Fast eccentric and inclined inspirals into a rotating black hole

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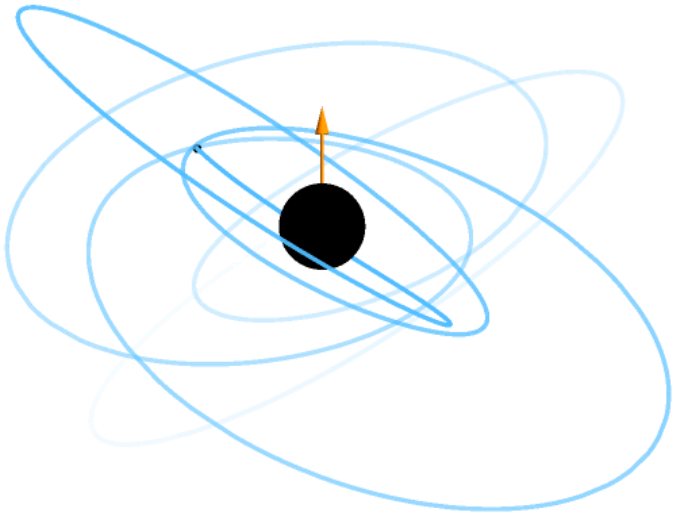
UCD RELATIVITY GROUP

CAPRA

Requirements for LISA

- Model must include precession and eccentricity
- Fast to evaluate <1s
- Phase accurate to $O(1/SNR)$

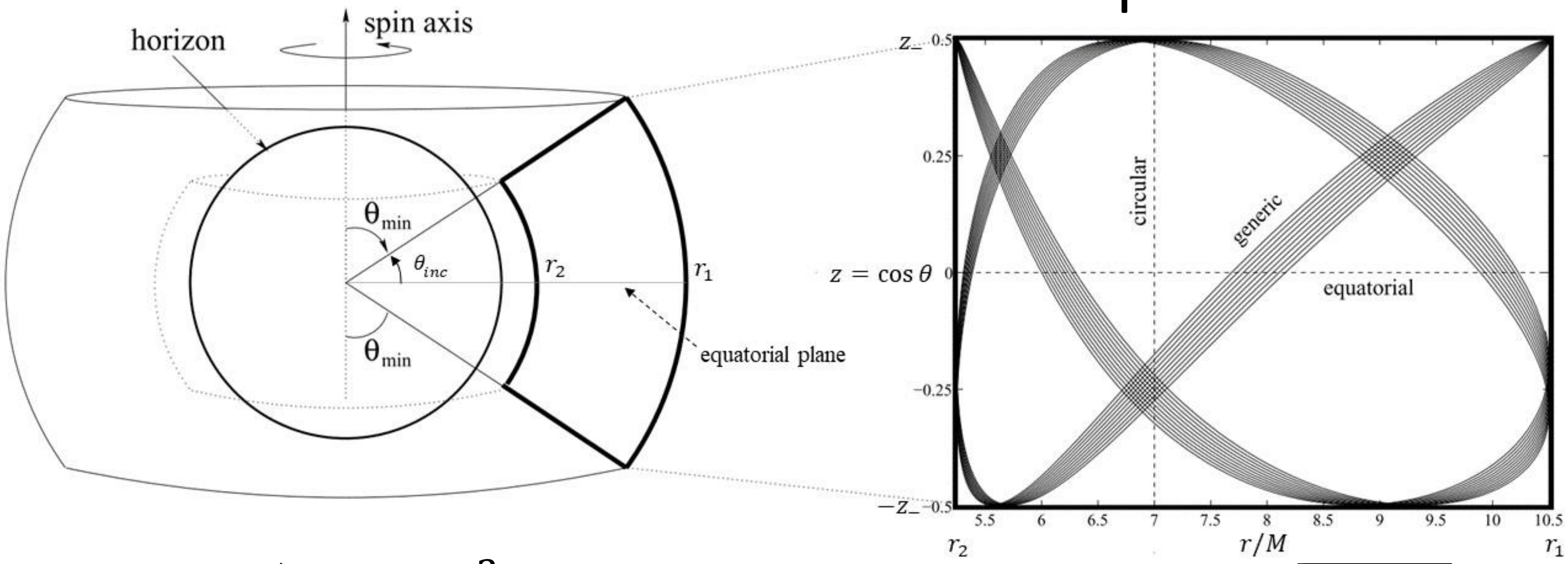


$$\frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau} = \epsilon a^\alpha_{(1)} + \epsilon^2 a^\alpha_{(2)} + \mathcal{O}(\epsilon^3)$$

$$\varphi = \epsilon^{-1} \varphi^{(0\mathrm{PA})} + \epsilon^{-1/2} \varphi^{(\mathrm{res})} + \varphi^{(1\mathrm{PA})} + \mathcal{O}(\epsilon)$$

- Adiabatic (0PA): $< a^{(1)}_{Diss} >$ or $\mathcal{E}$ & $\mathcal{L}$ Fluxes
- Post-Adiabatic (1PA): $a^{(1)}$ & $< a^{(2)}_{Diss} >$
- Orbital Resonances

Geodesic Motion in Kerr Spacetime



- Orbital Elements $\vec{P}$: $p = \frac{2r_1 r_2}{M(r_1+r_2)}$, $e = \frac{r_1 - r_2}{r_1+r_2}$, $x = \cos\theta_{inc} = \sqrt{1 - z_-^2}$

- Mino Time λ: $\frac{d\lambda}{d\tau} = (r^2 + a^2 \cos^2\theta)^{-1}$

- Phases $\vec{q}$: $q_r = \Upsilon_r^{(0)}(p, e, x)\lambda + q_{r,0}$ & $q_z = \Upsilon_z^{(0)}(p, e, x)\lambda + q_{z,0}$

Generic Kerr Inspirals

Equations of motion

- Osculating Geodesics (OG)
- $x^{\alpha}(\lambda) = x_G^{\alpha}(I^A(\lambda), \lambda)$
- $u^{\alpha}(\lambda) = u_G^{\alpha}(I^A(\lambda), \lambda)$
- $I^A = \{P_j, q_{i0}\}$
- $\dot{I}^A = 0 \rightarrow F(a^{\alpha})$

Force Model $\boldsymbol{a^{\alpha}}$

- Eccentric Equatorial [2112.05651]
- Spherical [2305.10533]
- Combine to create 1st order generic Kerr toy model $a_{(1)}^{\alpha}$
- Rescale $a_{(1)}^{\alpha}$ by PN orders to make 2nd order toy model $a_{(2)}^{\alpha}$
- $a^{\alpha} = \epsilon\, a_{(1)}^{\alpha} + \epsilon^2\, a_{(2)}^{\alpha}$

Osculating Geodesics (OG)

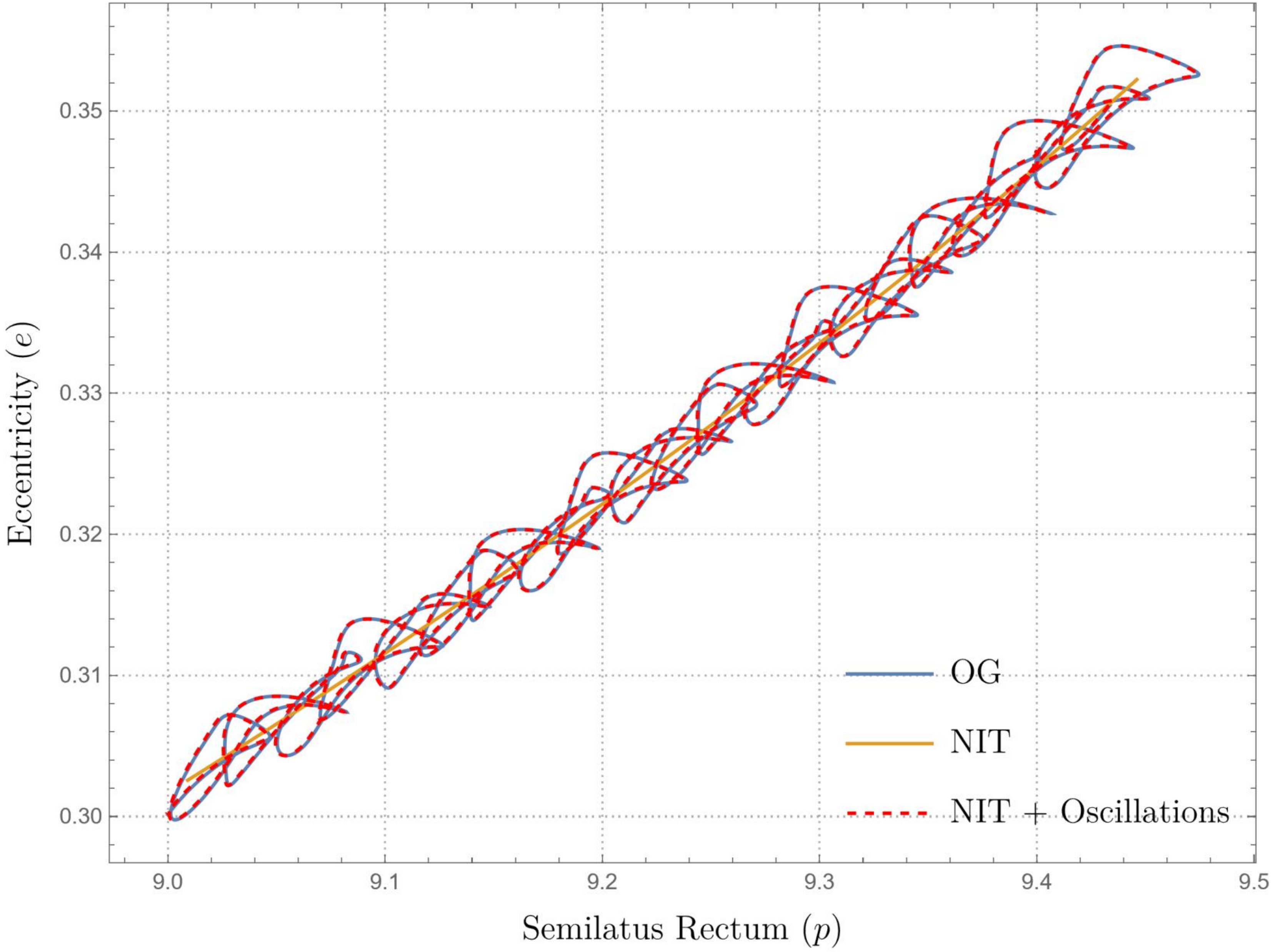
$$\dot{P}_j = 0 + \epsilon F_j^{(1)}(\vec{P},\vec{q}) + \epsilon^2 F_j^{(2)}(\vec{P},\vec{q})$$
$$\dot{q}_i = \Upsilon_i(\vec{P}) + \epsilon f_i^{(1)}(\vec{P},\vec{q})$$
$$\dot{t} = f_t^{(0)}(\vec{P},\vec{q})$$
$$\dot{\phi} = f_\phi^{(0)}(\vec{P},\vec{q})$$

Near Identity Transformations (NITs)

$$\tilde{P}_j = P_j + \epsilon Y_j^{(1)}(\vec{P},\vec{q}) + \epsilon^2 Y_j^{(2)}(\vec{P},\vec{q}) + \mathcal{O}(\epsilon^3)$$
$$\tilde{q}_i = q_i + \epsilon X_i^{(1)}(\vec{P},\vec{q}) + \epsilon^2 X_i^{(2)}(\vec{P},\vec{q}) + \mathcal{O}(\epsilon^3)$$
$$\tilde{t} = t + Z_t^{(0)}(\vec{P},\vec{q}) + \epsilon Z_t^{(1)}(\vec{P},\vec{q}) + \mathcal{O}(\epsilon^2)$$
$$\tilde{\phi} = \phi + Z_\phi^{(0)}(\vec{P},\vec{q}) + \epsilon Z_\phi^{(1)}(\vec{P},\vec{q}) + \mathcal{O}(\epsilon^2)$$

Averaged EOM

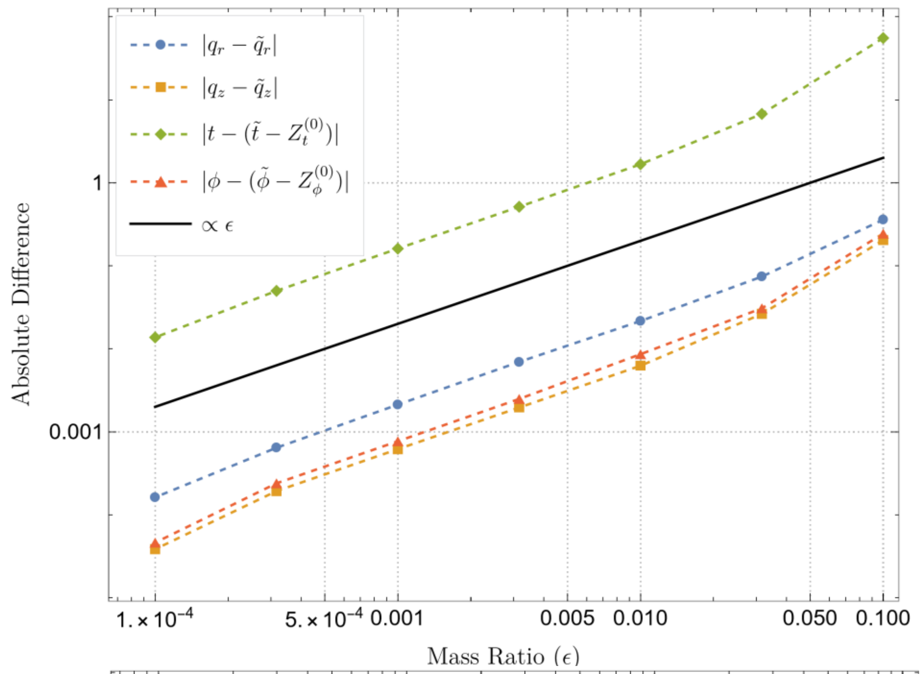
$$\dot{\tilde{P}}_j = 0 + \epsilon \tilde{F}_j^{(1)}(\vec{\tilde{P}}) + \epsilon^2 \tilde{F}_j^{(2)}(\vec{\tilde{P}})$$
$$\dot{\tilde{q}}_i = \Upsilon_i^{(0)}(\vec{\tilde{P}}) + \epsilon \Upsilon_i^{(1)}(\vec{\tilde{P}})$$
$$\dot{\tilde{t}} = \Upsilon_t^{(0)}(\vec{\tilde{P}}) + \epsilon \Upsilon_t^{(1)}(\vec{\tilde{P}})$$
$$\dot{\tilde{\phi}} = \Upsilon_\phi^{(0)}(\vec{\tilde{P}}) + \epsilon \Upsilon_\phi^{(1)}(\vec{\tilde{P}})$$

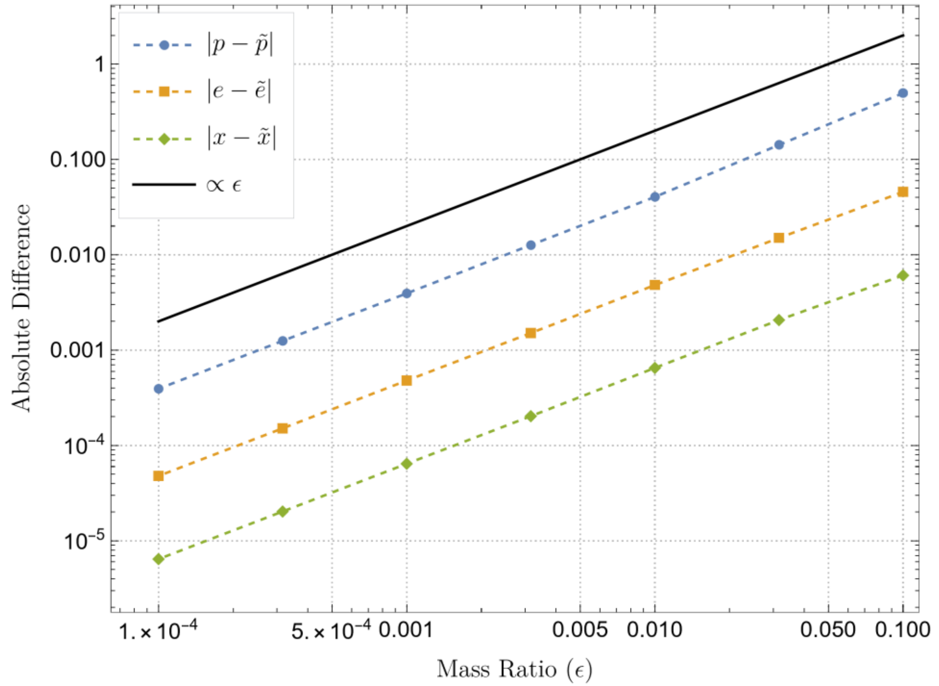

Eccentricity (e)
0.35
0.34
0.33
0.32
0.31
0.30
9.0
9.1
9.2
9.3
9.4
9.5
Semilatus Rectum (p)
OG
NIT
NIT + Oscillations

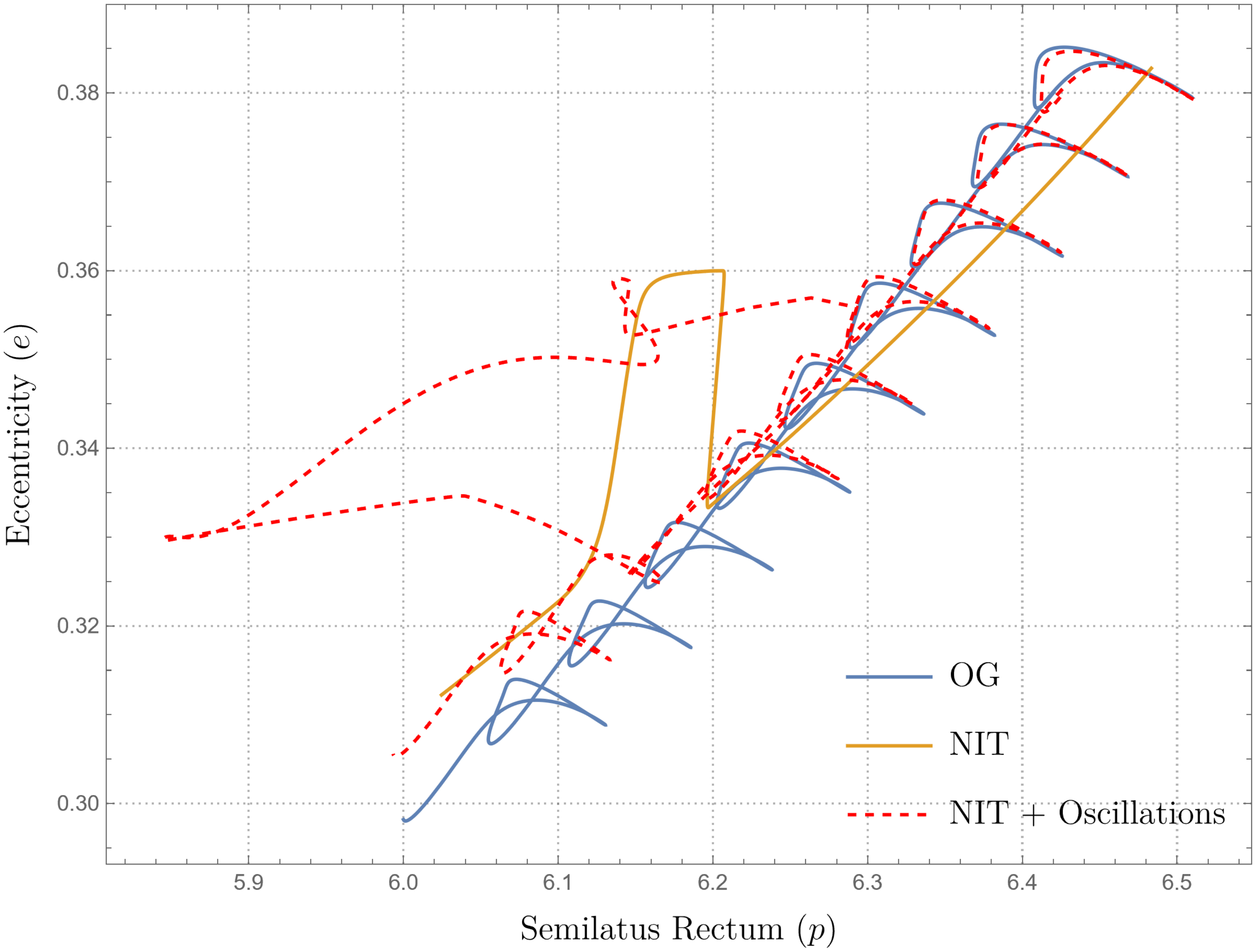
Generic Kerr NITs

ϵ	OG Inspiral	NIT Inspiral	Speed-up
10^{-1}	9.33s	0.127s	~ 73.5
$10^{-1.5}$	13.2s	0.099s	~ 133
10^{-2}	29.7s	0.088s	~ 338
$10^{-2.5}$	88.3s	0.103s	~ 857
10^{-3}	281s	0.117s	~ 2402
$10^{-3.5}$	860s	0.14s	~ 6143
10^{-4}	2697s	0.254s	$\sim 10,618$

Table 1: Computational time required to evolve an inspiral from its initial conditions of $(p_0, e_0, x_0) = (9.5, 0.38, 0.8)$ to $p = 9$ for different values of the mass ratio.



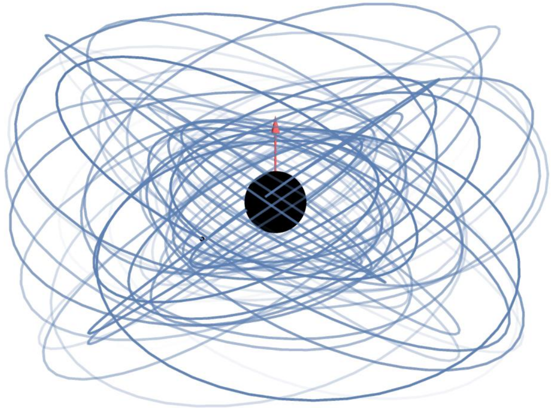


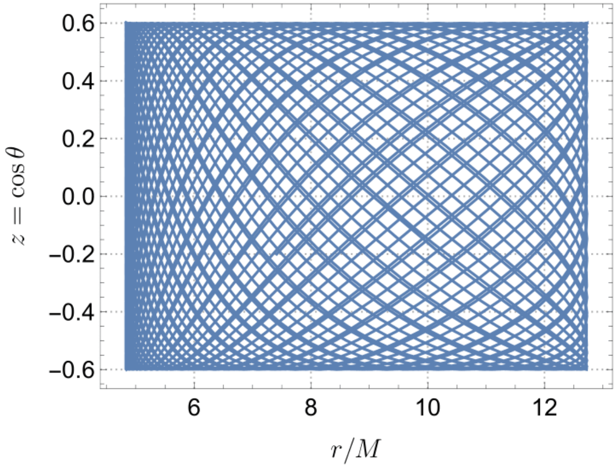


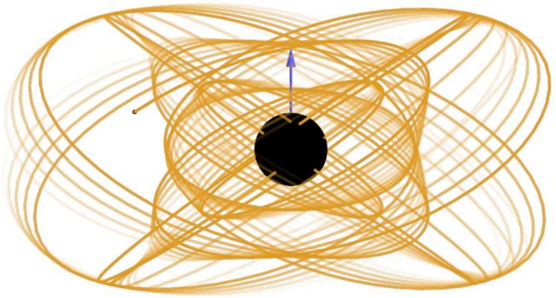
0.38
0.36
0.34
0.32
0.30
Eccentricity (e)
5.9
6.0
6.1
6.2
6.3
6.4
6.5
Semilatus Rectum (p)
OG
NIT
NIT + Oscillations

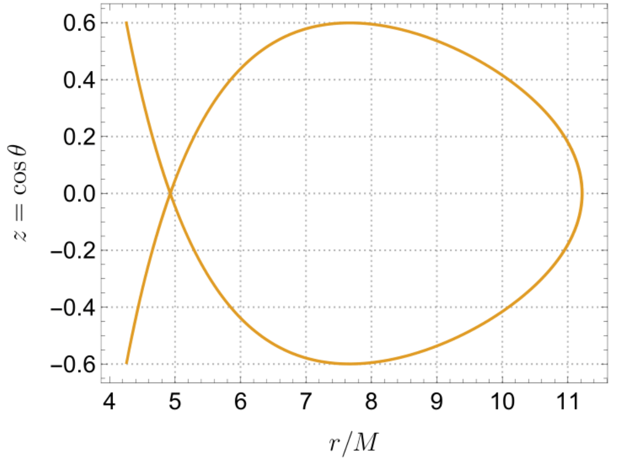
Orbital Resonances

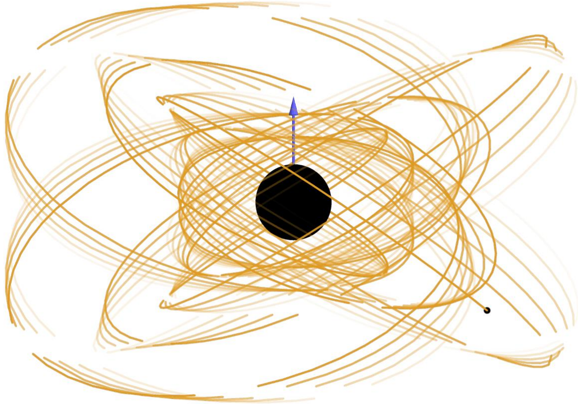
- NIT is singular when $\Upsilon_\perp = n\Upsilon_r + \mathrm{m}\,\Upsilon_\mathrm{z} = 0$
- Generic Average:
- $\langle A\rangle = \lim_{\lambda\to\infty} \frac{1}{2\Lambda}\int_{-\Lambda}^{\Lambda} A(\lambda)d\lambda = \frac{1}{(2\pi)^2}\oint\oint A(q_r,q_z)dq_r q_z = A_{(0,0)}$
- Resonance Average:
  - $\langle A\rangle_{\mathrm{res}} = \lim_{\lambda\to\infty} \frac{1}{2\Lambda}\int_{-\Lambda}^{\Lambda} A(\lambda)d\lambda = \sum_{N\in\mathbb{Z}} A_{(Nn,Nm)}e^{iNq_\perp}$
- Resonant phase: $q_\perp = n\mathrm{q}_r + \mathrm{m}\ \mathrm{q_z}$
- All other combinations: $q_\parallel$
- Cause “jumps” in $\vec{P} \propto \epsilon^{1/2}$
- Results in phase error $\propto \epsilon^{-1/2}$

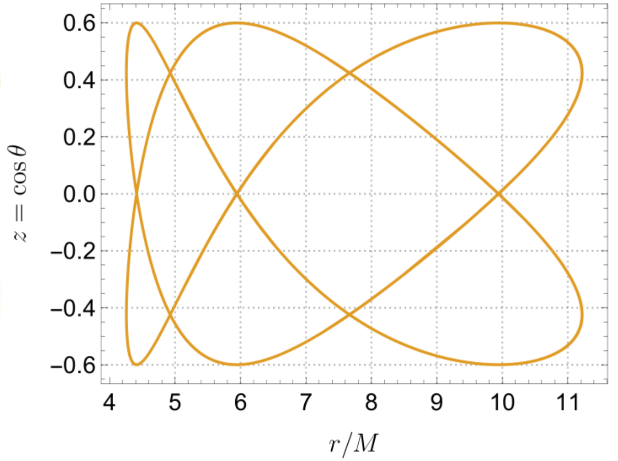




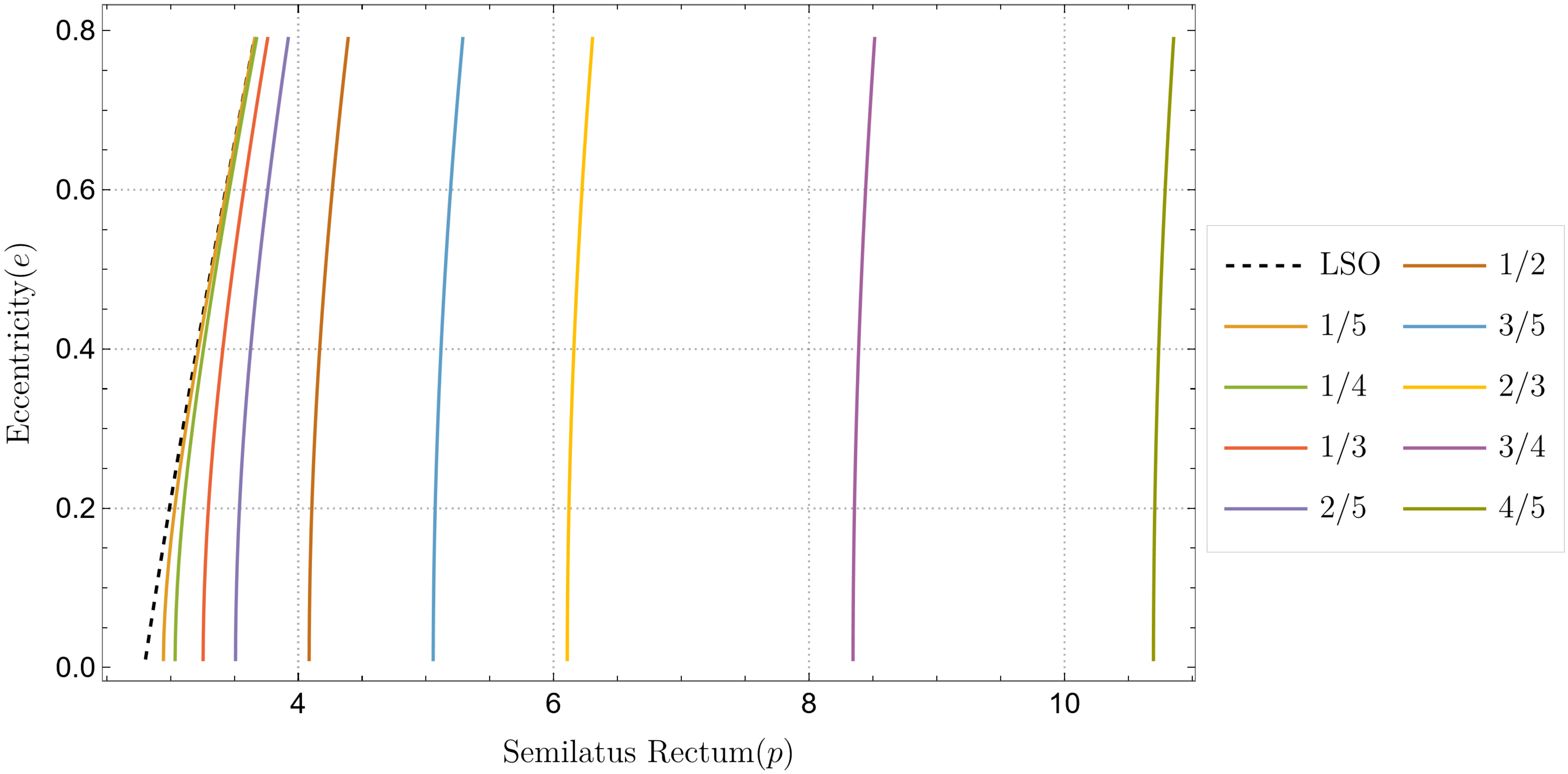








Transient Orbital Resonance Locations



Near-Resonant NIT

$$\tilde{P}_j = P_j + \epsilon Y_j^{(1)}(\vec{P}, q_{\|}) + \epsilon^2 Y_j^{(2)}(\vec{P}, q_{\|}) + \mathcal{O}(\epsilon^3)$$

$$\tilde{q}_i = q_i + \epsilon X_i^{(1)}(\vec{P}, q_{\|}) + \epsilon^2 X_i^{(2)}(\vec{P}, q_{\|}) + \mathcal{O}(\epsilon^3)$$

$$\tilde{t} = t + Z_t^{(0)}(\vec{P}, q_{\|}) + \epsilon Z_t^{(1)}(\vec{P}, q_{\|}) + \mathcal{O}(\epsilon^2)$$

$$\tilde{\phi} = \phi + Z_\phi^{(0)}(\vec{P}, q_{\|}) + \epsilon Z_\phi^{(1)}(\vec{P}, q_{\|}) + \mathcal{O}(\epsilon^2)$$

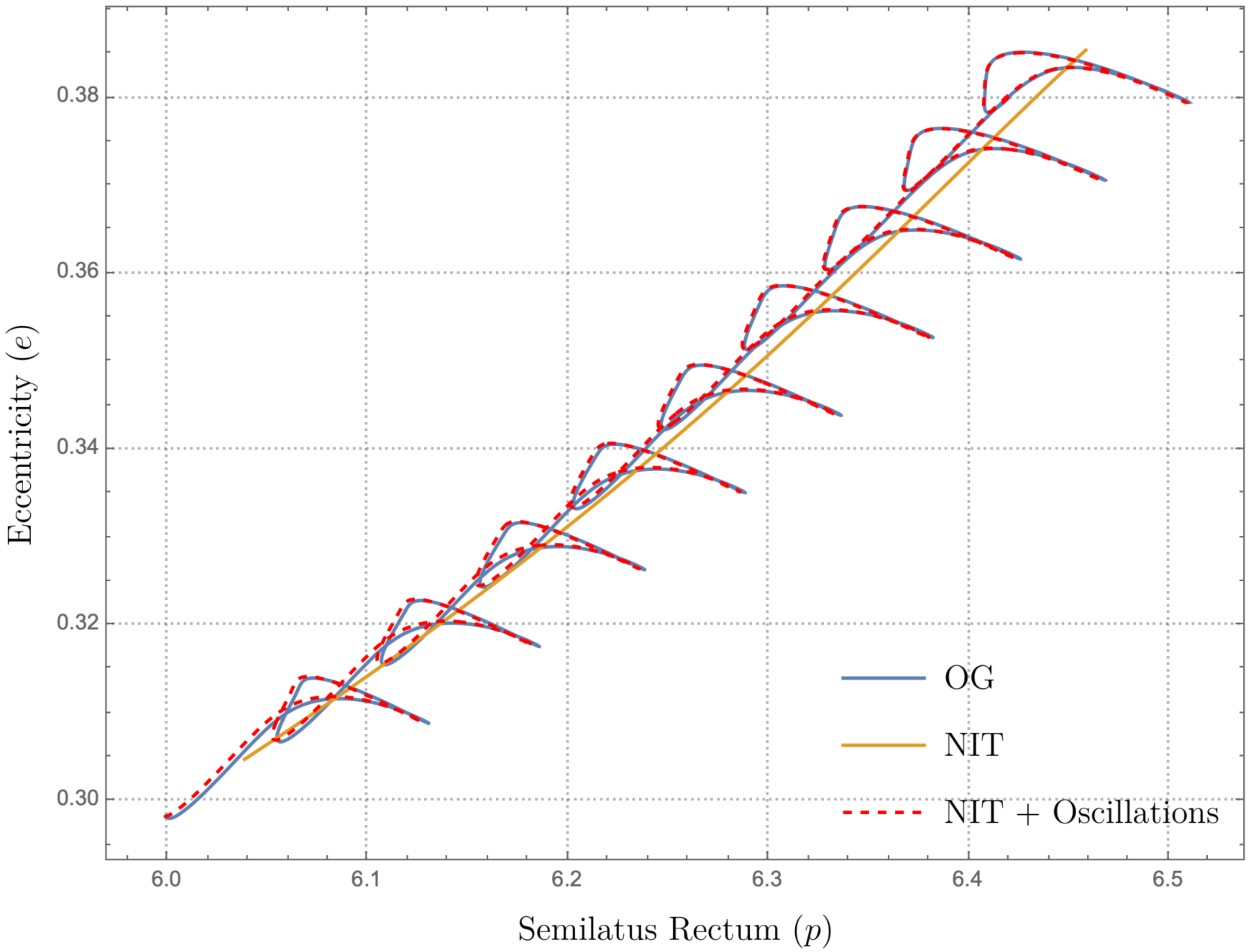
Averaged EoM

$$\dot{\tilde{P}}_j = 0 + \epsilon \tilde{F}_j^{(1)}(\vec{\tilde{P}}, \tilde{q}_\perp) + \epsilon^2 \tilde{F}_j^{(2)}(\vec{\tilde{P}}, \tilde{q}_\perp)$$

$$\dot{\tilde{q}}_i = \Upsilon_i^{(0)}(\vec{\tilde{P}}) + \epsilon \Upsilon_i^{(1)}(\vec{\tilde{P}}, \tilde{q}_\perp)$$

$$\dot{\tilde{t}}_k = \Upsilon_t^{(0)}(\vec{\tilde{P}}) + \epsilon \Upsilon_t^{(1)}(\vec{\tilde{P}})$$

$$\dot{\tilde{\phi}}_k = \Upsilon_\phi^{(0)}(\vec{\tilde{P}}) + \epsilon \Upsilon_\phi^{(1)}(\vec{\tilde{P}})$$



Eccentricity (e)
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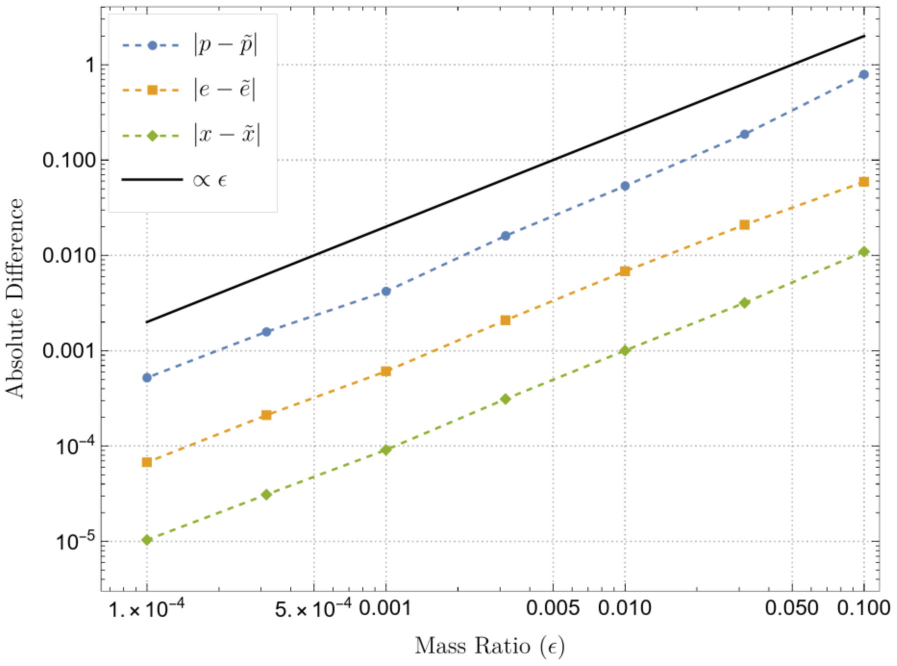
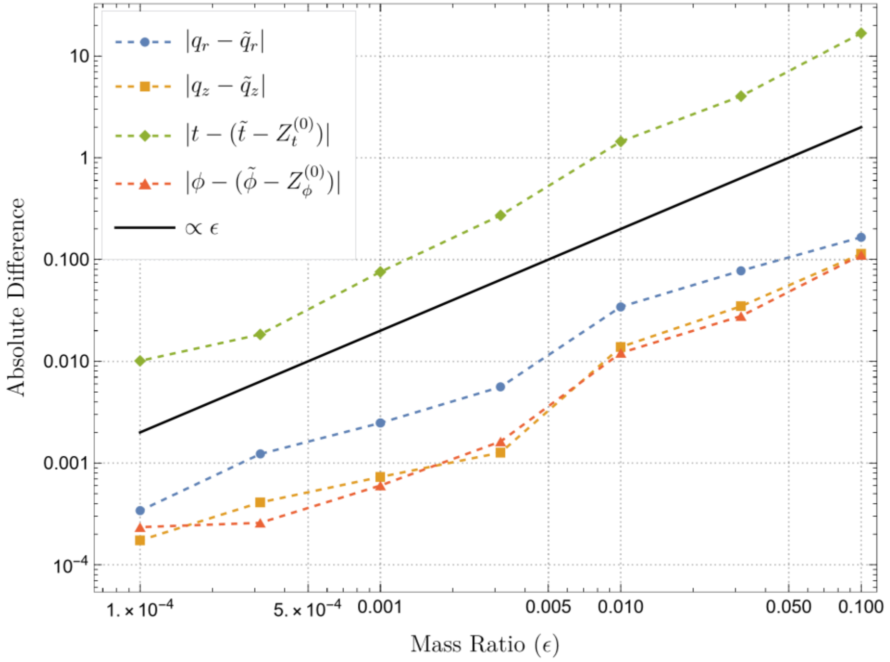
6.0
6.1
6.2
6.3
6.4
6.5
Semilatus Rectum (p)
OG
NIT
NIT + Oscillations

Near Resonant NIT Results

ϵ	OG Inspiral	NIT Inspiral	Speed-up
10^{-1}	3.44s	0.724s	~ 4.75
$10^{-1.5}$	7.21s	0.72s	~ 10
10^{-2}	18.7s	0.817s	~ 22.9
$10^{-2.5}$	48.7s	1.36s	~ 35.8
10^{-3}	160s	2.41s	~ 66.4
$10^{-3.5}$	516s	5.08s	~ 102
10^{-4}	1611s	14.35s	~ 112

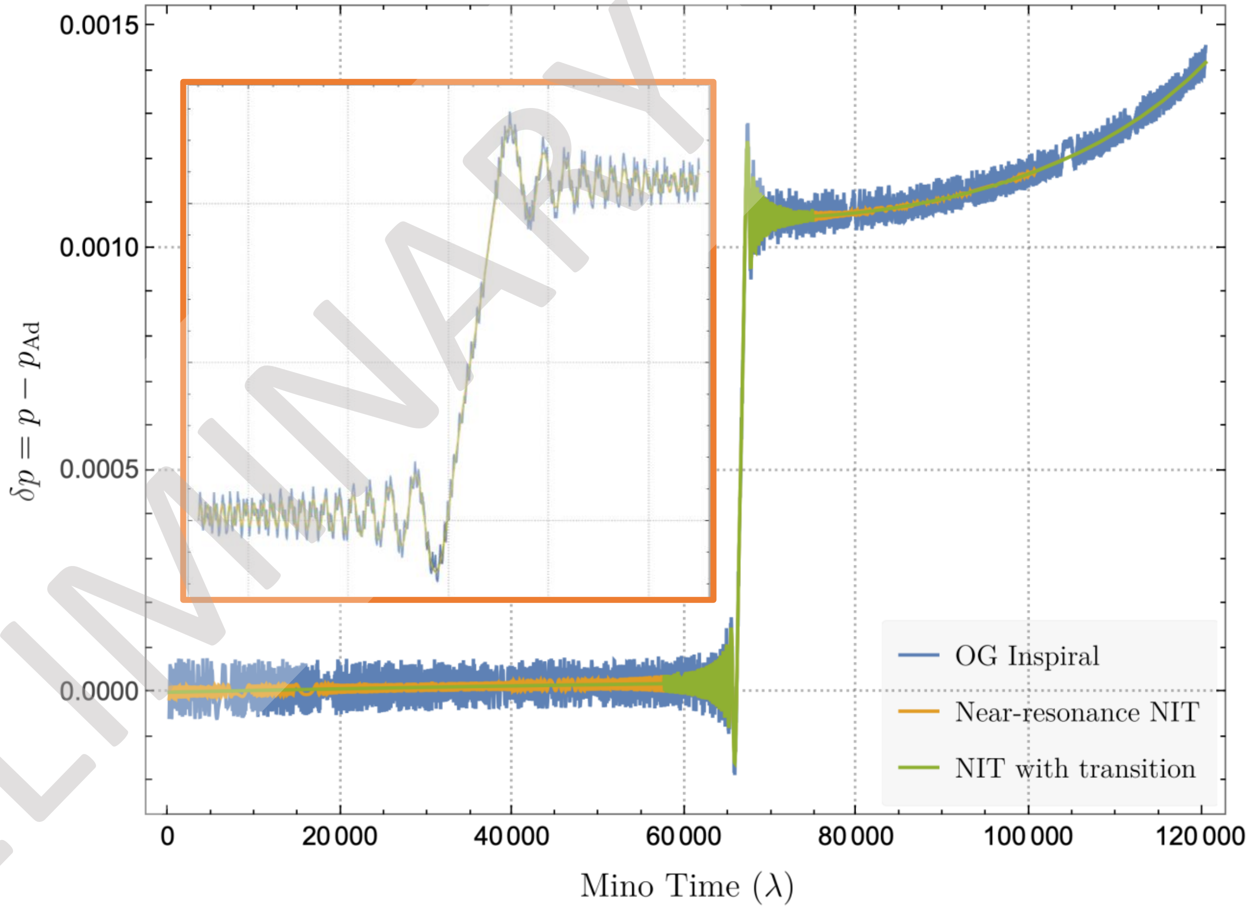
Table 1: Computational time required to evolve an inspiral from its initial conditions of $(p_0, e_0, x_0) = (6.5, 0.38, 0.8)$ to $p = 5.8$ for different values of the mass ratio.





Evolving through a single resonance (2/3)

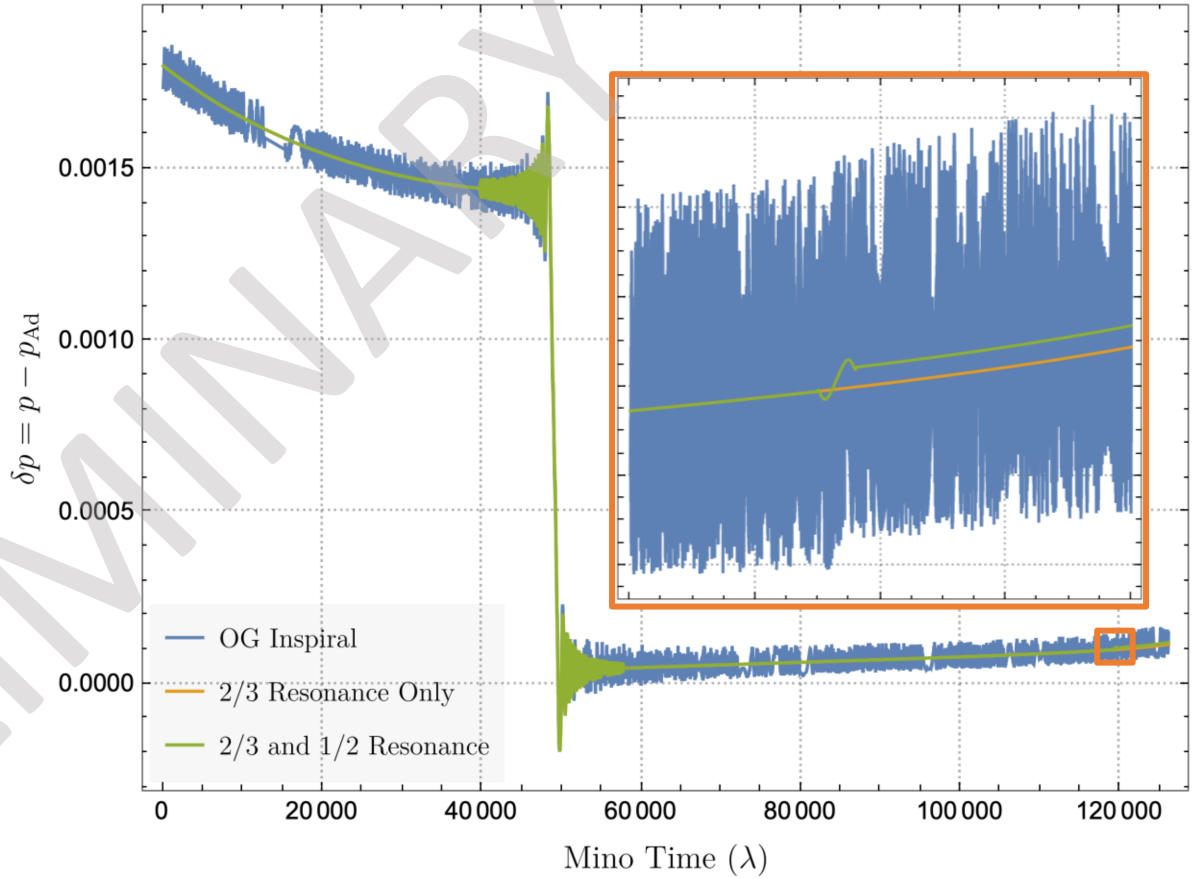
- Non-resonant NIT when $|n\Upsilon_r + \mathrm{m}\,\Upsilon_z| \geq \epsilon^{\frac{1}{7}}\,\delta^{\frac{2}{7}}\,\langle T_{Res}\rangle$
- Near-resonant NIT when $|n\Upsilon_r + \mathrm{m}\,\Upsilon_z| \leq \epsilon^{\frac{1}{7}}\,\delta^{\frac{2}{7}}\,\langle T_{Res}\rangle$
- OG inspiral takes ~ 2.5 days



Inspiral	Runtime	Mismatch
Non-resonant NIT	2.462s	0.376
Near-resonant NIT	312s	3.914×10^{-4}
NIT w/ transition	50.1s	3.917×10^{-4}

Evolving through multiple resonances (2/3 + 1/2)

- Non-resonant NIT when $|n\Upsilon_r + \mathrm{m}\,\Upsilon_z| \geq \epsilon^{\frac{1}{7}}\,\delta^{\frac{2}{7}}\,\langle T_{Res}\rangle$
- Near-resonant NIT when $|n\Upsilon_r + \mathrm{m}\,\Upsilon_z| \leq \epsilon^{\frac{1}{7}}\,\delta^{\frac{2}{7}}\,\langle T_{Res}\rangle$
- OG inspiral takes ~ 3 days

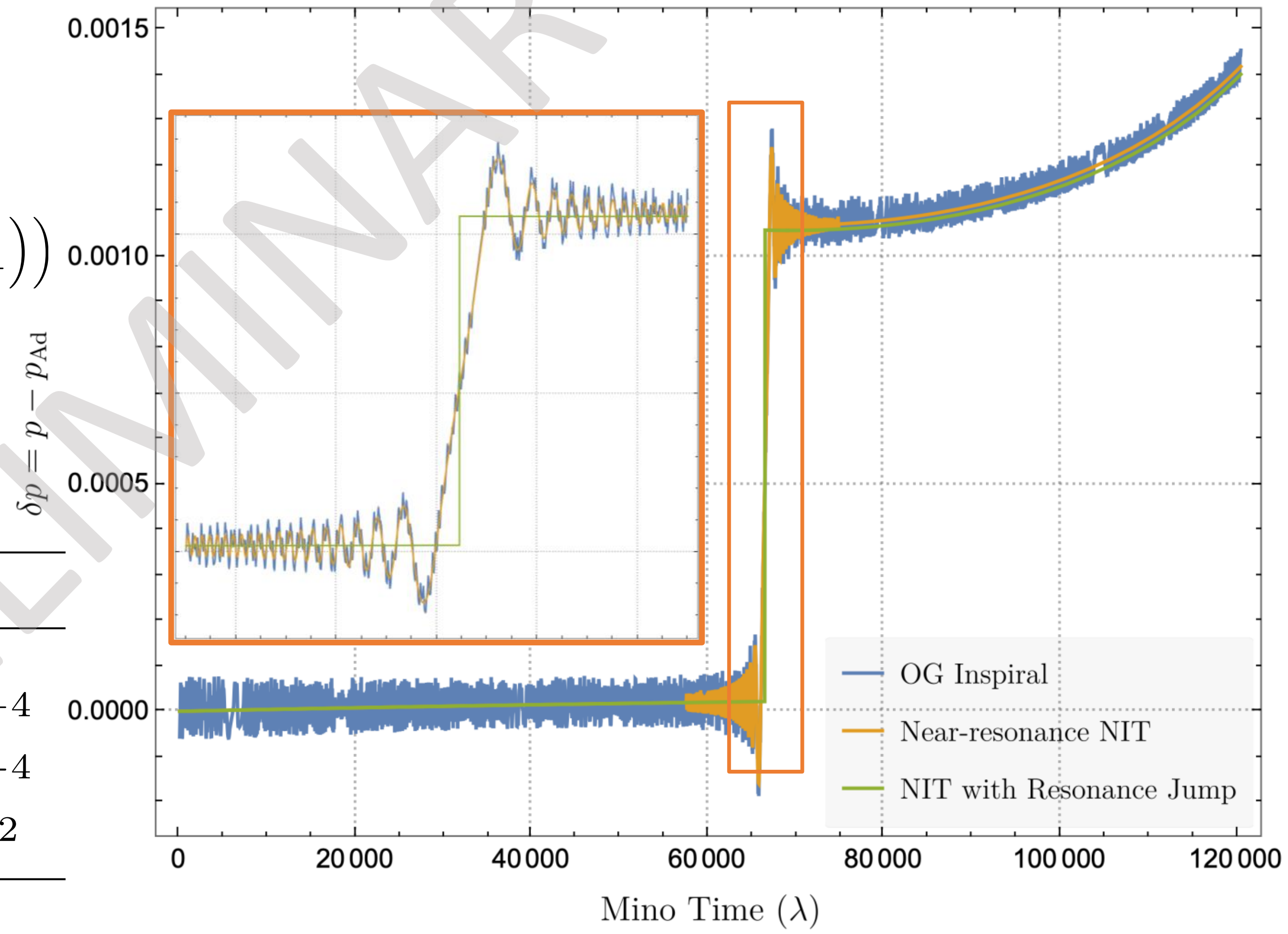


Inspiral	Runtime	Mismatch
No resonance transition	4.68s	0.648
Single resonance transition	50.2s	2.34×10^{-4}
Double resonance transition	103.5s	2.25×10^{-4}

What about a Resonance Jump?

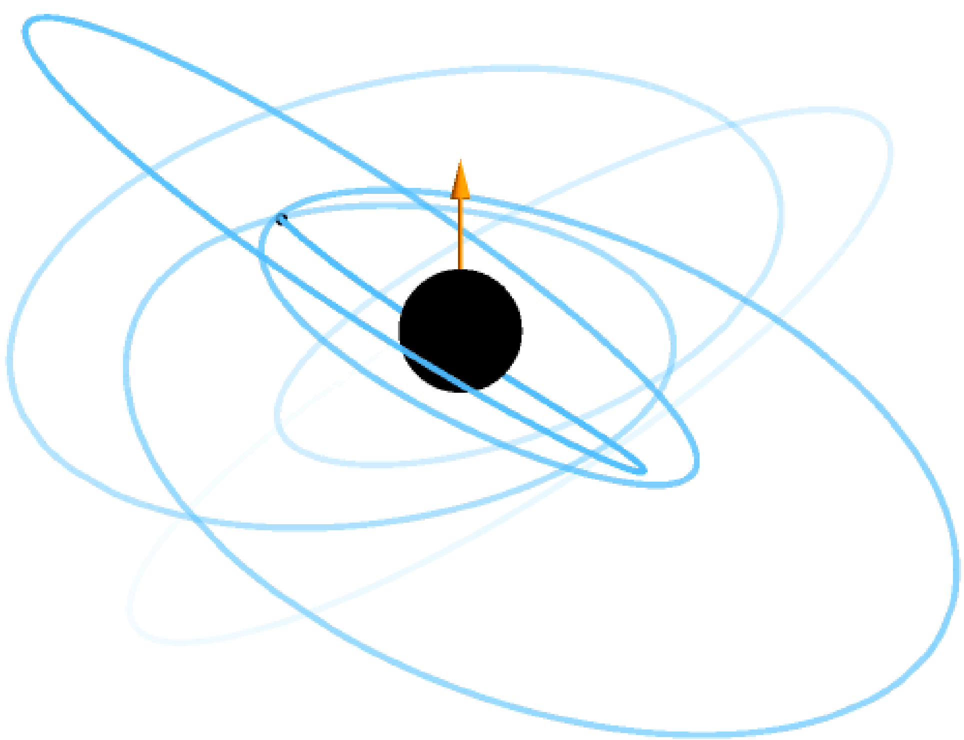
$$\Delta_{\rm res} P_j = \sum_{N=0} \sqrt{\frac{2\pi}{|N\dot{\Upsilon}_\perp|}} F^{(1)}_{j,N\vec{\kappa}_{\rm res}} \exp\left(i\left(\mathrm{sgn}(N\dot{\Upsilon}_\perp)\frac{\pi}{4} + Nq_\perp\right)\right)$$

Inspiral	Runtime	Mismatch
Non-resonant NIT	2.462s	0.376
Near-resonant NIT	312s	3.914×10^{-4}
NIT w/ transition	50.1s	3.917×10^{-4}
NIT w/ Jump	24.4s	3.15×10^{-2}



Conclusions

- Using NITs lets you calculate EMRI trajectories in than a second by removing oscillations from EoM
- Break down near resonances, which are likely to occur
- Modify NIT to remove all oscillations except the resonant ones
- Fastest solution so far is to switch between full and near-resonant NIT



Near-Identity (Averaging) Transformations

Transform to new variables

$$\tilde{P}_j = P_j + \epsilon Y_j^{(1)}(\vec{P},\vec{q}) + \epsilon^2 Y_j^{(2)}(\vec{P},\vec{q}) + \mathcal{O}(\epsilon^3)$$

$$\tilde{q}_i = q_i + \epsilon X_i^{(1)}(\vec{P},\vec{q}) + \epsilon^2 X_i^{(2)}(\vec{P},\vec{q}) + \mathcal{O}(\epsilon^3)$$

$$\tilde{S}_k = S_k + Z_k^{(0)}(\vec{P},\vec{q}) + \epsilon Z_k^{(1)}(\vec{P},\vec{q}) + \mathcal{O}(\epsilon^2)$$

To obtain equations of motion independent of $\vec{q}$

$$\dot{\tilde{P}}_j = 0 + \epsilon \tilde{F}_j^{(1)}(\vec{\tilde{P}}) + \epsilon^2 \tilde{F}_j^{(2)}(\vec{\tilde{P}})$$

$$\dot{\tilde{q}}_i = \Upsilon_i^{(0)}(\vec{\tilde{P}}) + \epsilon \Upsilon_i^{(1)}(\vec{\tilde{P}})$$

$$\dot{\tilde{S}}_k = \Upsilon_k^{(0)}(\vec{\tilde{P}}) + \epsilon \Upsilon_k^{(1)}(\vec{\tilde{P}})$$

Geodesic

Adiabatic

Post-Adiabatic