Radiation Reaction in Post-Minkowskian Approximation

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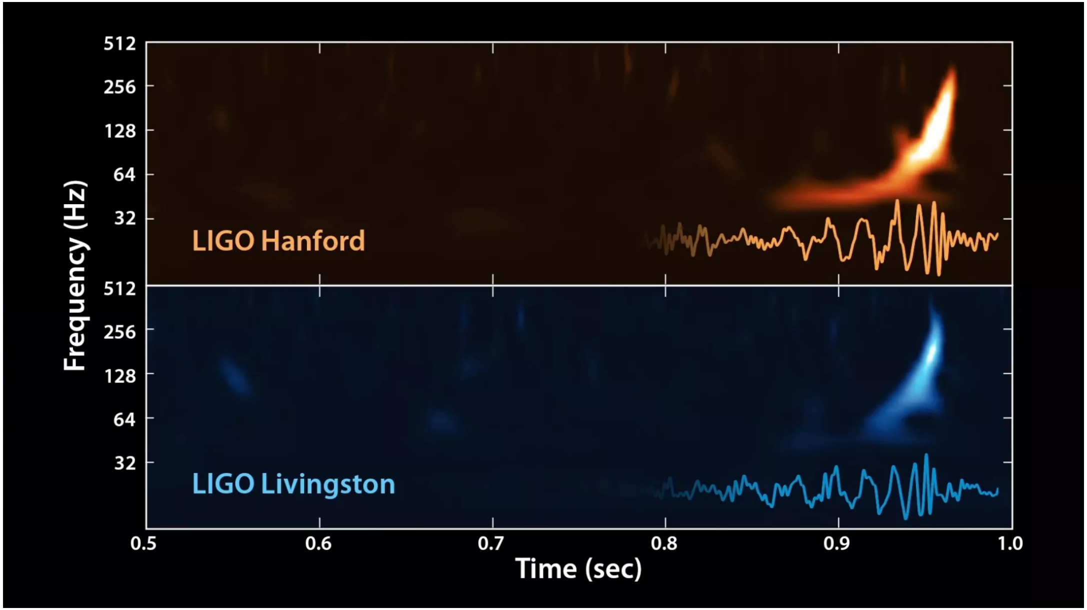
26th Capra Meeting, Copenhagen

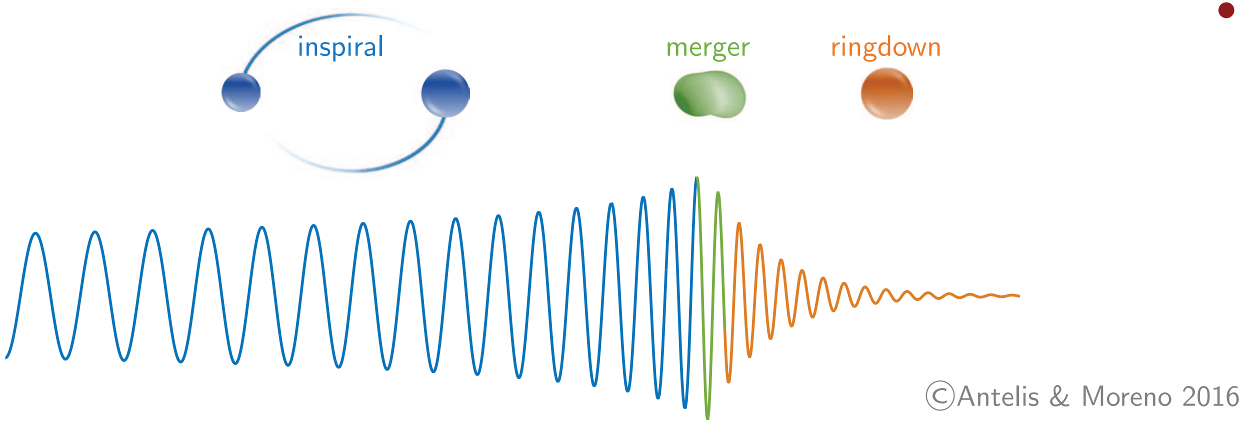
First observation of gravitational waves



One hundred years after the publication of his General Relativity,
Einstein's dream came true!

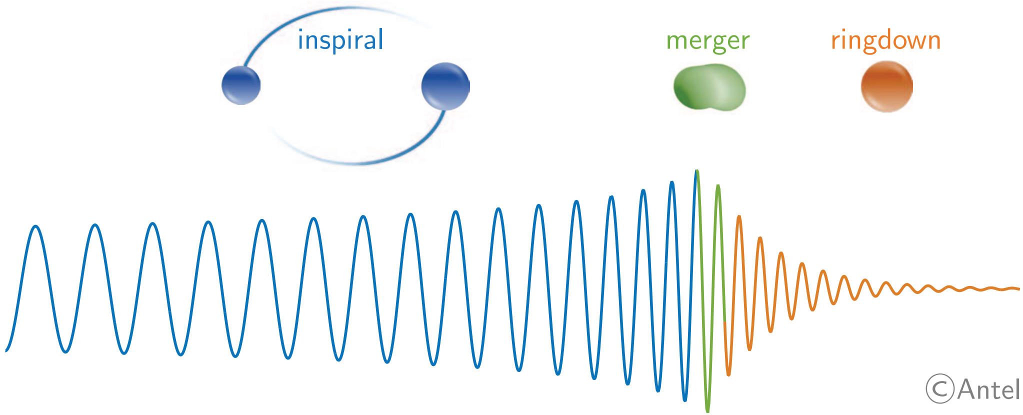
Gravitational waves provide an entirely new way to understand the Universe.

Gravitational-wave science



"Waveforms will be far more complex and carry more information than expected. Improved modeling will be needed for extracting the GW's information." Kip Thorne, Last 3 minutes, 1993

Gravitational-wave science



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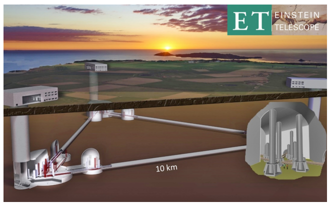
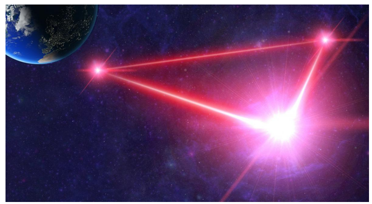
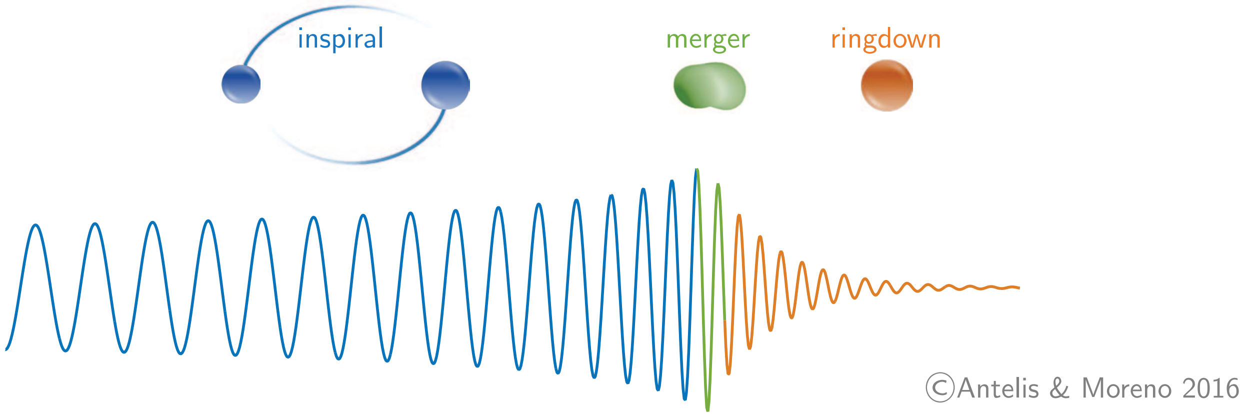
LIGO VIRGO KAGRA





Precise theoretical predictions for the motion of GW sources are crucial in interpreting data and maximizing discovery potential for present and future observations.

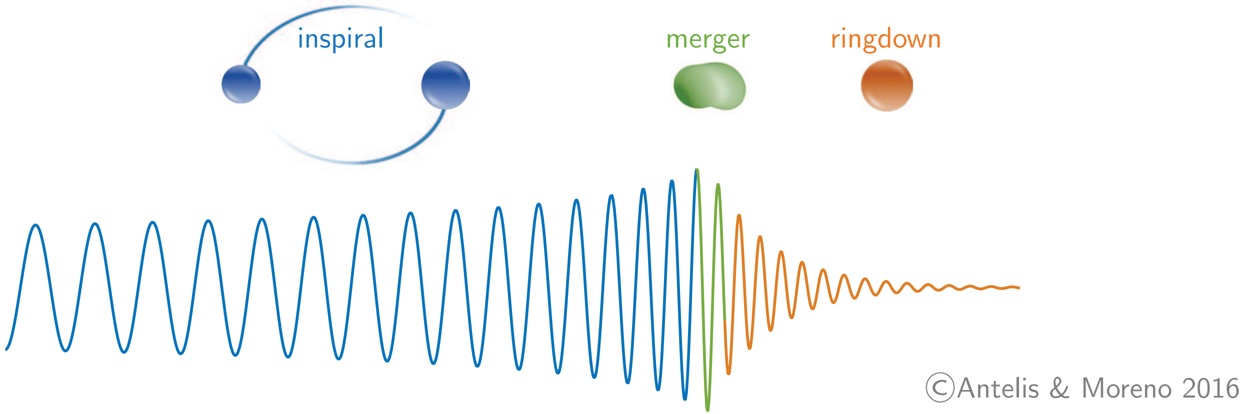
Gravitational waves from binary coalescences



Merger: Numerical Relativity

Ringdown: Black Hole Perturbation theory

Gravitational waves from binary coalescences



Merger: Numerical Relativity

Ringdown: Black Hole Perturbation theory

Inspiral: the relative velocity v is small

$$v^2 \sim \frac{GM}{r} \ll 1$$

- Numerical Relativity: accurately, but computationally expensive
- Analytic methods: corrections in v or G are perturbatively calculable

Effective Field Theory

- In the inspiral phase

$$g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{32\pi G}\, h_{\mu\nu}$$

- Effective action for gravitational binary systems Goldberger-Rothstein 2004

$$e^{i\mathcal{S}_{\text{eff}}[x_a(\tau)]} = \int \mathcal{D}h_{\mu\nu}\, e^{i\mathcal{S}_{\text{WL}}+i\mathcal{S}_{\text{GR}}}$$

where

$$\mathcal{S}_{\text{WL}} = \sum_{i=1,2}\left[-\frac{m_i}{2}\int dt\, g_{\mu\nu}\dot{x}_i^{\mu}\dot{x}_i^{\nu} + \cdots\right], \quad \mathcal{S}_{\text{GR}} = \frac{-1}{16\pi G}\int d^4x\,\sqrt{-g}\,R + \cdots$$

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- Post-Minkowskian expansion in powers of G

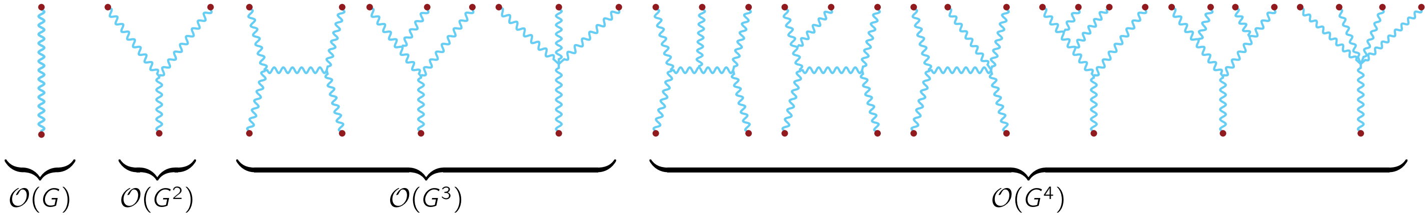
$$\mathcal{L}_{\text{eff}} = \mathcal{L}_0 + G\mathcal{L}_1 + G^2\mathcal{L}_2 + \cdots \qquad \mathcal{L}_0 = -\sum_i \frac{m_i}{2}\eta_{\mu\nu}\dot{x}_i^\mu \dot{x}_i^\nu$$

The equations of motion for trajectories:

$$m_i \ddot{x}_i^\mu = -\eta^{\mu\nu}\sum_{n=1}^{\infty}\left(\frac{\partial \mathcal{L}_n}{\partial x_i^\nu} - \frac{d}{d\tau_i}\frac{\partial \mathcal{L}_n}{\partial \dot{x}_i^\nu}\right) \qquad x_i^\mu = b_i^\mu + u_i^\mu \tau + \delta x_i^\mu(\tau) + \cdots$$

Effective Field Theory

- Classical physics: we use the saddle-point approximation in path integrals



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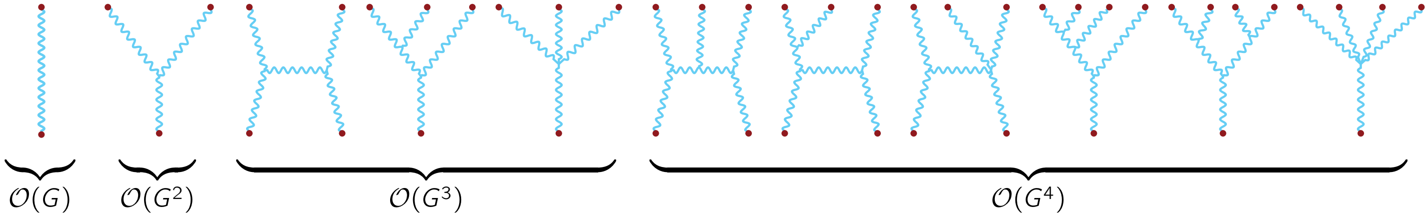


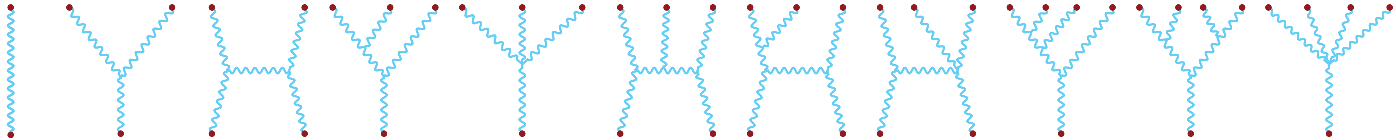
- Observables at $\mathcal{O}(G^N)$ Kälin-**ZL**-Porto 2007.04977 Dlapa-Kälin-**ZL**-Porto 2106.08276 2210.05541 2304.01275

$$\Delta p_i^\mu \sim \int d^D q \, \frac{e^{iq\cdot b}\,\delta(q\cdot u_1)\delta(q\cdot u_2)}{|q^2|^\sharp} \int \left(\prod_{i=1}^{N-1} d^D\ell_i \, \frac{\delta(\ell_j\cdot u_a)}{(\ell_i\cdot u_b - i0)^{\nu_i}} \right) \frac{\mathcal{N}^\mu(q,u_a)}{D_1 D_2 D_3\cdots}$$

  - Graviton propagators: $$\frac{1}{D_i} \longrightarrow \frac{1}{(\ell^0 \pm i0)^2 - \vec{\ell}^{\,2}} \quad \text{or} \quad \frac{1}{\ell^2 + i0}$$
  - Cut: always one delta function $\delta(\ell_i\cdot u_a)$ for each loop
  - Kinematics: $q\cdot u_a = 0,\ u_a^2 = 1,\ u_1\cdot u_2 = \gamma \implies$ single scale γ to all orders

- Analytic technology in high-energy physics and can be used to solve gravitational problems!

Inspiralling dynamics at NNNLO



$\mathcal{O}(G^4)$: three-loop integrals   Dlapa-Kälin-**ZL**-Porto 2106.08276 2112.11296 2210.05541 2304.01275

$$\int d^D\ell_1 d^D\ell_2 d^D\ell_3 \frac{\delta(\ell_1\cdot u_1)\,\delta(\ell_2\cdot u_2)\,\delta(\ell_3\cdot u_2)}{[\ell_1\cdot u_2]^{\alpha_1}\,[\ell_2\cdot u_1]^{\alpha_2}\,[\ell_3\cdot u_1]^{\alpha_3}}\,\frac{D_8^{-\nu_8}D_9^{-\nu_9}}{D_1^{\nu_1}D_2^{\nu_2}\cdots D_7^{\nu_7}}$$

$$\{\ell_1^2, \ell_2^2, (\ell_1-q)^2, (\ell_2-q)^2, (\ell_3-q)^2, \ell_3^2, (\ell_1-\ell_2)^2, (\ell_2-\ell_3)^2, (\ell_3-\ell_1)^2\}$$

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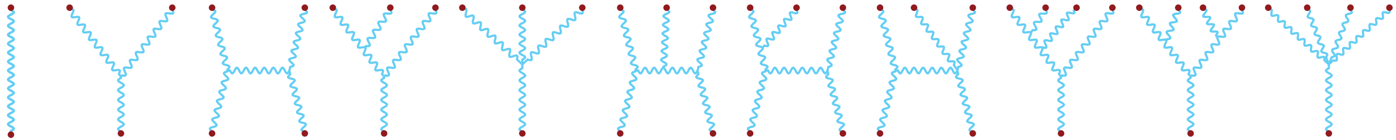

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$$\int d^D\ell_1 d^D\ell_2 d^D\ell_3 \frac{\delta(\ell_1\cdot u_1)\,\delta(\ell_2\cdot u_2)\,\delta(\ell_3\cdot u_2)}{[\ell_1\cdot u_2]^{\alpha_1}\,[\ell_2\cdot u_1]^{\alpha_2}\,[\ell_3\cdot u_1]^{\alpha_3}} \frac{D_8^{-\nu_8} D_9^{-\nu_9}}{D_1^{\nu_1} D_2^{\nu_2}\cdots D_7^{\nu_7}}$$

$\{\ell_1^2, \ell_2^2, (\ell_1-q)^2, (\ell_2-q)^2, (\ell_3-q)^2, \ell_3^2, (\ell_1-\ell_2)^2, (\ell_2-\ell_3)^2, (\ell_3-\ell_1)^2\}$

IBP reduction: any integral = a linear combination of a small number of basis integrals

$$\vec{f} = \{I_1, I_2, \ldots\}$$

Differential Equations

$$\frac{d\vec{f}(\gamma,\epsilon)}{d\gamma} = M(\gamma,\epsilon)\,\vec{f}(\gamma,\epsilon) \qquad D = 4-2\epsilon$$

- The majority can be solved in terms of multiple polylogarithms   Henn 2013 Lee 2014
- Elliptic integrals appear in post-Minkwskian gravity for the first time
- Expansion in PN ($v \ll 1$): potential $\ell^\mu \sim (v,1)$, radiation $\ell^\mu \sim (v,v)$ Beneke-Smirnov 1997
- Post-Minkowskian physics can be bootstrapped from Post-Newtonian data!

Inspiralling dynamics at NNNLO

The full impulse at $\mathcal{O}(G^4)$: Dlapa-Kälin-**ZL**-Porto 2106.08276 2112.11296 2210.05541 2304.01275

$$\Delta p_1^\mu\Big|_{\text{NNNLO}} = \frac{G^4}{|b|^4}\left(c_b \frac{b^\mu}{|b|} + c_1 \frac{\gamma u_2^\mu - u_1^\mu}{\gamma^2-1} + c_2 \frac{\gamma u_1^\mu - u_2^\mu}{\gamma^2-1} \right)$$

$$\begin{aligned}
\frac{c_b}{\pi} = &-\frac{3h_{34}m_2m_1(m_1^3+m_2^3)}{64v_\infty^5} + \frac{m_1^2m_{12}m_2^2}{4}\Bigg[\frac{3h_6\mathrm{K}^2(w_2)}{4v_\infty^3} - \frac{3h_8\mathrm{K}(w_2)\mathrm{E}(w_2)}{4v_\infty^3} + \frac{21h_5w_3\mathrm{E}^2(w_2)}{8v_\infty^3} - \frac{\pi^2h_{16}v_\infty}{4(\gamma+1)} + \frac{3\gamma h_{10}(\mathrm{Li}_2(w_2)-4\mathrm{Li}_2(\sqrt{w_2}))}{w_3v_\infty^2} \\
&+ \log(v_\infty)\left(\frac{h_{32}}{2v_\infty^3} - \frac{3h_{14}\log(\frac{w_3}{2})}{v_\infty} - \frac{3\gamma h_{22}\log(w_1)}{2v_\infty^4}\right)\Bigg] + m_2^2m_1^3\Bigg[\frac{h_{52}}{48v_\infty^6} - \frac{h_{63}}{768\gamma^9w_3v_\infty^5} - \frac{3v_\infty(h_{40}\mathrm{Li}_2(w_2)+2w_3h_{33}\mathrm{Li}_2(-w_2))}{64w_3} \\
&+ \frac{3h_{14}\log(\frac{w_3}{2})\log(w_3)}{4v_\infty} + \frac{\gamma h_{39}\log(w_1)}{8w_3^3v_\infty^2} + \frac{3\gamma h_{22}\log(w_3)\log(w_1) - h_{35}\log(\frac{w_3}{2})}{8v_\infty^4} + \frac{h_{56}\log(2) - h_{57}\log(w_3) + 2\gamma h_{55}\log(\gamma)}{32v_\infty^5} - \frac{\gamma h_{51}\log(w_1)}{16v_\infty^7}\Bigg] \\
&+ m_1^2m_2^3\Bigg[\frac{h_{58}}{192\gamma^7v_\infty^5} + \frac{h_{53}}{48v_\infty^6} + \frac{\gamma h_{49}\log(w_1)}{16v_\infty^6} - \frac{2\gamma h_{50}\log(w_1) + 3\gamma^2h_{13}\log^2(w_1)}{32v_\infty^7} - \frac{h_{41}\log(\frac{w_3}{2})}{8v_\infty^4} + \frac{3\gamma\log(w_1)(5h_{26}\log(2)+8h_{12}\log(w_3))}{8v_\infty^4} \\
&- \frac{h_{36}\log(w_3)}{4v_\infty^3} + \frac{\gamma h_{30}\log(\gamma)}{2v_\infty^3} + \frac{h_{37}\log(2)}{8v_\infty^3} + \frac{3(h_{17}w_3\mathrm{Li}_2(w_2) - 2h_{23}\mathrm{Li}_2(-w_2) + h_{15}\log^2(w_3) - h_9\log^2(2))}{8v_\infty} - \frac{3h_7\log(2)\log(w_3)}{v_\infty}\Bigg] \\
c_1 = &\, m_1m_2^2\left(\frac{2h_{46}m_{12s}}{v_\infty^6} + \frac{9\pi^2h_1m_{12}^2}{32v_\infty^2}\right) + m_1^2m_2^3\left(\frac{4\gamma h_{47}}{3v_\infty^6} - \frac{8\gamma h_2\log(w_1)}{v_\infty^6} + \frac{16h_{25}\log(w_1)}{v_\infty^3} - \frac{8h_3}{3v_\infty^5}\right) \\
c_2 = &-m_1^4m_2\left(\frac{9\pi^2h_1}{32v_\infty^2} + \frac{2h_{46}}{v_\infty^6}\right) + m_2^2m_1^3\Bigg[+\frac{h_{60}}{705600\gamma^8v_\infty^5} - \frac{4\gamma h_{48}}{3v_\infty^6} + \frac{3h_{38}(\mathrm{Li}_2(w_2)-4\mathrm{Li}_2(\sqrt{w_2})) - \gamma h_{21}(\mathrm{Li}_2(-w_1^2)+2\log(\gamma)\log(w_1))}{16v_\infty^4} \\
&+ \frac{3\gamma h_{31}(2\mathrm{Li}_2(-w_1)+\log(w_1)\log(w_3))}{8v_\infty^4} + \frac{h_{62}\log(w_1)}{6720\gamma^9v_\infty^6} + \frac{32\gamma^2h_{44}\log^2(w_1)}{v_\infty^7} + \frac{8\gamma(2h_4\log(2) - h_{27}\log(w_1))\log(w_1)}{v_\infty^4} - \frac{32h_{29}\log(w_1)}{3v_\infty^3} + \frac{\pi^2h_{42}}{192v_\infty^4}\Bigg] \\
&+ m_2^3m_1^2\Bigg[\frac{h_{59}}{1440\gamma^7v_\infty^5} - \frac{h_{19}(\mathrm{Li}_2(-w_1^2)+2\log(\gamma)\log(w_1))}{8v_\infty^4} + \frac{h_{43}(\mathrm{Li}_2(w_2)-4\mathrm{Li}_2(\sqrt{w_2}))}{32v_\infty^4} - \frac{h_{20}(2\mathrm{Li}_2(-w_1)+\log(w_1)\log(w_3))}{4v_\infty^4} \\
&- \frac{h_{61}\log(w_1)}{480\gamma^8v_\infty^6} - \frac{16\gamma^2h_{11}\log^2(w_1)}{v_\infty^4} - \frac{32\gamma h_{45}\log^2(w_1)}{v_\infty^7} + \frac{16\gamma h_{28}\log(w_1)}{5v_\infty^3} - \frac{32h_{24}\log(2)\log(w_1)}{v_\infty^4} - \frac{\pi^2h_{18}}{48v_\infty^4} - \frac{2h_{54}}{45v_\infty^6}\Bigg]
\end{aligned}$$

with $\gamma \equiv u_1\cdot u_2$, $v_\infty = \sqrt{\gamma^2-1}$, $w_1 = \gamma - v_\infty$, $w_2 = \frac{\gamma-1}{\gamma+1}$, $w_3 = \gamma+1$, $h_i =$ polynomial in γ.

$$\mathrm{Li}_2(z) \equiv \int_0^z dx \frac{\log(1-x)}{-x}$$

$$K(z) \equiv \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-zx^2)}}$$

$$E(k) \equiv \int_0^1 dx \frac{\sqrt{1-zx^2}}{\sqrt{1-x^2}}$$

Inspiralling dynamics at NNNLO

The full impulse at $\mathcal{O}(G^4)$: Dlapa-Kälin-**ZL**-Porto 2106.08276 2112.11296 2210.05541 2304.01275

$$\Delta p_1^\mu\Big|_{\text{NNNLO}} = \frac{G^4}{|b|^4}\left(c_b\,\frac{b^\mu}{|b|} + c_1\,\frac{\gamma u_2^\mu - u_1^\mu}{\gamma^2-1} + c_2\,\frac{\gamma u_1^\mu - u_2^\mu}{\gamma^2-1}\right)$$

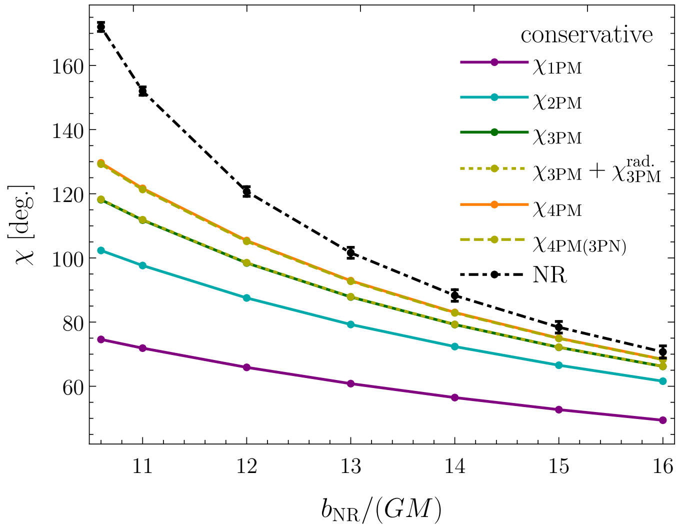
- We obtained the full dynamics for binary system at $\mathcal{O}(G^4)$ for the first time!
- We computed the radiated energy:

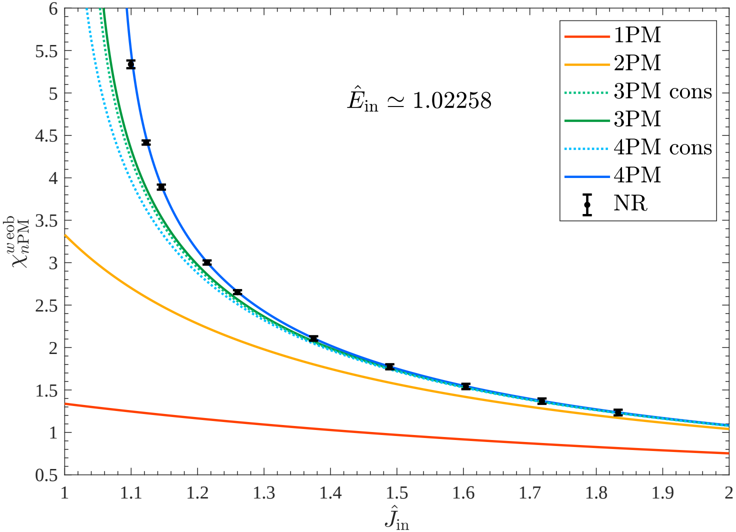
$$\begin{aligned}\Delta E^{\text{4PM}}_{\text{hyp}} \sim\; & \frac{1568}{45 v_\infty} + \left(\frac{18608}{525} - \frac{1136\nu}{45}\right) v_\infty + \frac{3136 v_\infty^2}{45} + \left(\frac{764\nu^2}{45} - \frac{356\nu}{63} + \frac{220348}{11025}\right) v_\infty^3 \\ & + \left(\frac{1216}{105} - \frac{2272\nu}{45}\right) v_\infty^4 + \left(-\frac{622\nu^3}{45} + \frac{3028\nu^2}{1575} - \frac{199538\nu}{33075} - \frac{151854}{13475}\right) v_\infty^5 \\ & + \left(\frac{1528\nu^2}{45} - \frac{8056\nu}{1575} + \frac{117248}{1575}\right) v_\infty^6 + O\left(v_\infty^7\right) \qquad v_\infty = \sqrt{\gamma^2 - 1} \ll 1\end{aligned}$$

- Perfect agreement with the state-of-the-art PN computations

Cho-Dandapat-Gopakumar 2021 Cho 2022 Bini-Geralico 2021 2022 Bini-Damour 2022

Comparison with Numerical Simulations





Khalil-Buonanno-Steinhoff-Vines 2204.05047

Damour-Rettegno 2211.01399

Conclusions

Modern techniques from particle physics and mathematics have already proven useful to improve theoretical predictions for gravitational-wave observables.

- Effective Field Theory & Multi-loop techniques
- Scattering amplitudes (see Andres Luna's talk)
- On-shell approaches (see Andrea Cristofoli's talk)

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We have developed an efficient framework and made breakthroughs to NNNLO.

	0PN	1PN	2PN	3PN	4PN	5PN	6PN	$\cdots$	
$G($	1	$+ v^2$	$+ v^4$	$+ v^6$	$+ v^8$	$+ v^{10}$	$+ v^{12}$	$+ \cdots)$	1PM
$G^2($		1	$+ v^2$	$+ v^4$	$+ v^6$	$+ v^8$	$+ v^{10}$	$+ \cdots)$	2PM
$G^3($			1	$+ v^2$	$+ v^4$	$+ v^6$	$+ v^8$	$+ \cdots)$	3PM
$G^4($				1	$+ v^2$	$+ v^4$	$+ v^6$	$+ \cdots)$	4PM
$G^5($					1	$+ v^2$	$+ v^4$	$+ \cdots)$	5PM
$G^6($						1	$+ v^2$	$+ \cdots)$	6PM

STATE OF THE ART

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	0PN	1PN	2PN	3PN	4PN	5PN	6PN	⋯	
	$G(1$	v^2	v^4	v^6	v^8	v^{10}	v^{12}	$\cdots)$	1PM
		$G^2(1$	v^2	v^4	v^6	v^8	v^{10}	$\cdots)$	2PM
			$G^3(1$	v^2	v^4	v^6	v^8	$\cdots)$	3PM
STATE OF THE ART				$G^4(1$	v^2	v^4	v^6	$\cdots)$	4PM
underway					$G^5(1$	v^2	v^4	$\cdots)$	5PM
						$G^6(1$	v^2	$\cdots)$	6PM

Next-generation observations need more precise predictions. We are progressing!

Thanks for your attention!

Radiation Reaction: in-in formalism

Keldysh representation:

$$x_a^+ = \frac{1}{2}\left(x_a^{(1)} + x_a^{(2)}\right) \qquad h_{\mu\nu}^+ = \frac{1}{2}\left(h_{\mu\nu}^{(1)} + h_{\mu\nu}^{(2)}\right)$$
$$x_a^- = x_a^{(1)} - x_a^{(2)} \qquad h_{\mu\nu}^- = h_{\mu\nu}^{(1)} - h_{\mu\nu}^{(2)}$$

Closed-time-path integral

$$e^{i\mathcal{S}_{\text{eff}}[x_a^\pm]} = \int \mathcal{D}h^+ \mathcal{D}h^- \, e^{i\mathcal{S}_{\text{GR}} + i\mathcal{S}_{\text{WL}}}$$

backup

Impulse

$$\Delta p_i^\mu = -\eta^{\mu\nu} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} d\tau \left(\frac{\partial \mathcal{L}_{\text{eff}}}{\partial x_i^{-,\nu}(\tau)} \right) \Bigg|_{x_a^- \to 0, x_a^+ \to x_a}$$

Causal propagator matrix

$$i \begin{pmatrix} 0 & -\Delta_{\text{adv}}(x-y) \\ -\Delta_{\text{ret}}(x-y) & 0 \end{pmatrix}$$