# Flux-balance laws from an effective stress-energy tensor

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based on earlier, preliminary work with Éanna Flanagan, Zeyd Sam, and Jonathan Thompson)

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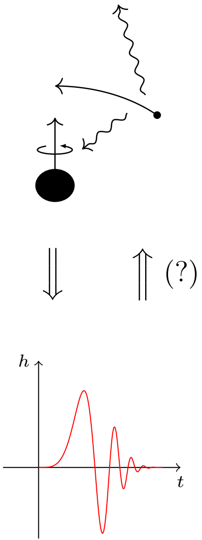
# Flux-balance laws

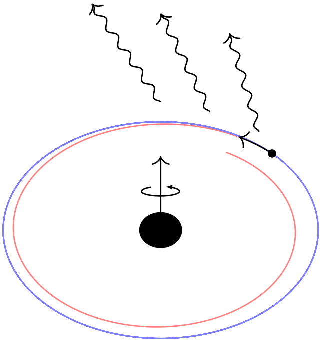
- Goal of self-force program: radiation during inspiral
- BH perturbation theory: motion $\implies$ radiation
- Motion depends on (regularized) fields at body: these are hard to compute!
- Goal of flux-balance:
  - Derive *some aspects* of motion w/ asymptotic fields (easier to compute)
  - Exploit symmetries of background spacetime: conserved quantities and currents



# Two-timescale formalism (old-fashioned style)

- Geodesic motion: constants of motion $Q$
- Non-geodesic motion: "conserved quantities" $Q(t, \tilde{t})$
  - Slow decay of orbit (this talk): dependence on $\tilde{t} = \varepsilon t$ ("slow time")
  - Rapid oscillations (average over): dependence on $t$ ("fast time")
- Q: Can one determine $\langle \partial Q / \partial \tilde{t} \rangle$ from fluxes of a conserved current?

  A: Yes, for $E_\xi \equiv p_a \xi^a$, up to second order (this talk)



# Conserved current

- General functional of $g_{ab}$:

$$(J_\xi)_{abc}[\boldsymbol{g}] \equiv G^{de}[\boldsymbol{g}] g_{ef} \xi^f \epsilon_{dabc}[\boldsymbol{g}]$$

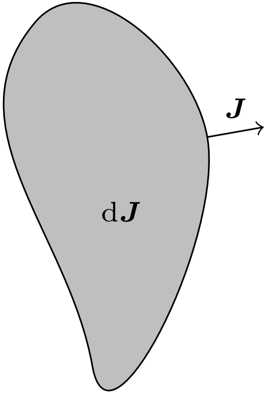
- Perform two perturbations:

$$\delta_1 \delta_2 \boldsymbol{J}_\xi \equiv \underset{(2)}{\boldsymbol{J}}_\xi\{\delta_1 \boldsymbol{g}, \delta_2 \boldsymbol{g}\} + \underset{(1)}{\boldsymbol{J}}_\xi\{\delta_1 \delta_2 \boldsymbol{g}\}.$$

- Properties of $\underset{(2)}{\boldsymbol{J}}_\xi$:
  - Symmetric & bilinear in $\delta_1 g_{ab}$, $\delta_2 g_{ab}$
  - Conserved for vacuum perturbations:

$$\mathrm{d}\underset{(2)}{\boldsymbol{J}}_\xi\{\delta_1 \boldsymbol{g}, \delta_2 \boldsymbol{g}\} = \frac{1}{2}\left[\underset{(1)}{G}^{ab}\{\delta_1 \boldsymbol{g}\} \pounds_\xi \delta_2 g_{ab} + (\delta_1 \longleftrightarrow \delta_2)\right] \boldsymbol{\epsilon}$$

  - Current arises from "effective stress-energy tensor"



# Two-timescale equations of motion (first order)

- Represent particle motion with $\gamma(t; \tilde{t}, \varepsilon)$:
  - $\gamma(t; \tilde{t}) \equiv \gamma(t; \tilde{t}, 0)$ geodesic
  - $\gamma(t; \tilde{t}, \varepsilon)$ stays close to $\gamma(t; \tilde{t})$
  - $\gamma(t; \varepsilon t, \varepsilon)$ geodesic in "effective metric"

$$\tilde{g}_{ab} \equiv g_{ab} + \varepsilon \mathfrak{h}^R_{ab} + O(\varepsilon^2)$$

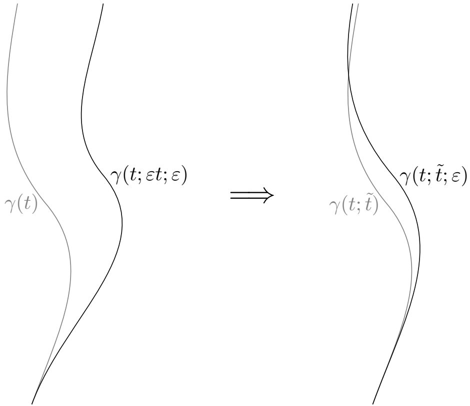
- $\mathfrak{h}^1_{ab}$ retarded solution to

$$\underset{(1)}{G}{}^{ab}\{\mathfrak{h}^1\} = 8\pi \mathfrak{T}^{ab}_1,$$

split into regular $\mathfrak{h}^R_{ab}$ and singular $\mathfrak{h}^S_{ab}$:

$$\underset{(1)}{G}{}^{ab}\{\mathfrak{h}^R\} = 0, \qquad \underset{(1)}{G}{}^{ab}\{\mathfrak{h}^S\} = 8\pi \mathfrak{T}^{ab}_1,$$

for distributional $\mathfrak{T}^{ab}_1$ supported on $\gamma(t; \tilde{t})$



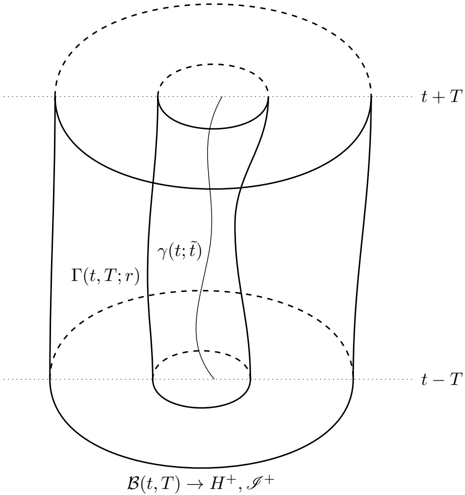
# Flux-balance law

- Asymptotic (average) flux:

$$\mathcal{F}_\xi[\mathfrak{h}^1] \equiv \lim_{T\to\infty} \frac{1}{2T} \int_{\underbrace{\mathcal{B}(t,T)}_{\to\Gamma(t,T;r)}} \underset{(2)}{\boldsymbol{J}}_\xi\{\mathfrak{h}^1, \mathfrak{h}^1\}$$

- Decompose $\mathfrak{h}^1_{ab} = \mathfrak{h}^R_{ab} + \mathfrak{h}^S_{ab}$:
  - $\mathfrak{h}^R, \mathfrak{h}^R$: exactly conserved, gives nothing
  - $\mathfrak{h}^R, \mathfrak{h}^S$: $\propto \langle \partial E_\xi / \partial \tilde{t} \rangle + O(\varepsilon)$ ("local" flux-balance)
  - $\mathfrak{h}^S, \mathfrak{h}^S$: naïvely divergent $I^{SS}$
- Result:

$$\left\langle \frac{\partial E_\xi}{\partial \tilde{t}} \right\rangle = \frac{1}{16\pi} \left\{ \mathcal{F}_\xi[\mathfrak{h}^1] - I^{SS} + O(\varepsilon) \right\}$$



# Residual term $I^{SS}$

This term vanishes, for two reasons:

Parity:

- Form of $\mathfrak{h}^S_{ab}$ near worldline:

$$\mathfrak{h}^S \sim m/r$$

- Integrand contains odd $n^a$'s
- $\int \mathrm{d}\Omega \, n^{a_1} \cdots n^{a_{2k+1}} = 0$

Dimensional analysis:

- Has same units as $[\mathrm{d}(p_a \xi^a)/\mathrm{d}t] = [\boldsymbol{\xi}]$
- Must be:
  - Linear in $\xi^a$
  - Proportional to $m^2$ from $\mathfrak{h}^S \propto m$
  - Remainder polynomial in $\dot{\gamma}^a(\tilde{t})$ and $\mathcal{E}_{ab}$, $\mathcal{B}_{ab}$
- *No* such scalar!

$$\boxed{\left\langle \frac{\partial E_\xi}{\partial \tilde{t}} \right\rangle = \frac{1}{16\pi} \left\{ \mathcal{F}_\xi[\mathfrak{h}^1] + O(\varepsilon) \right\}}$$

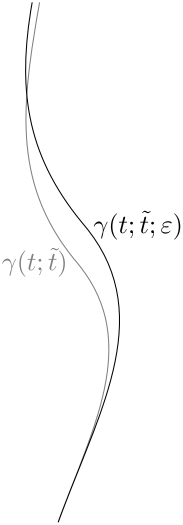
# Two-timescale (second order)

- Effective metric: $\tilde{g}_{ab} = g_{ab} + \varepsilon \mathfrak{h}^{R}_{ab} + \varepsilon^2 \mathfrak{h}^{RR}_{ab} + O(\varepsilon^3)$
- $\mathfrak{h}^2_{ab}$ decomposed into solutions to [Upton & Pound, 2021]

$$\underset{(1)}{G}^{ab}\{\mathfrak{h}^{RR}\} = -\frac{1}{2}\underset{(2)}{G}^{ab}\{\mathfrak{h}^{R}, \mathfrak{h}^{R}\} + \underset{(1,1)}{G}^{ab}\{\mathfrak{h}^{R}\},$$
$$\underset{(1)}{G}^{ab}\{\mathfrak{h}^{SR}\} = -\underset{(2)}{G}^{ab}\{\mathfrak{h}^{S}, \mathfrak{h}^{R}\} + 8\pi \mathfrak{T}^{ab}_{2} + \underset{(1,1)}{G}^{ab}\{\mathfrak{h}^{S}\},$$
$$\underset{(1)}{G}^{ab}\{\mathfrak{h}^{SS}\} = -\frac{1}{2}\underset{(2)}{G}^{ab}\{\mathfrak{h}^{S}, \mathfrak{h}^{S}\}, \qquad \text{[only need off } \gamma(t;\tilde{t})\text{]}$$

where $\underset{(1,1)}{G}^{ab}$ contains $\tilde{t}$-derivatives

- $\mathfrak{T}^{ab}_{2}$ once again only supported on $\gamma(t;\tilde{t})$



# Further splitting

Further divide $\mathfrak{h}^2_{ab}$ into "nonlinearity"- and "two-timescale"-sourced pieces:

Nonlinearity: $\hat{\mathfrak{h}}^2_{ab}$, decomposed as

$$\underset{(1)}{G}^{ab}\{\hat{\mathfrak{h}}^{RR}\} = -\frac{1}{2}\underset{(2)}{G}^{ab}\{\mathfrak{h}^R, \mathfrak{h}^R\},$$

$$\underset{(1)}{G}^{ab}\{\hat{\mathfrak{h}}^{SR}\} = -\underset{(2)}{G}^{ab}\{\mathfrak{h}^S, \mathfrak{h}^R\} + 8\pi\hat{\mathfrak{T}}^{ab}_2,$$

$$\underset{(1)}{G}^{ab}\{\hat{\mathfrak{h}}^{SS}\} = -\frac{1}{2}\underset{(2)}{G}^{ab}\{\mathfrak{h}^S, \mathfrak{h}^S\},$$

Two-timescale: $\check{\mathfrak{h}}^2_{ab}$, decomposed as

$$\underset{(1)}{G}^{ab}\{\check{\mathfrak{h}}^{RR}\} = \underset{(1,1)}{G}{}^{ab}\{\mathfrak{h}^R\},$$

$$\underset{(1)}{G}^{ab}\{\check{\mathfrak{h}}^{SR}\} = 8\pi\check{\mathfrak{T}}^{ab}_2 + \underset{(1,1)}{G}{}^{ab}\{\mathfrak{h}^S\}$$

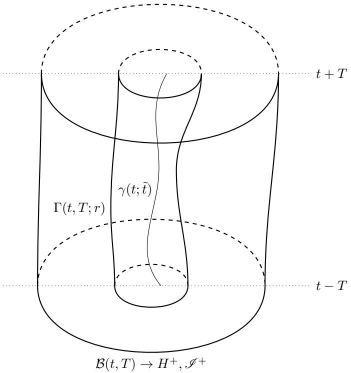
Note: axistationary part of $\hat{\mathfrak{h}}^2_{ab}$ diverges, but should not affect final answer

# Second order flux-balance

- Flux has higher-order terms:

$$\mathcal{F}_\xi[\mathfrak{h}^1, \hat{\mathfrak{h}}^2] \equiv \lim_{T\to\infty} \frac{1}{2T} \int_{\underbrace{\mathcal{B}(t,T)}_{\to\Gamma(t,T;r)}} \Big[ \underset{(2)}{\boldsymbol{J}}_\xi\{\mathfrak{h}^1, \mathfrak{h}^1 + 2\varepsilon\hat{\mathfrak{h}}^2\} + \frac{\varepsilon}{3} \underset{(3)}{\boldsymbol{J}}_\xi\{\mathfrak{h}^1, \mathfrak{h}^1, \mathfrak{h}^1\} \Big]$$

- Decompose $\mathfrak{h}^1_{ab}$, $\hat{\mathfrak{h}}^2_{ab}$:
  - All $\mathfrak{h}^R$ and $\hat{\mathfrak{h}}^{RR}$: exactly conserved, gives nothing
  - One $\mathfrak{h}^S$ or $\hat{\mathfrak{h}}^{SR}$, rest $\mathfrak{h}^R$ or $\hat{\mathfrak{h}}^{RR}$: related to $\langle \partial E_\xi / \partial \tilde{t} \rangle$ (complications on next slide)
  - Remaining terms: naïvely divergent, vanish like $I^{SS}$



# Flux-balance laws: less useful at second order?

$$\left\langle \frac{\partial E_\xi}{\partial \tilde{t}} \right\rangle = \frac{1}{16\pi} \overbrace{\mathcal{F}_\xi[\mathfrak{h}^1, \hat{\mathfrak{h}}^2]}^{\text{asymptotic fields}} + \varepsilon \underbrace{\left\langle \tilde{\mathcal{F}}^{\text{local}}_\xi[\mathfrak{h}^R] + \check{\mathcal{F}}^{\text{local}}_{h,\xi}[\check{\mathfrak{h}}^{RR}] - \check{\mathcal{F}}^{\text{local}}_{T,\xi}[\check{\mathfrak{T}}_2] \right\rangle}_{\text{local fields}} + O(\varepsilon^2)$$

- Where do extra terms come from:
  - $\tilde{\mathcal{F}}^{\text{local}}_\xi[\mathfrak{h}^R]$: two-timescale expansion/corrections to $E_\xi$ definitions
  - $\check{\mathcal{F}}^{\text{local}}_{h,\xi}[\check{\mathfrak{h}}^{RR}]$: $\check{\mathfrak{h}}^{RR}_{ab}$ (*not* $\hat{\mathfrak{h}}^{RR}_{ab}$) occurs in effective metric
  - $\check{\mathcal{F}}^{\text{local}}_{T,\xi}[\check{\mathfrak{T}}_2]$: $\check{\mathfrak{T}}^{ab}_2$ (*not* $\hat{\mathfrak{T}}^{ab}_2$) directly related to worldline stress-energy
- How easy is it to compute local fields:
  - $\mathfrak{h}^R_{ab}$: needs to be computed anyway (for full first-order self force)
  - $\check{\mathfrak{h}}^{RR}_{ab}$: more complicated, as it is sourced by $\mathfrak{h}^1_{ab}$ around worldline
  - $\check{\mathfrak{T}}^{ab}_2$: probably easy?

# Conclusions and future work

- Flux-balance law exists for second-order self-force for conserved quantities from isometries
  - Schwarzschild: *all* conserved quantities
  - Kerr: energy and (axial) angular momentum (still can handle equatorial/quasispherical orbits)
- Future work:
  - Turn into *practical* flux-balance law (i.e., determine how bad previous slide is, compute fluxes in terms of Teukolsky variables)
  - Carter constant evolution: Capra talk from 2021 [arXiv:2209.13829] w/ Jordan Moxon (only first order, only "local flux-balance")