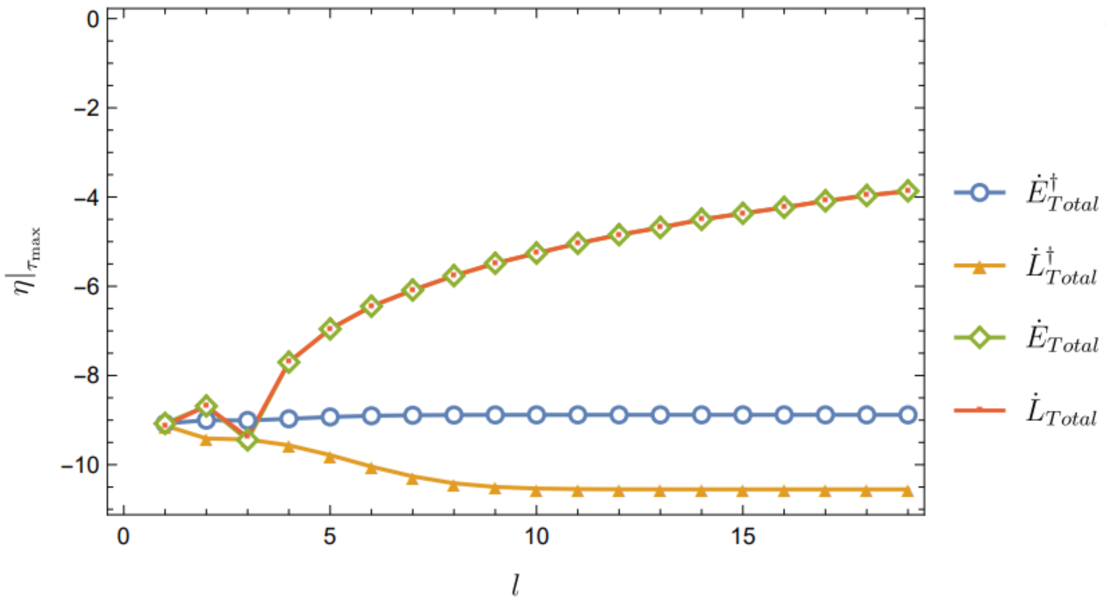


Fig 4 – Numerical error associated with the individual modal contributions E˙ † Total/L˙ † Total and the error associated with the increase of modal contributions to the final value of the energy and angular momentum fluxes.

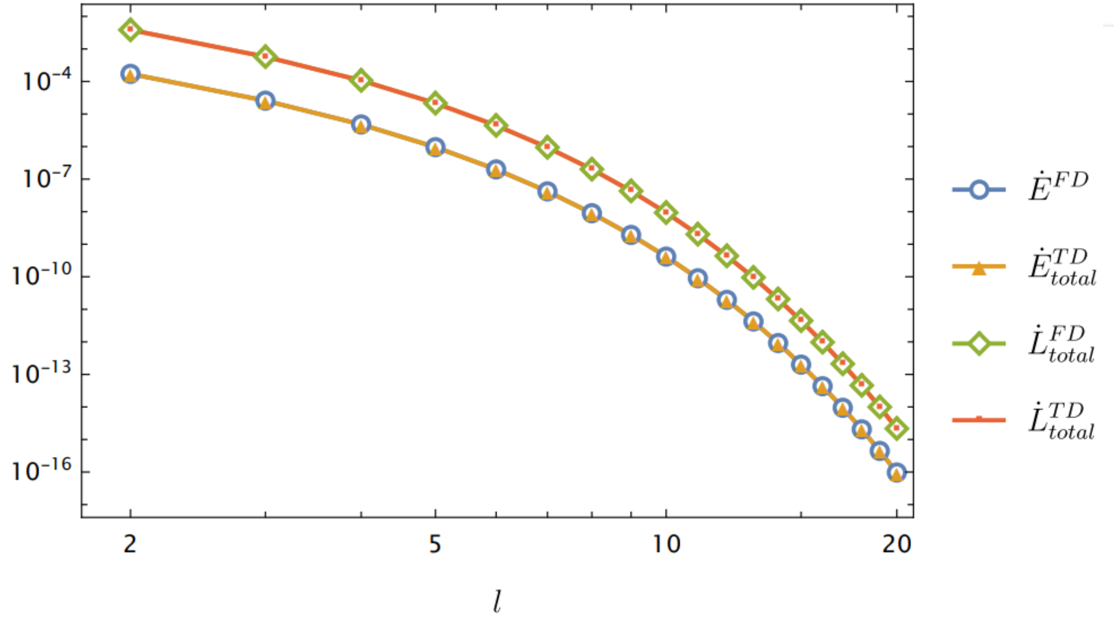


Fig 5 – E˙ Total and L˙ Total modes convergence. Around the l_{max} = 20 mode the individual contributions become increasingly less significant, where in the last few l– modes is at around $10^{-15/16}$.

Q	Our TD Algorithm	Q	Our TD algorithm	η
$\frac{dE_{\text{total}}}{dt}$	**2.03294497 × 10^{-4}**	$\mathcal{F}^{t}_{diss}$	**3.85636795 × 10^{-4}**	**1.3× 10^{-9}**
$\frac{dL_{\text{total}}}{dt}$	**4.5531889267×10^{-3}**	$\mathcal{F}^{\phi}_{diss}$	**-8.6371112157×10^{-3}**	**2.8×10^{-11}**

Table I – Energy/angular momentum fluxes and respective components for the dissipative GSF, for a circular orbit (p = 7.9456, e=0) on a Schwarzschild background obtained from solving the RW/Z equations for a total of l_{max} = 20 modes

Main Observations from test 1:

- ✓ Significant improvement from previous TD work
- ✓ Competitive accuracy to FD methods when solving a distributionally sourced PDE like the RWZ equations is possible to at most l = 20 modes
- ✓ Minimal time steady-state evolution and hereby extraction requires careful consideration to ensure junk radiation from ID choices does not contaminate our results

Numerical optimisation factors for test 2

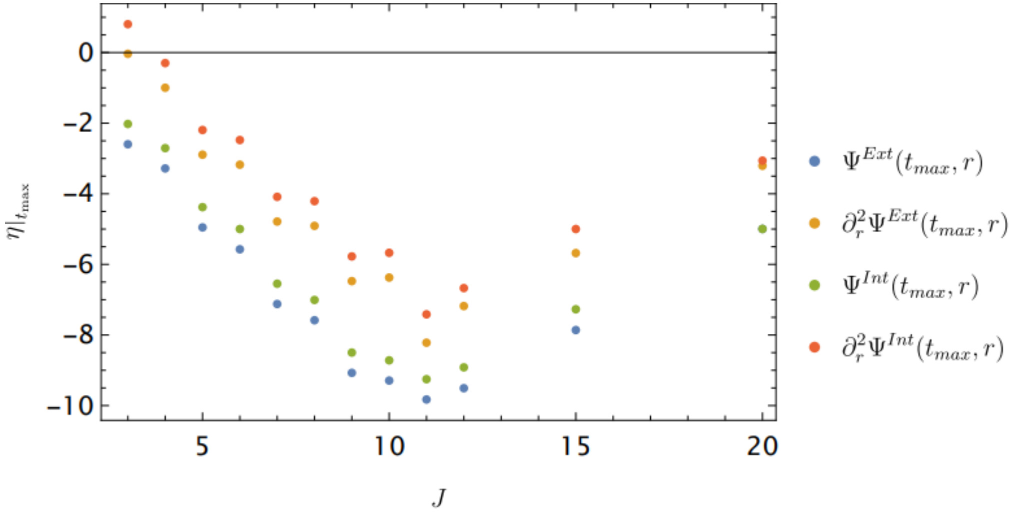
https://arxiv.org/pdf/2306.13153.pdf

You now need to interpolate at wanted position use:

$$\boldsymbol{\phi}(\boldsymbol{\sigma}) = \sum_{j=0}^{N} \left[\boldsymbol{\phi}_j + \boldsymbol{\Delta}(\boldsymbol{\sigma}_j - \boldsymbol{\sigma}_p; \boldsymbol{\sigma} - \boldsymbol{\sigma}_p)\right]$$

Or

$$\partial_\sigma^n \boldsymbol{\phi}(\boldsymbol{\tau}, \boldsymbol{\sigma})\Big|_{\sigma=\sigma_i} \approx \sum_{j=0}^{N} \boldsymbol{D}_{ij=0}^{(n)} \left[\boldsymbol{\phi}_j + \boldsymbol{\Delta}(\boldsymbol{\sigma}_j - \boldsymbol{\sigma}_p; \boldsymbol{\sigma}_i - \boldsymbol{\sigma}_p)\right]$$



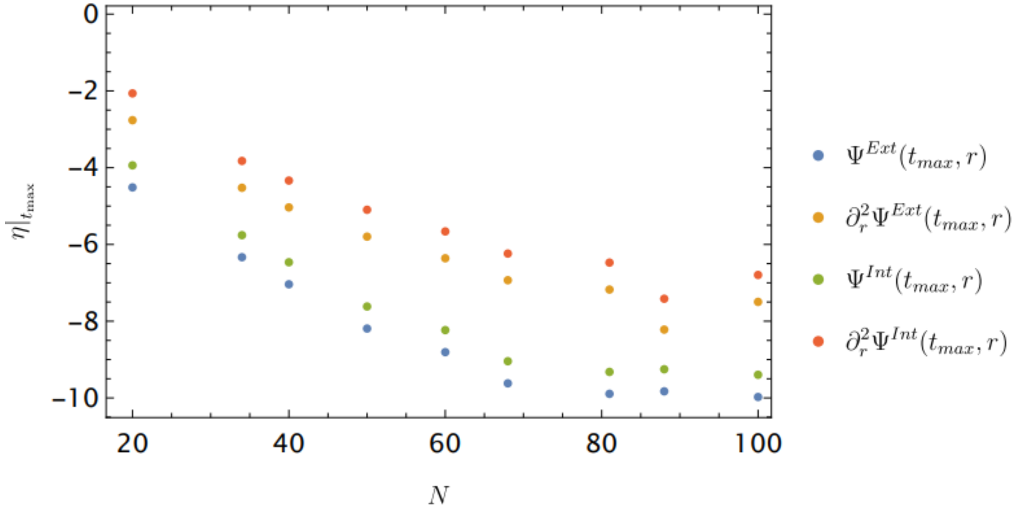


Fig 6 – Convergence studies to determine the optimisation factors **Top:** Convergence study determining the optimal number of jumps required for accurate evolution **Bottom:** Convergence study determining the optimal number of nodes required for accurate evolution at the final evolution time τ_{max}

Numerics:

Collocation nodes: Pseudospectral
Integration scheme: Hermite 4

Numerical Optimisation factors:

i. N= 89
ii. J = 11
iii. Delta t = 0.02
iv. T extraction = 10,000

Phase 1 – Test 2 – Direct evaluations at the particle

https://arxiv.org/pdf/2306.13153.pdf

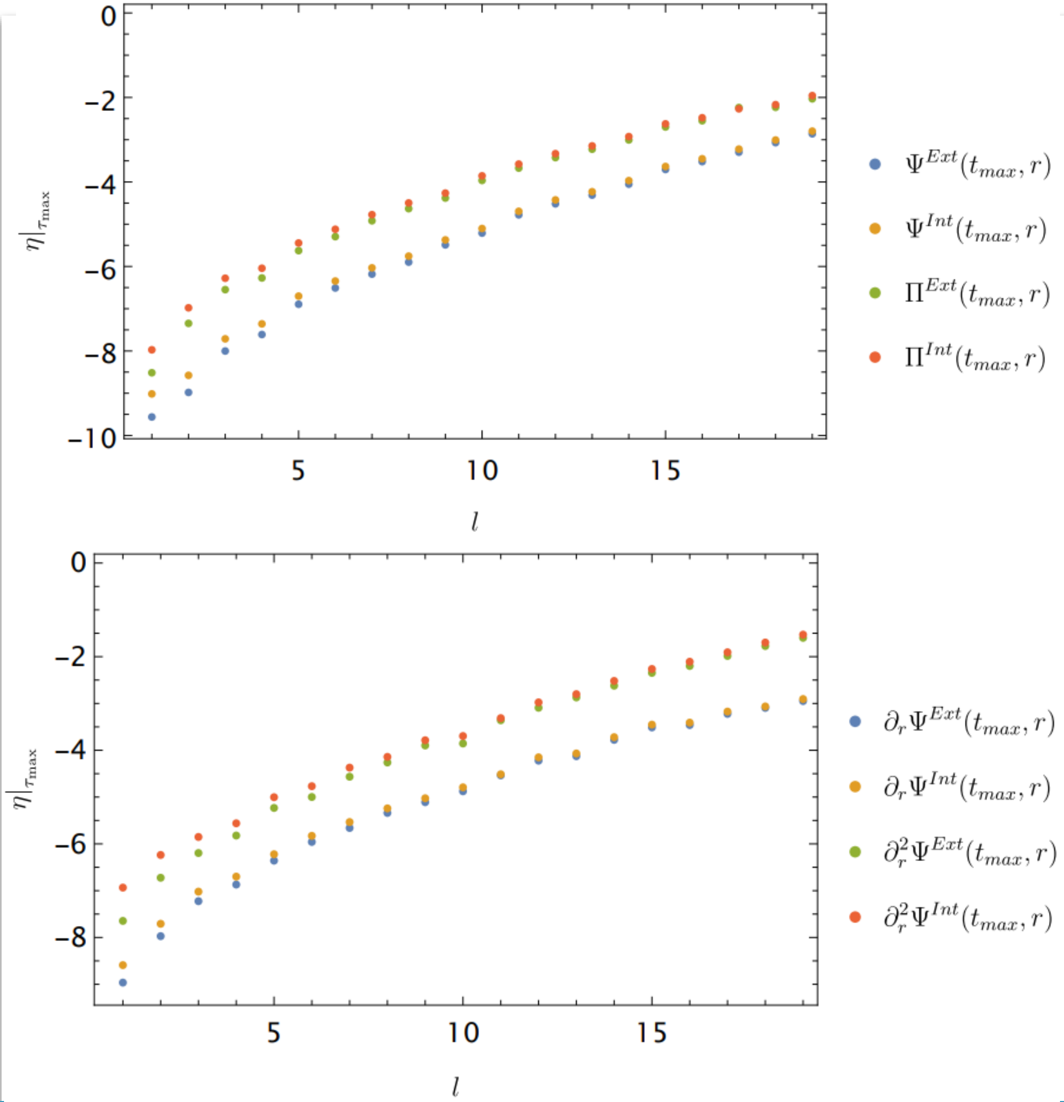


Fig 7 – Individual l-mode contributions to the numerical error between our time-domain results against the reference values of Thompson et al for the fields and its first order t,r derivatives.

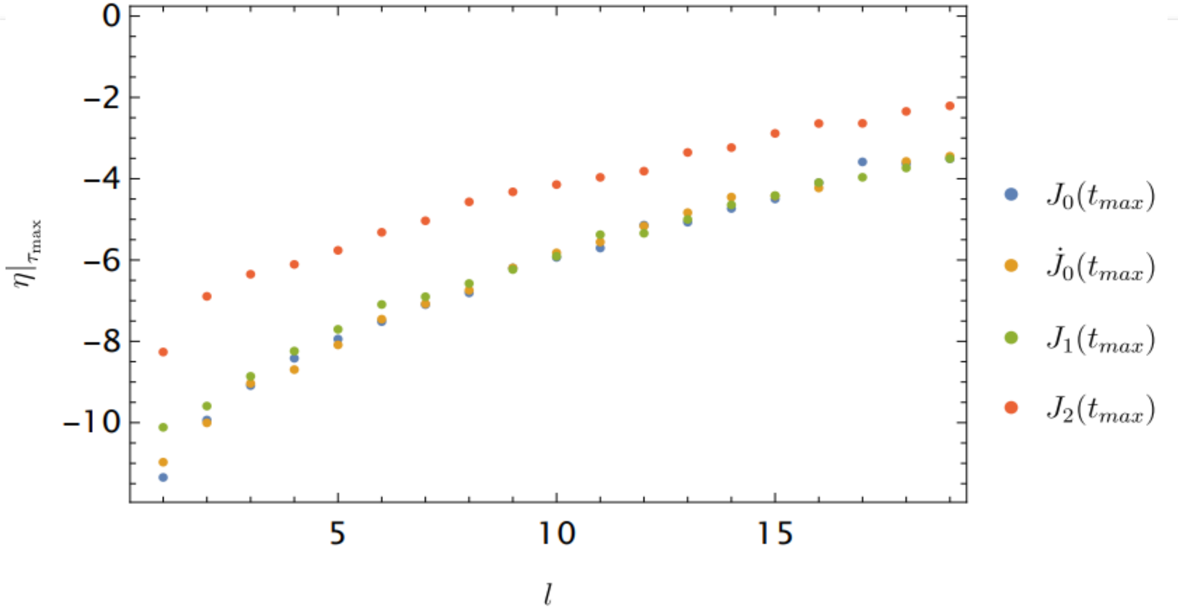


Fig 8 – Internal Error - Individual l-mode contribution to the numerical error associated with the difference of our TD numerical approach versus the reference FD approach [38]

Main Observations from test 2:

- ✓ What is optimal for test 1, in your framework is not necessarily optimal for the evaluations at the particle
- ✓ Algorithm shows suitability to either computations
- ✓ Gauge optimal factors based on most complex higher order derivative
- ✓ Competitive accuracy to FD methods when solving a distributionally sourced PDE like the RWZ equations is possible to at most l = 20 modes
- ✓ Minimal time steady-state evolution and hereby extraction requires careful consideration to ensure junk radiation from ID choices does not contaminate our results

Phase 2 – SSF – Self-force Calculation – Results attained without adequate calibrations before our publication

SSF from fluxes balance laws:

- Compute combined total energy coming in (horizon) and out (infinity) of the BH

$$F_t = \mu\, u^t \;\frac{dE}{dt}$$

r_p	Our TD result	Reference FD [1]	Numerical Error
6 M	3.6090×10^{-4}	3.6090×10^{-4}	4.82437×10^{-6}

Table II – SSF results from our time domain code compared against reference FD values computed through dissipative approaches. **[1] – BHPT toolkit;**

Mode sum

- Solve wave equation mode-by-mode, subtracting adequate regularisation parameters, the sum over all modes

$$F_{lt} = \sum_{m=-l}^{l} \partial_t\, \psi(t, r_p) Y^{*,lm}\left(\frac{\pi}{2}, \phi_p\right)$$

r_p	Our TD result	TD [2]	TD [3]	Numerical Error
6M	3.6088×10^{-4}	3.60778×10^{-4}	3.60339×10^{-4}	8.33809×10^{-5}

Table III – SSF results from our time domain code compared against reference FD values computed through dissipative approaches; **[1] – BHPT toolkit; [2] - https://arxiv.org/abs/0903.0505 ; [3] - https://arxiv.org/pdf/0704.0797.pdf**

Paper II: Hyperboloidal Discontinuous Numerical Algorithm for the Computation of the SSF on a Charged Particle on a Schwarzschild background in the TD – Lidia J. Gomes Da Silva and Rodrigo Panosso Macedo, Oliver Long, Juan Valiente Kroon and Leor Barack (to also include Southampton's code results led by Rodrigo, not here included)

Phase 2 & 3 – GSF – Gravitational Self-force in TD @ the particle

- Required metric perturbation for GSF odd calculation

$$\beta(t,r) = K(r)[f\phi^{RW} + rf\partial_r\phi^{RW} - r^3 E_c]$$

- GSF odd expression in RW Gauge (same as Eazy Gauge)

$$F^{r,lm}_{RW,odd} = K(r)[\beta^{lm}(t,r) + r_p\partial_r\beta^{lm}(t,r)]\partial_\theta Y_{lm}\left(\frac{\pi}{2},\Omega\, t\right)$$

- Final GSF result requires both parities

$$F_{lm} = F_{lm}^{Odd} + F_{lm}^{Even}$$

- Final expression shall account for full regularization

$$F^r_{lm} = \sum_{l=0}^{l_{max}}\left[F^{r,l}_{RW,ret} - (2l+1)F^r_{[-1]} + F^r_{[0],RW}\right] - D^r_{RW}$$

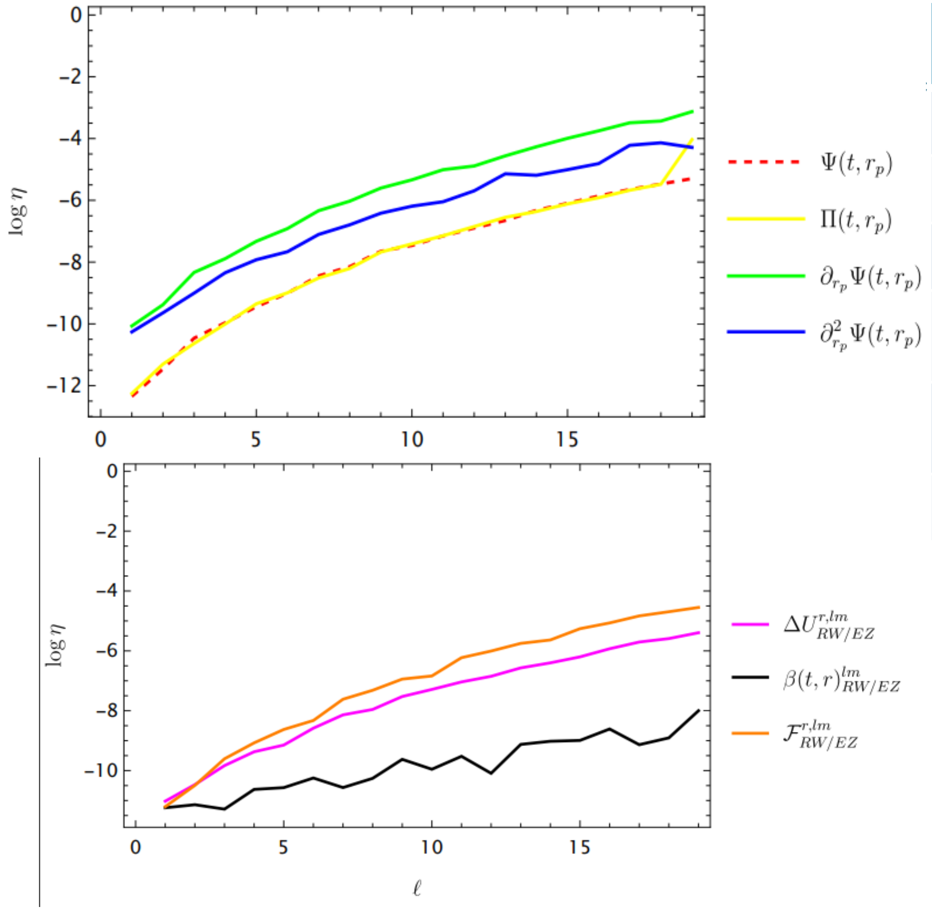


Fig 9 – Numerical convergence for the field and relevant physical quantities for the all l_{max} = 20 modes against reference frequency domain values of Thompson et al at r_p = 7.9456M.

Quantity	Accuracy relative to FD work of [1]
$\phi(t,r_p)$	3.3526 x 10^{-15}
$\partial_r\phi(t,r_p)$	5.00657 x 10^{-11}
$\partial_r^2\phi(t,r_p)$	7.71624 x 10^{-11}
Redshift, $U(t,r_p)$	4.13964 x 10^{-13}-2.93961 x 10^{-12} I
Metric, $h(t,r_p)$	-7.33721 x 10^{-13} + 3.31624 x 10^{-12} I
$F(t,r_p)$	2.05052 x 10^{-12}

Table IV – Comparisons for all relevant physical quantities for the odd component of the SF against [1].

[1] – Thompson et al, https://arxiv.org/pdf/1811.04432.pdf

Paper III : GSF computation in the RW and EZ gauges in the TD – Lidia J. Gomes Da Silva, Rodrigo Panosso Macedo and Jonathan Thompson, Juan Valiente Kroon, Leanne Durkan, Oliver Long

Summary & Outlook & Thanks

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- Accurate physical computations **are possible in the TD**
- Test 2 – Evaluations at the particle **should be the paramount test when determining if a TD method is or not feasible for complete SF computations**
- Adequate hyperboloidal, discontinuous interpolation and time-symmetric frameworks **significantly** simplify the numerical recipe
- Quadruple precision HPC codes are needed to aid further computations in both TD and FD (specially if using hyperboloidal methods …………..)

- Help complete research programme **at the very least** in Schwarzschild in the RW and EZ gauges and compute the GSF first order calculations
  - Eccentric Orbits – implementation of our work and complete regularization framework (to date effort been by Josh Mathews, Jonathan Thompson et al);
  - Radial Infall to compare with Barack et al;
  - Corroborate Josh's spinning secondary work in the TD

- Compute hyperbolic scattering angles (for comparison with PN/PM/EOB)
- **Self-consistent evolution** under the influence of GSF in TD

- Very special thank you to my collaborators.
- Very special thank you to the mentorship, PhD support and encouragement from QMUL's **Juan, Pau, Alex, Katie and Katy.**

*** Thanks for listening ***

Always happy to address science queries, lidiajoana@pm.me

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Some reserve slides from here - Phase 2 – GSF – Metric Reconstruction

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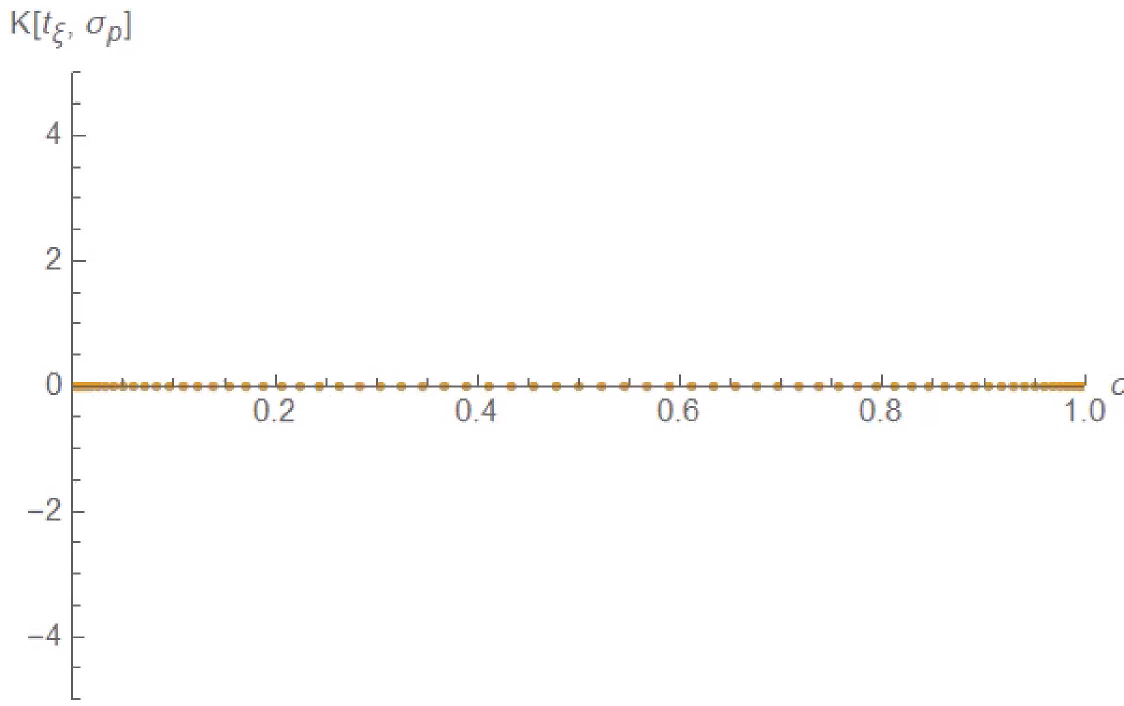
For example, one of the 4 perturbations in the even case is:

$$K^{(t,r)} = f(r)\,\frac{\partial \phi_{ZM}}{\partial r} + A(r)\phi_{ZM} - \frac{r^2 f(r)^2}{(\Lambda + 1)(\Lambda)}\,Q^{tt}$$

Generically we can write the perturbations as,

$$Pert(t,r) = Pert_{Smo}(t,r) + Pert_{Sin}\,(t,r)$$



Upcoming Paper III : Hyperboloidal Discontinuous Numerical Algorithm for the Computation of the GSF on Schwarzschild background in the TD – Lidia J. Gomes Da Silva, Rodrigo Panosso Macedo and Jonathan Thompson, Juan Valiente Kroon, Leanne Durkan, Oliver Long

Discontinuous collocation method - Lagrange interpolation

/***

N^{th} order polynomial

$$p(x) = \sum_{j=0}^{N} c_j \pi_j \qquad (3)$$

Collocation conditions

$$p(x) = f_i, \;\; i = 0,1, \ldots N \qquad (4)$$

Solution: Lagrange interpolating polynomial

$$p(x) = \sum_{j=0}^{N} f_j(x)\pi_j(x)\,, \pi_j = \prod_{k=0,k\neq j}^{N} \frac{x-x_k}{x_j-x_k} \qquad (5)$$

***/

Discontinuous CM: Discontinuous Lagrange Interpolation

/***

Nth order piecewise polynomial

$$p(x) = \sum_{j=0}^{N} \left[\theta(x-\xi) c_j^+ + \theta(\xi - x) c_j^- \right] x^j \quad (6)$$

Collocation conditions

$$p(x) = f_i, \;\; i = 0,1, \ldots., N \quad (7)$$

Jump conditions

$$p^{(k)}(\xi^+) - p^{(k)}(\xi^-) = J_k, \;\; k = 0,1,2, \ldots.. \quad (8)$$

Solution: Interpolating piecewise polynomial

$$p(x) = \sum_{j=0}^{N} \left[f_j(x) + \Delta\left(x_j - \xi; x - \xi\right) \right] \pi_j(x) \quad (9)$$

***/

Discontinuous CM: Examples

/**

Interpolation

$$p(x) = \sum_{j=0}^{N} \left[f_j(x) + \Delta\left(x_j - \xi; x_i - \xi\right) \right], \qquad D_{ij}^{n} = \pi_j^{(n)}(x_i) \tag{10}$$

Differentiation

$$p^{(n)}(x_i) = \sum_{j=0}^{N} D_{ij}^{n} \left[f_j(x) + \Delta\left(x_j - \xi; x_i - \xi\right) \right] \tag{11}$$

where

$$\Delta\left(x_j - \xi; x_i - \xi\right) = \left[\theta(x_i - \xi) - \theta\left(x_j - \xi\right) \right] \sum_k \frac{J_k}{k!} \left(x_j - \xi\right)^k \tag{12}$$

**/

Discontinuous “Conformal” RWZ Solution - Static Solution

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Boundary Conditions - Avoided by using conformal theory

Initial Data – Exact Static Even and Odd Solutions

- Solve

$$\phi^{RW(Odd)}(\sigma) = \frac{1}{\sqrt{-g}}\partial_\beta\left(\sqrt{-g}\, g^{\alpha\beta}\left(\partial_\alpha\phi(\sigma)\right)\right) - V^{RW(Odd)}(\sigma)\phi^{RW(Odd)}(\sigma) = 0$$

To get hypergeometric odd solutions

$$\phi^{RW(Odd)}(\sigma) = \left[(-1)^{l+1}\sigma^{-(l+1)}c_1\left[Hypergeometric[-(l+2,(2-l),-2l),\sigma]\right],(-\sigma)^{l+1}c_2\left[Hypergeometric[-1+l,3+l,2(1+l),\sigma]\right]\right]$$

- Get even hypergeometric solution by applying Darboux transformation

$$DT(\sigma) = \frac{n(n+1)}{3M} + \frac{3M(r(\sigma)-2M)}{r^2(\sigma)(n\, r(\sigma)+3M)}$$

- To get the equivalent two solutions for the even case

$$\phi^{even(1/2)}(\sigma)\left(1-\frac{2M}{r(\sigma)}\right)\frac{1}{\partial_\sigma r(\sigma)}\partial_\sigma\phi^{odd(1/2)}(\sigma) + DT(\sigma)\phi^{odd(1/2)}(\sigma)$$

See https://arxiv.org/pdf/1702.06459.pdf

Spatial Discontinuous Discretization (apply recurrence jump relation) ***+ Time Discontinuous*** Integration as given in

***/

Time Integration Methods – Demystifying GI Explicit Methods

/***

In the late 90's** implicit methods start being explored through the **Iterated Crank Nicholson** (ICN) Scheme written in an explicit form.

Misconceptions seemed to originate here with two major conclusions:

- **Proposition I** - The ICN scheme **retains stability for at most two iterations**, any higher number of iterations will result in a less stable method.

- **Proposition II** - The accuracy of the ICN scheme remains second order in Δt and Δx from the first iteration on. **Doing more iterations changes the stability behavior, but not the accuracy**. Since the smallest number of iterations for which the method is stable is two, there is no point in carrying out more iterations than this.

We can solve the Iterated Crank-Nicholson Scheme explicitly as:

- Initializing Step

$$u^{n+1} = u^n + (\Delta t\, \boldsymbol{L}).\, u^n$$

- Then Iterate n times

$$u^{n+1} = u^n + \frac{\Delta t}{2} \boldsymbol{L}.\, (u^{n+1} + u^n)$$

**** Sources:**
https://arxiv.org/pdf/gr-qc/9909026.pdf, https://arxiv.org/pdf/0807.0734.pdf

Symmetry, Symplecticity and Stability of ICN and Hermite Schemes – Lidia J. Gomes Da Silva, Michael F. O'Boyle, Charalampos Markakis

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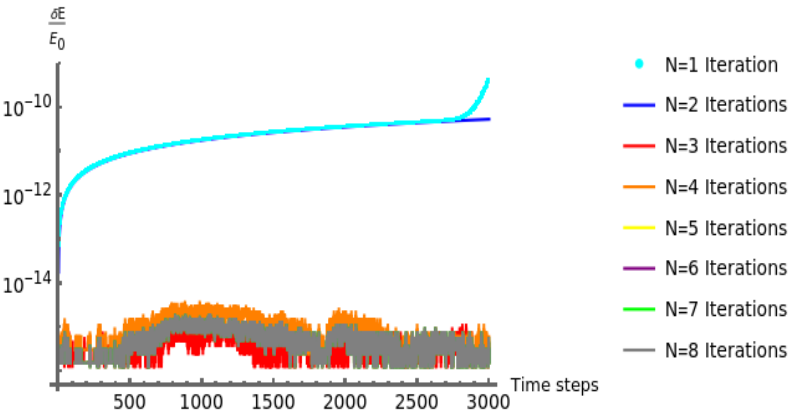
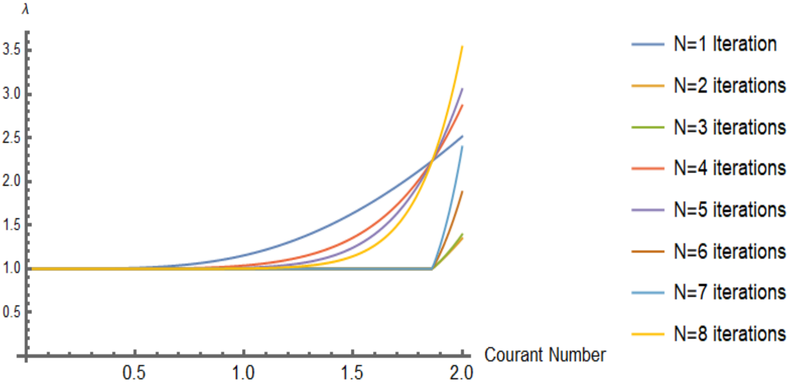
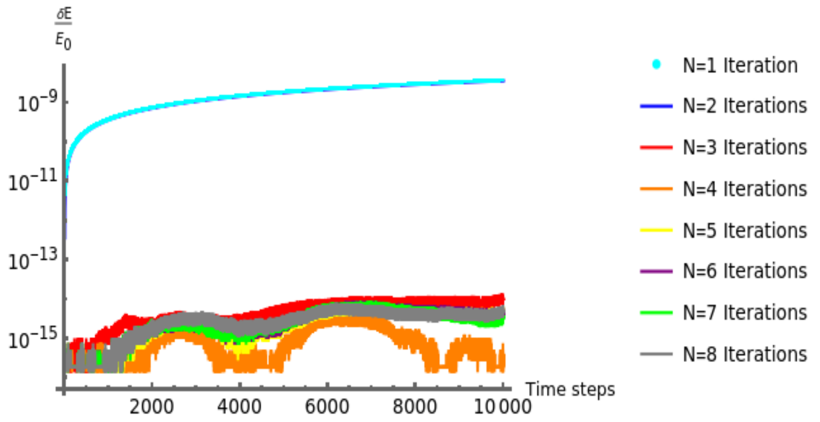
ICN Time Integration – Examples

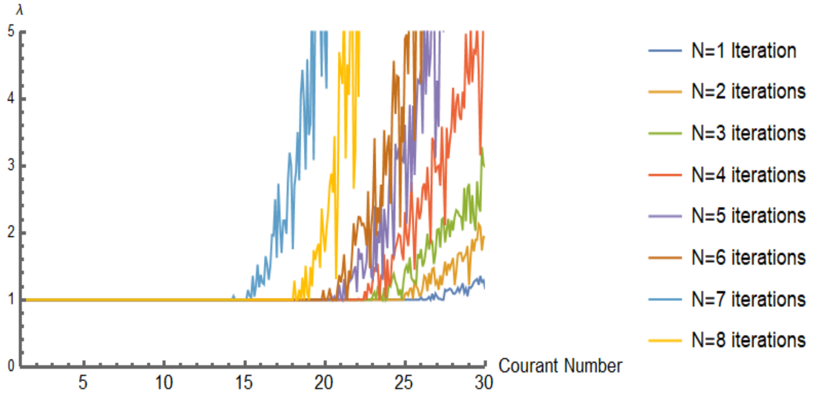
EX.1 - Wave Equation with **Freezing BCs**



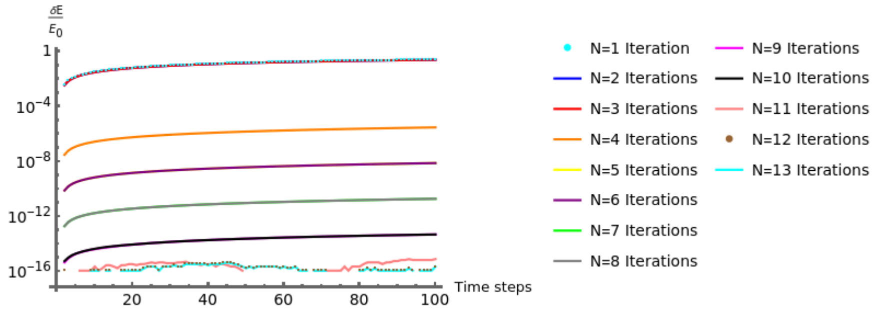


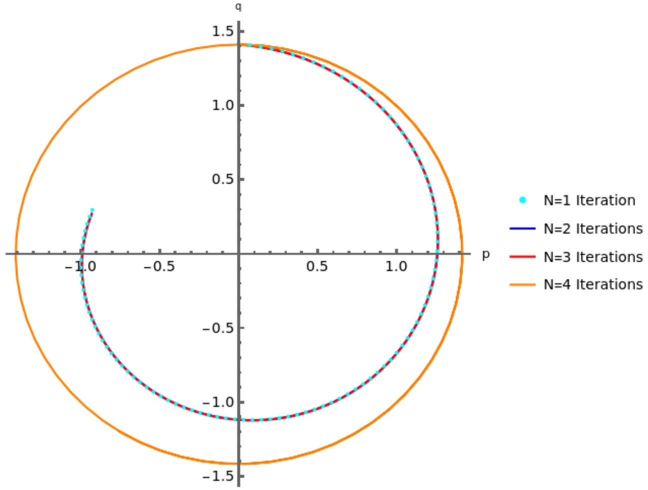
EX.2 – Wave Equation in a ***Conformal Setting***





Ex.3 – Harmonic Oscillator





Symmetry, Symplecticity and Stability of ICN and Hermite Schemes – Lidia J. Gomes Da Silva, Michael F. O'Boyle, Charalampos Markakis

Results

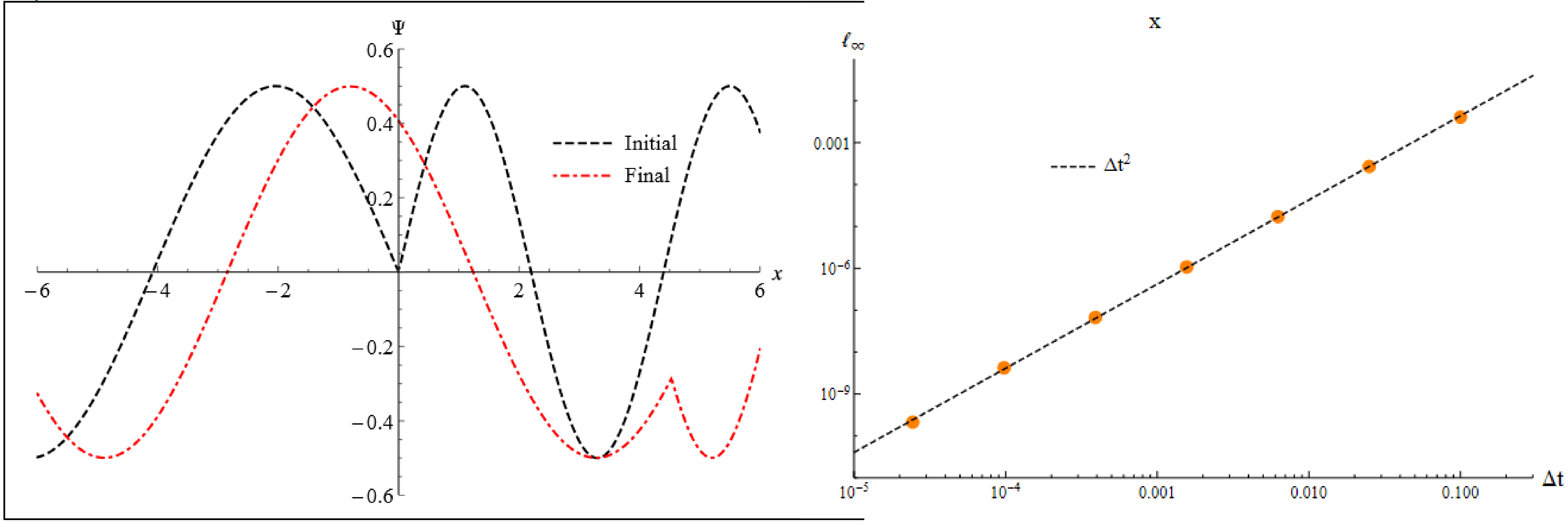


Figure 2: Left: Numerical solution using the discontinuous collocation methods with an order 2 Hermite integration time stepper. We selected initial function to be the exact solution provided in https://arxiv.org/abs/0902.1287 .
Right: The l^{∞} error norm scales as Δt^2, as expected of our method.

This work can be found in https://arxiv.org/abs/1406.4865.

Results - Conformal Homogeneous Wave Equation

/***

We want to solve numerically:

$$\eta^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi = 0 \qquad (28)$$

transformed with the conformal slice of eq.(19)

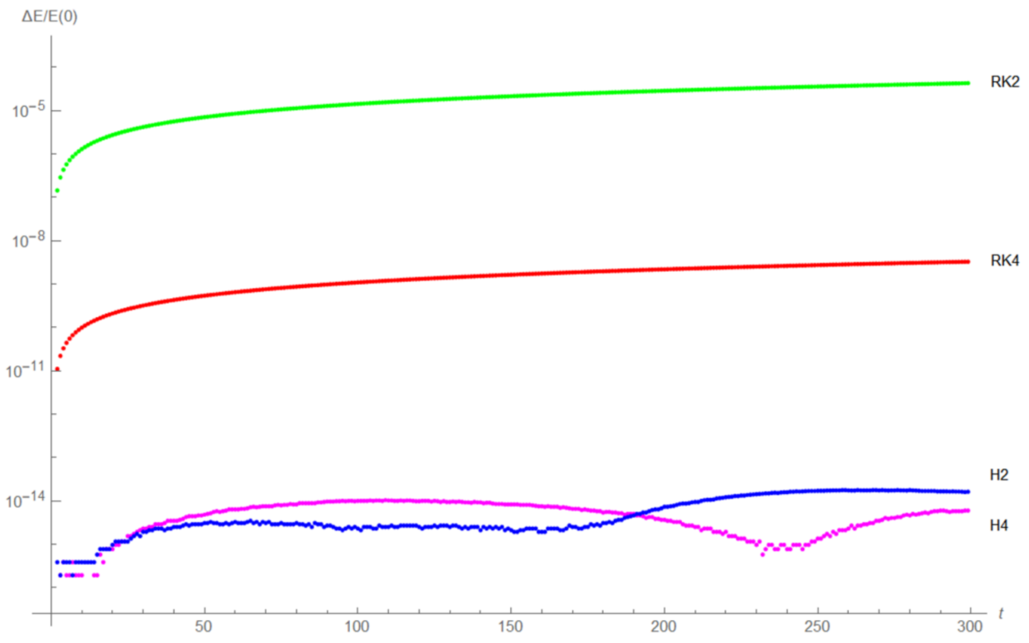


Figure 3 : Relative error in energy for the flat homogeneous conformal wave equation

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Results - Conformal Homogeneous Wave Equation

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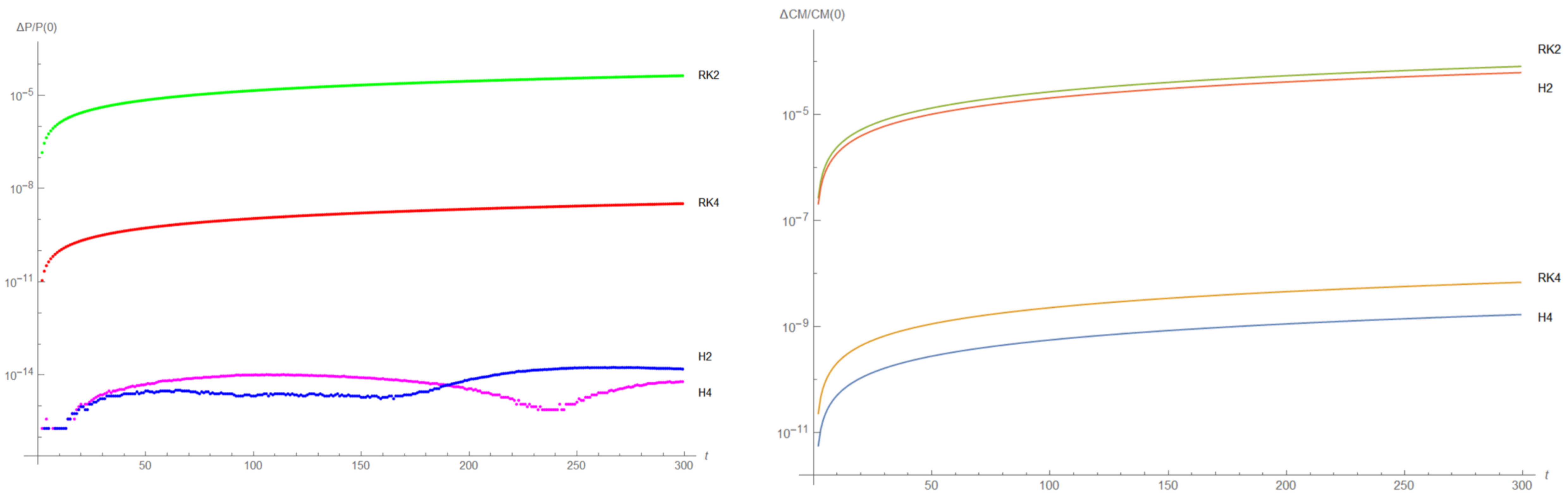

Figure 4 Left: Relative error in momentum for the flat homogeneous conformal wave equation.
Right: Relative error in CoM for the flat homogeneous conformal wave equation.

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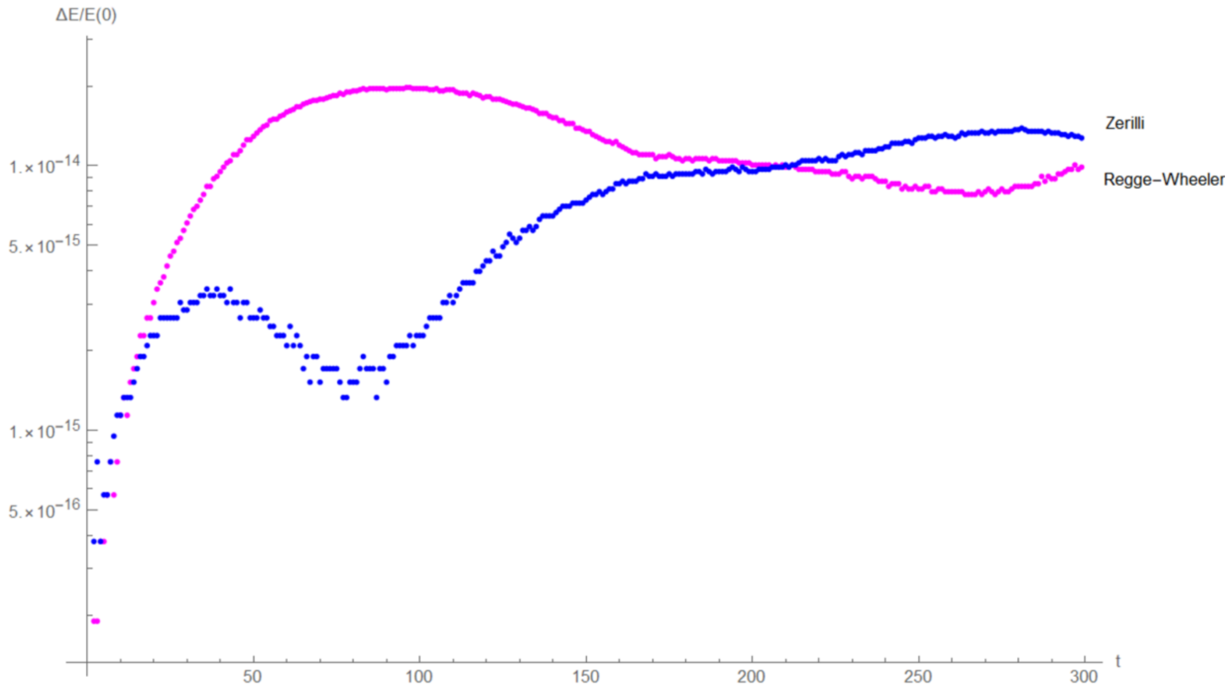
Results - Conformal Homogeneous RWZ Equation

/**



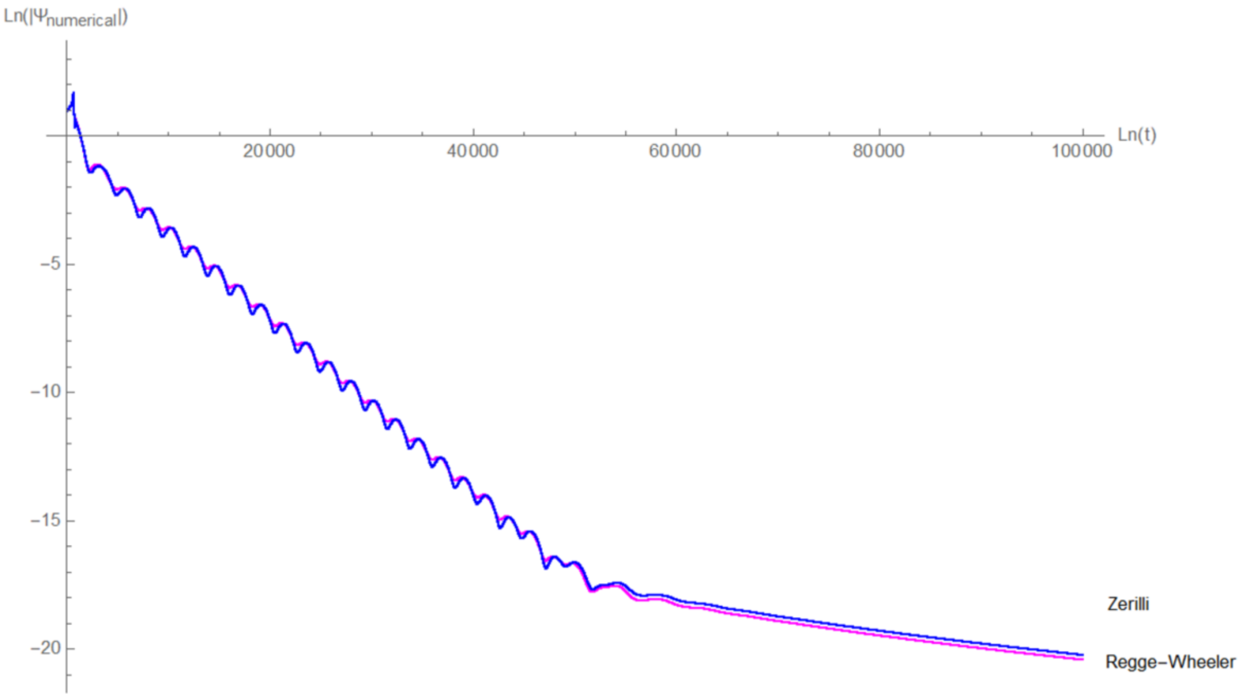


Figure 4 Left: Relative error in momentum for the flat homogeneous RWZ equation.
Right: Late time evolution tail computation for the RWZ equation.

This discussion will be on ***Paper I.***

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Discontinuous Conformal RWZ Solution – Circular Orbits

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Boundary Conditions - Avoided by using conformal theory as given by eqs.(20-21)

Initial Data – Two options

=> Choose zero initial data (as previously done in the literature)

=> Compute exact solution using known frequency domain solutions and known orbital frequency

Spatial Discontinuous Discretization (apply recurrence jump relation) ***+ Time Discontinuous*** Integration as given in eqs.(29)

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Previous work – Groups working towards calculation the GSF in the TD in a RWZ Gauge

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- 2 order Finite difference + grid decomposition (Martel,Poisson)
  - https://arxiv.org/pdf/gr-qc/0107104.pdf, https://arxiv.org/pdf/gr-qc/0311017.pdf
  - https://arxiv.org/pdf/gr-qc/9705071.pdf

- Finite element representation of δ
  - https://arxiv.org/abs/gr-qc/0512028 (Sopuerta, Laguna)

- Hyperboloidal layer + approximate δ as a Gaussian pulse (Bernuzzi, Nagar, Zenginoglu)
  - https://arxiv.org/abs/1008.3809
  - https://arxiv.org/abs/1012.2456
  - https://arxiv.org/abs/1107.5402

- Domain decomposition + time dependent grid + jump conditions across the particle
  - https://arxiv.org/abs/0902.1287, https://arxiv.org/abs/1001.2578 (Field et al)
  - https://arxiv.org/pdf/0811.0294.pdf, https://arxiv.org/pdf/0903.0505.pdf, https://arxiv.org/pdf/1006.3201.pdf (Canizares et al)
  - Upcoming work referenced in https://arxiv.org/pdf/1712.01098.pdf (scalar case though) (Diener et al)

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