Applying the effective-source approach to frequency-domain self-force calculations for eccentric orbits

Benjamin Leather, Niels Warburton

Paper released on the arXiv today!

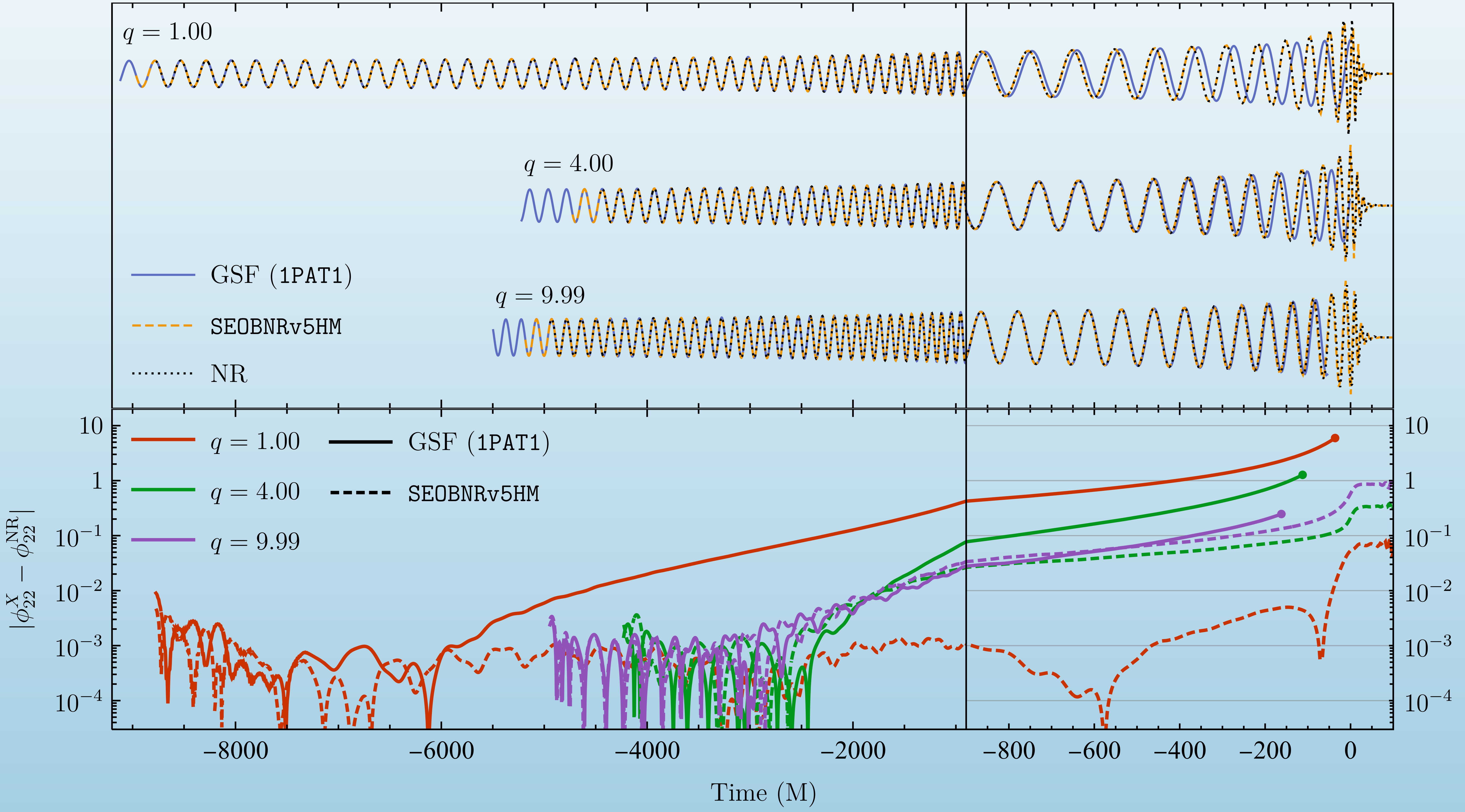
26th Capra Meeting on Radiation Reaction in General Relativity

arXiv: 2306.17221

CAPRA

MAX PLANCK INSTITUTE
FOR GRAVITATIONAL PHYSICS
(Albert Einstein Institute)

Motivation

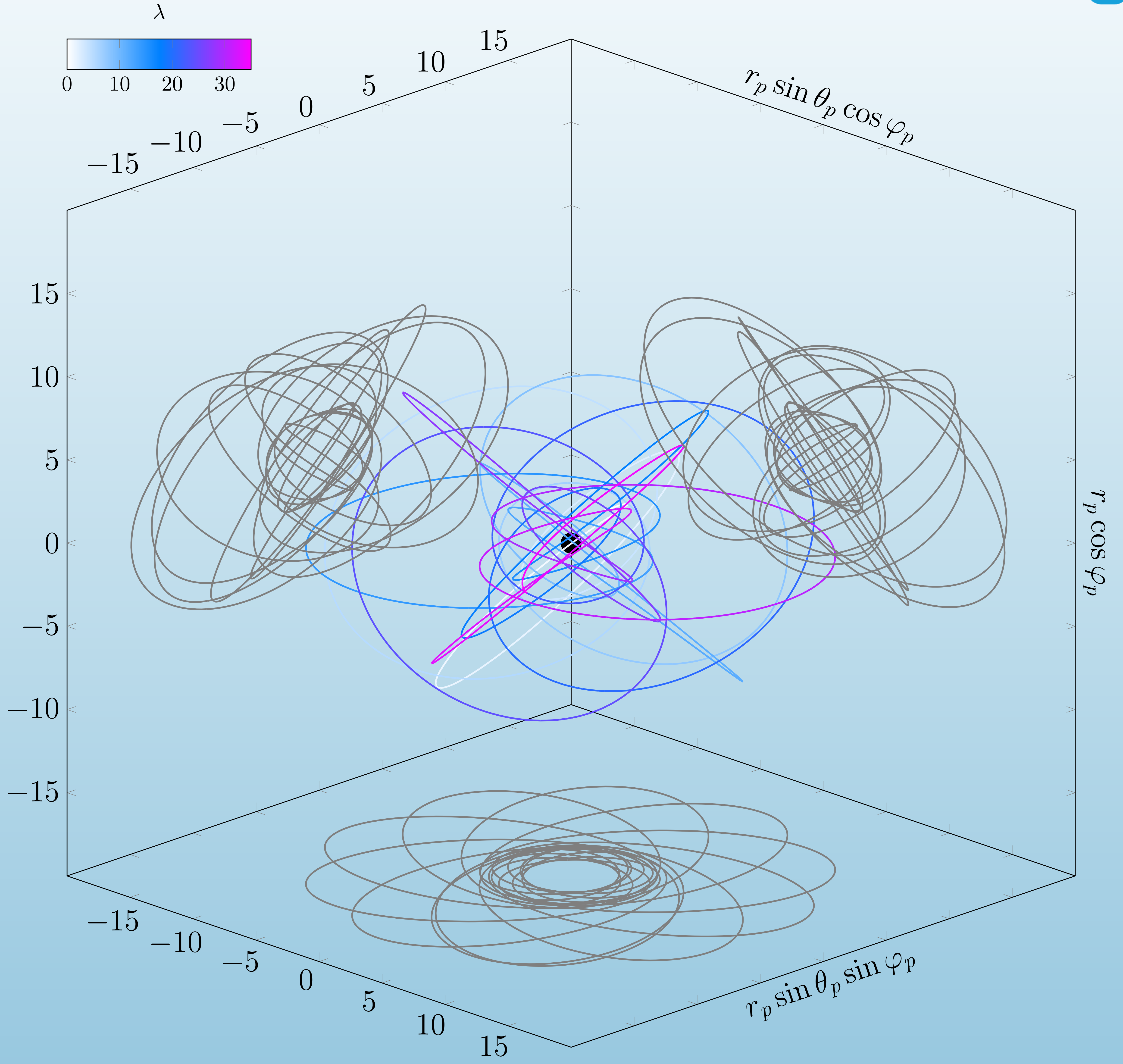


Enhancing the SEOBNRv5 effective-one-body waveform model with second-order gravitational self-force fluxes

Maarten van de Meent, Alessandra Buonanno, Deyan P. Mihaylov, Serguei Ossokine, Lorenzo Pompili, Niels Warburton, Adam Pound, Barry Wardell, Leanne Durkan, Jeremy Miller

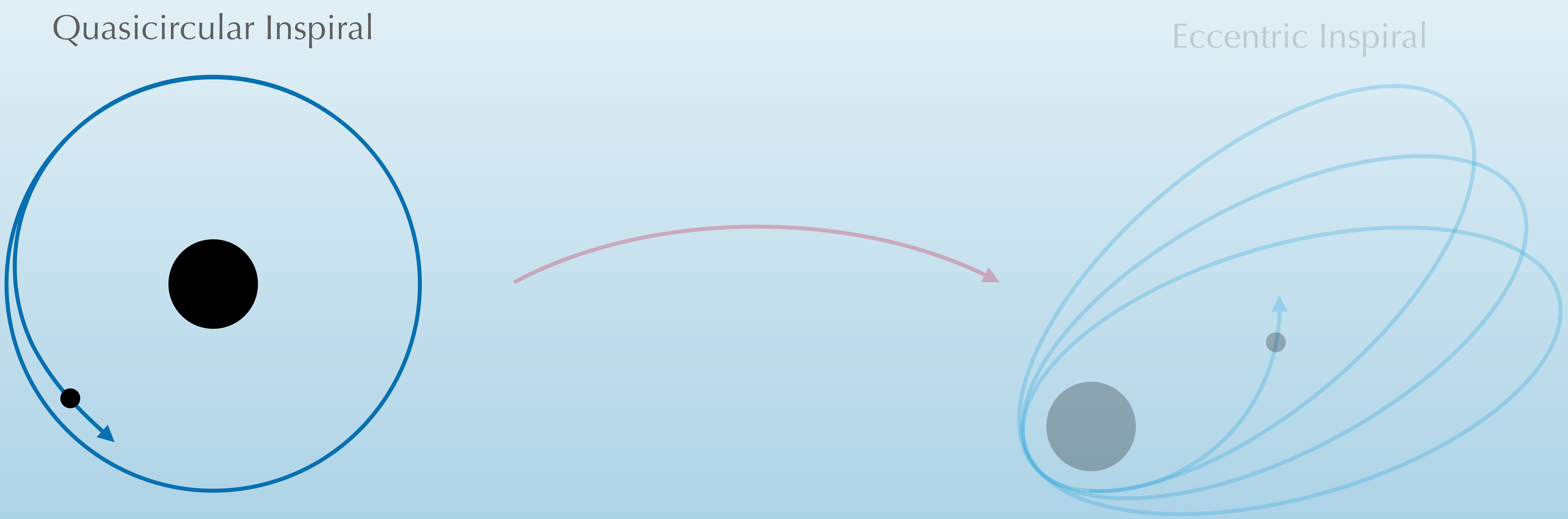
arXiv:2303.18026

Motivation

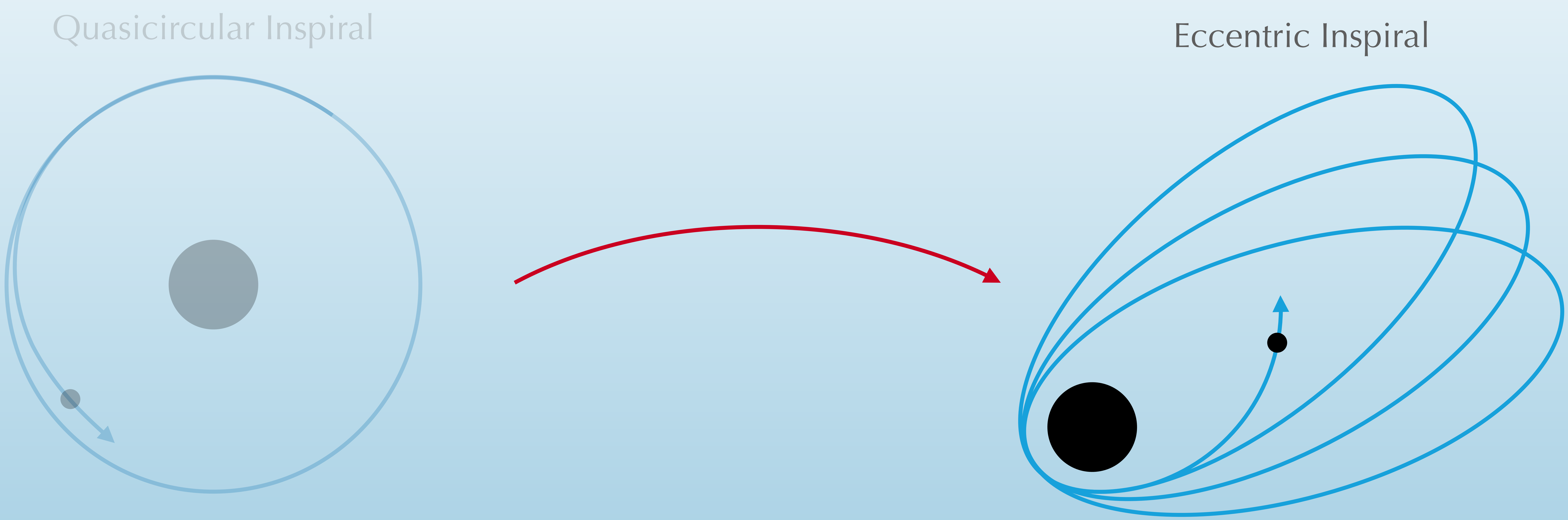


λ
0 10 20 30
$r_p \sin\theta_p \cos\varphi_p$
$r_p \sin\theta_p \sin\varphi_p$
$r_p \cos\varphi_p$
−15 −10 −5 0 5 10 15

Motivation

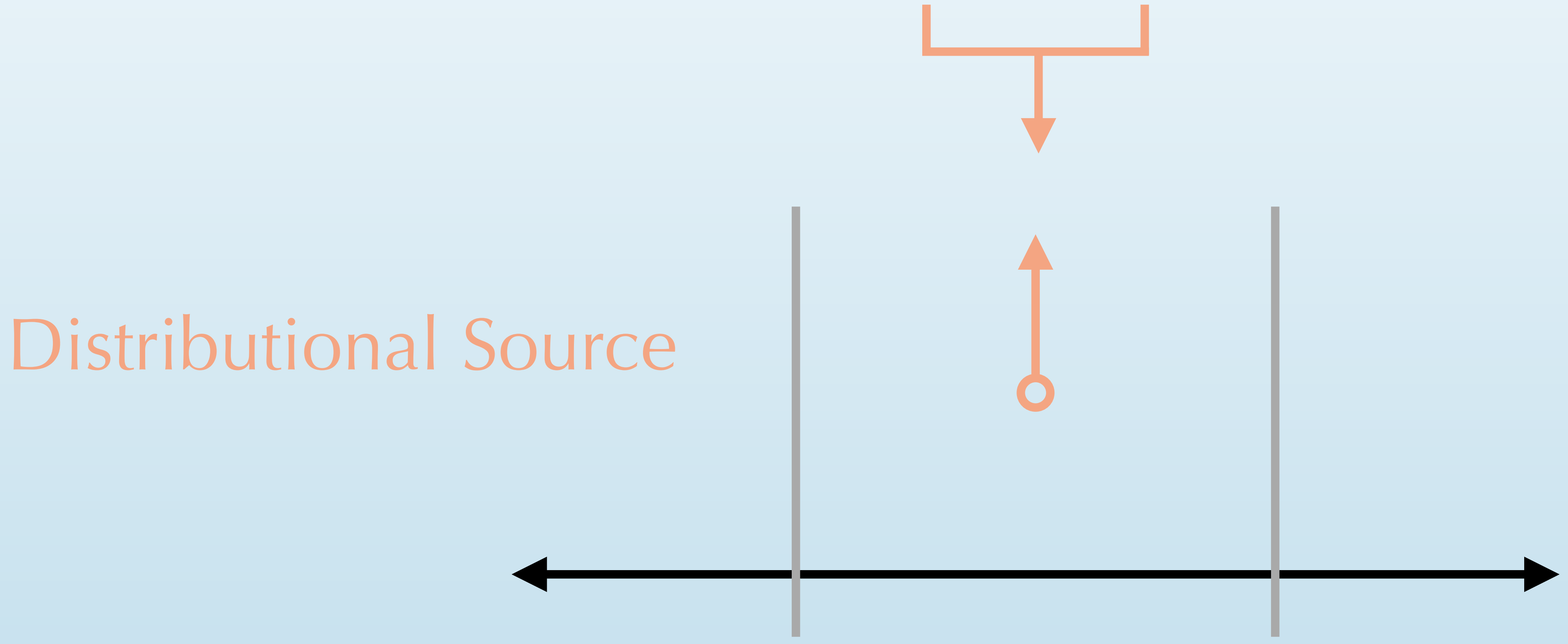


Motivation

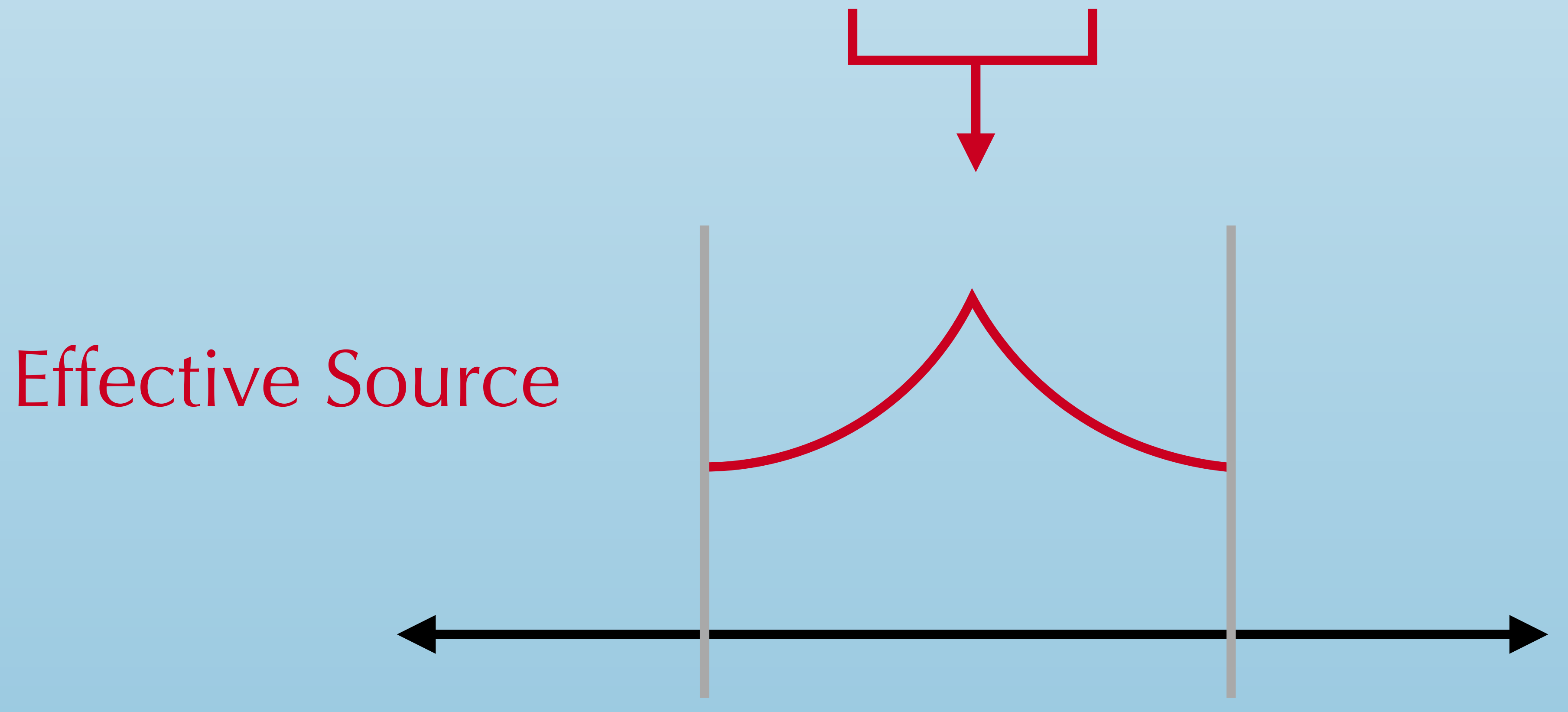


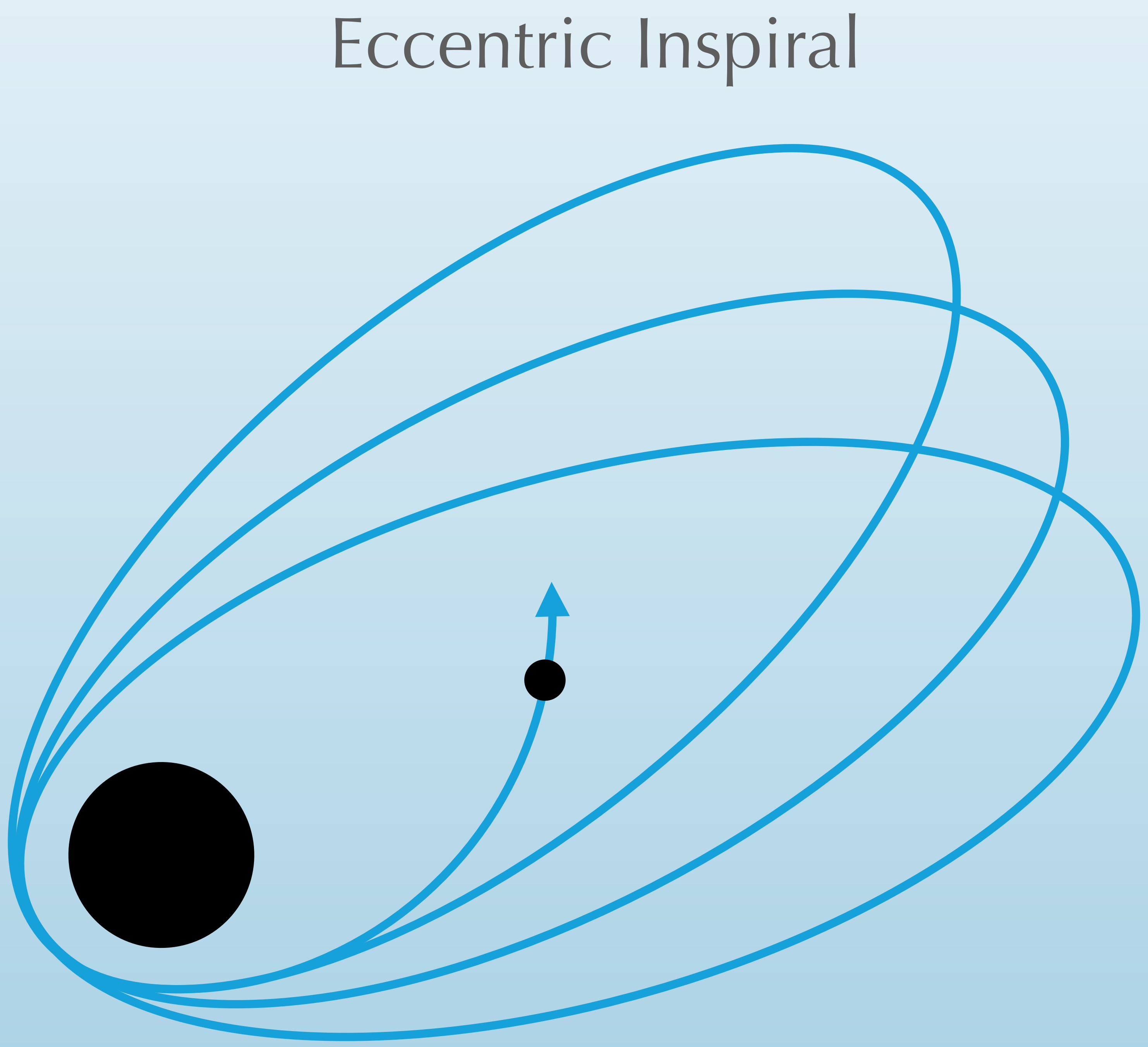
Motivation

First-order $\mathbf{g}_{\alpha\beta} = g_{\alpha\beta} + \epsilon h^{(1)}_{\alpha\beta} + \mathcal{O}(\epsilon^2)$

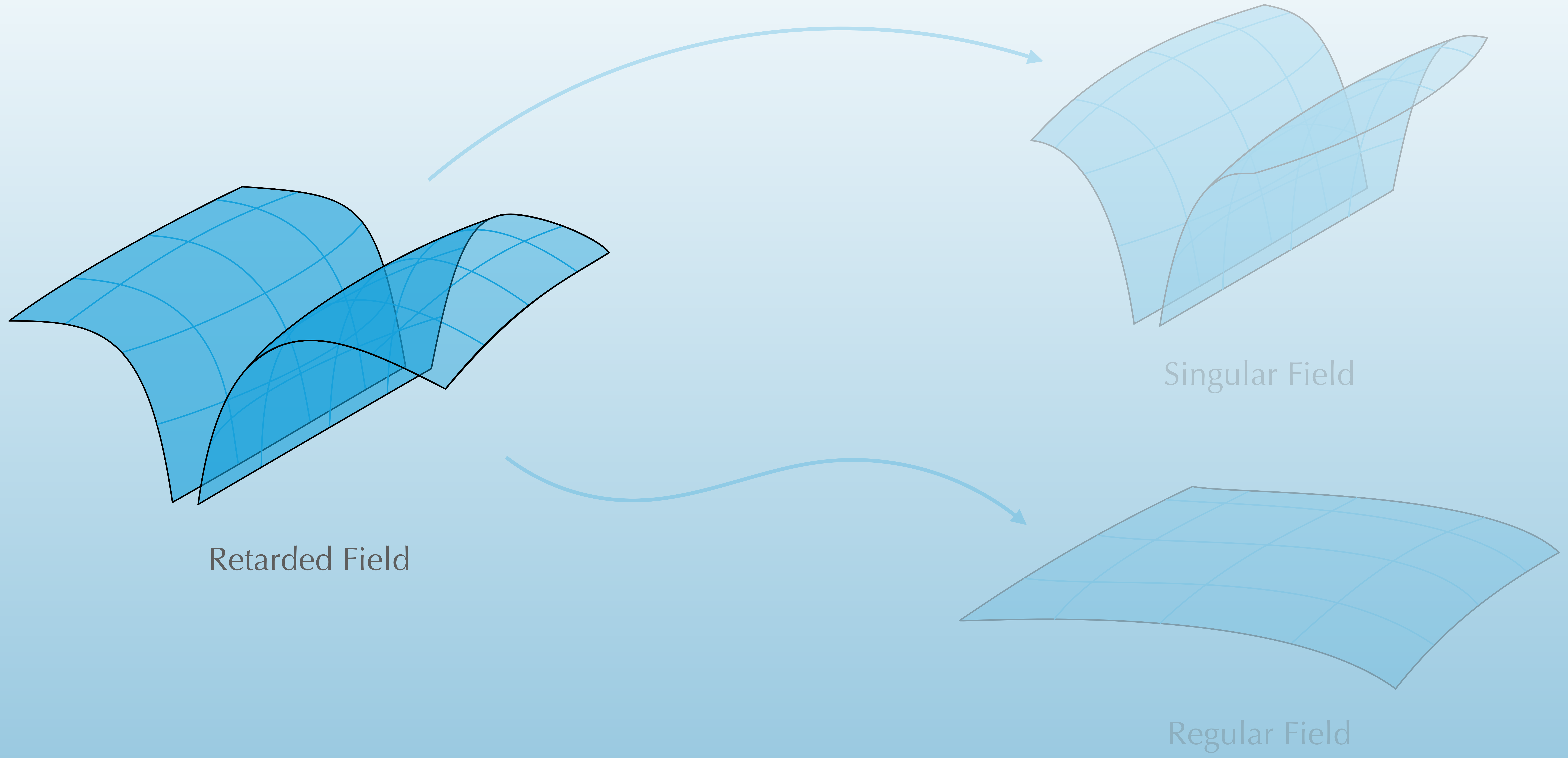


Second-order $\mathbf{g}_{\alpha\beta} = g_{\alpha\beta} + \epsilon h^{(1)}_{\alpha\beta} + \epsilon^2 h^{(2)}_{\alpha\beta} + \mathcal{O}(\epsilon^3)$

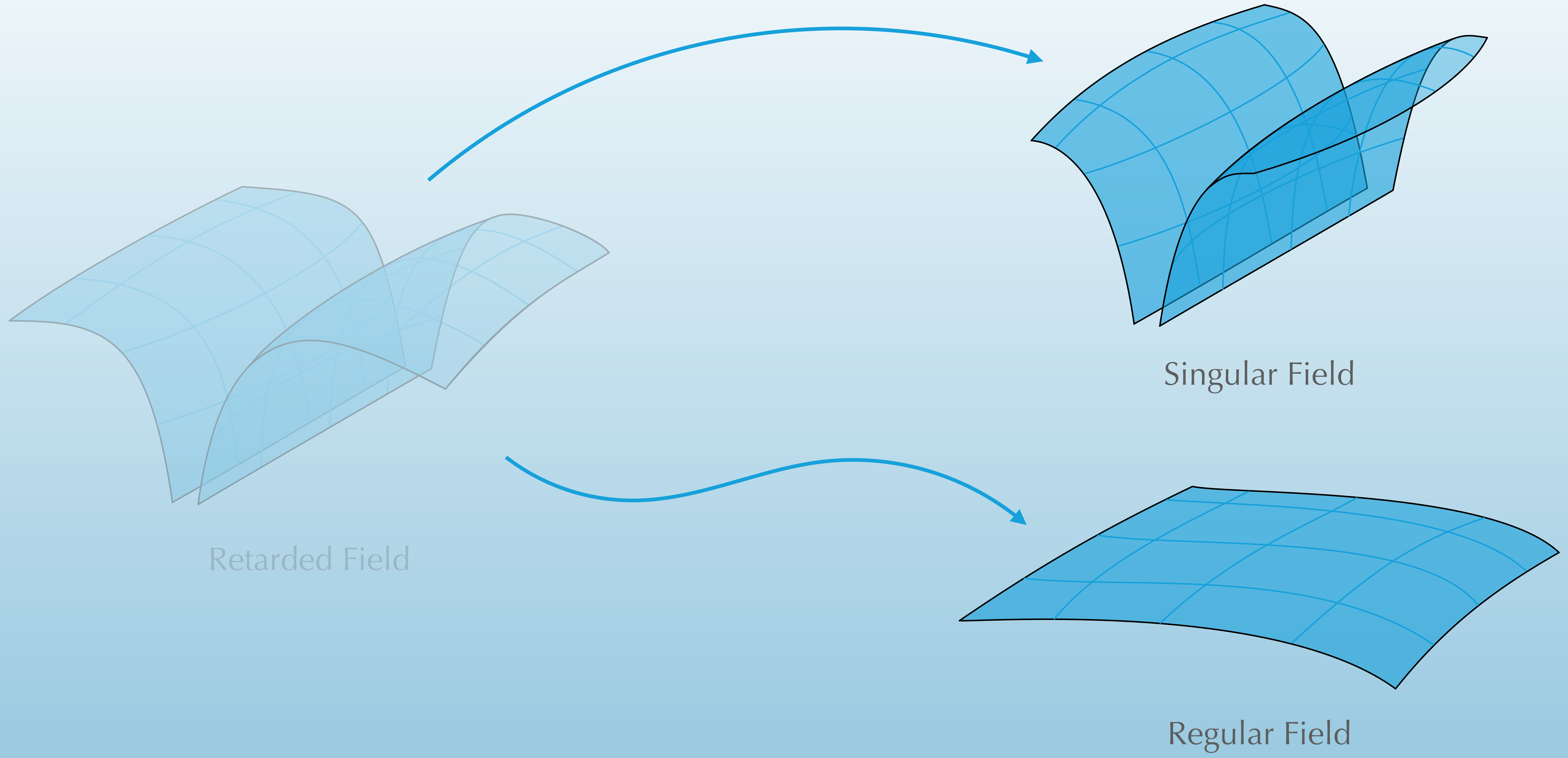




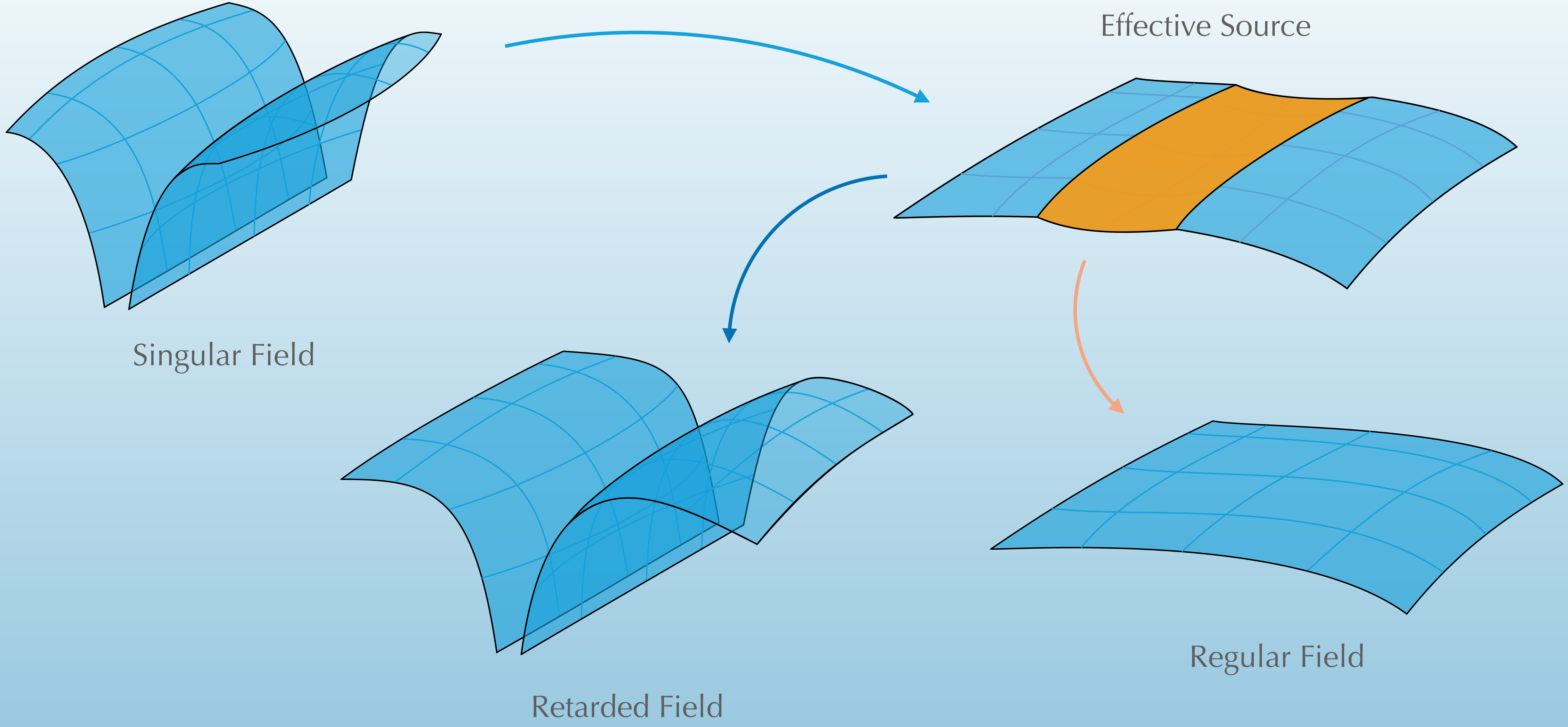
Worldtube and Effective Source



Worldtube and Effective Source



Worldtube and Effective Source



The Frequency Domain

Time Domain		Frequency Domain
$\Box_{\ell m}\psi_{\ell m}(t,r) = -4\pi r \rho_{\ell m}(t,r)$	$\longrightarrow$	$\Box_{\ell mn}\psi_{\ell mn}(r) = J_{\ell mn}(r)$
$\rho_{\ell m}(t,r) = \dfrac{q\,\hat{c}_{\ell m} P_\ell^m(0)}{r_p(t)^2 u^t}\delta[r - r_p(t)]e^{-im\varphi_p(t)}$	$\longrightarrow$	$J_{\ell mn}(r) = \dfrac{2q\,\hat{c}_{\ell m} P_\ell^m(0)}{T_r r\lvert u^r(r)\rvert f(r)^2}\cos[\omega_{mn}t_p(r) - m\varphi_p(r)] \times \Theta[r - r_{\min}] \times \Theta[r_{\max} - r]$
$\Box_{\ell m}\psi^{\mathcal{R}}_{\ell m} = -4\pi r \rho_{\ell m}(t,r) - \Box_{\ell m}\psi^{\mathcal{P}}_{\ell m} := S^{\text{eff}}_{\ell m}(t,r)$	$\longrightarrow$	$\Box_{\ell mn}\psi^{\mathcal{R}}(r) = J_{\ell mn}(r) - \Box_{\ell mn}\psi^{\mathcal{P}}_{\ell mn}(r) := S^{\text{eff}}_{\ell mn}(r)$
$S^{\text{eff}}_{\ell m}(r)$	$\longrightarrow$	$S^{\text{eff}}_{\ell mn}(r)$ $\longrightarrow$ $\psi^{\mathcal{R}}_{\ell mn}(r)$

$$\omega_{mn} := m\Omega_\varphi + n\Omega_r, \qquad m, n \in \mathbb{Z}$$

The Frequency Domain

Time Domain		Frequency Domain
$\Box_{\ell m}\psi_{\ell m}(t,r) = -4\pi r\rho_{\ell m}(t,r)$	$\longrightarrow$	$\Box_{\ell mn}\psi_{\ell mn}(r) = J_{\ell mn}(r)$
$\rho_{\ell m}(t,r) = \dfrac{q\,\hat{c}_{\ell m}P_\ell^m(0)}{r_p(t)^2u^t}\delta[r-r_p(t)]e^{-im\varphi_p(t)}$	$\longrightarrow$	$J_{\ell mn}(r) = \dfrac{2q\,\hat{c}_{\ell m}P_\ell^m(0)}{T_r r\lvert u^r(r)\rvert f(r)^2}\cos[\omega_{mn}t_p(r) - m\varphi_p(r)] \times \Theta[r - r_{\min}] \times \Theta[r_{\max} - r]$
$\Box_{\ell m}\psi^{\mathcal{R}}_{\ell m} = -4\pi r\rho_{\ell m}(t,r) - \Box_{\ell m}\psi^{\mathcal{P}}_{\ell m} := S^{\rm eff}_{\ell m}(t,r)$	$\longrightarrow$	$\Box_{\ell mn}\psi^{\mathcal{R}}(r) = J_{\ell mn}(r) - \Box_{\ell mn}\psi^{\mathcal{P}}_{\ell mn}(r) := S^{\rm eff}_{\ell mn}(r)$
$S^{\rm eff}_{\ell m}(t,r)$	$\longrightarrow$	$S^{\rm eff}_{\ell mn}(r)$

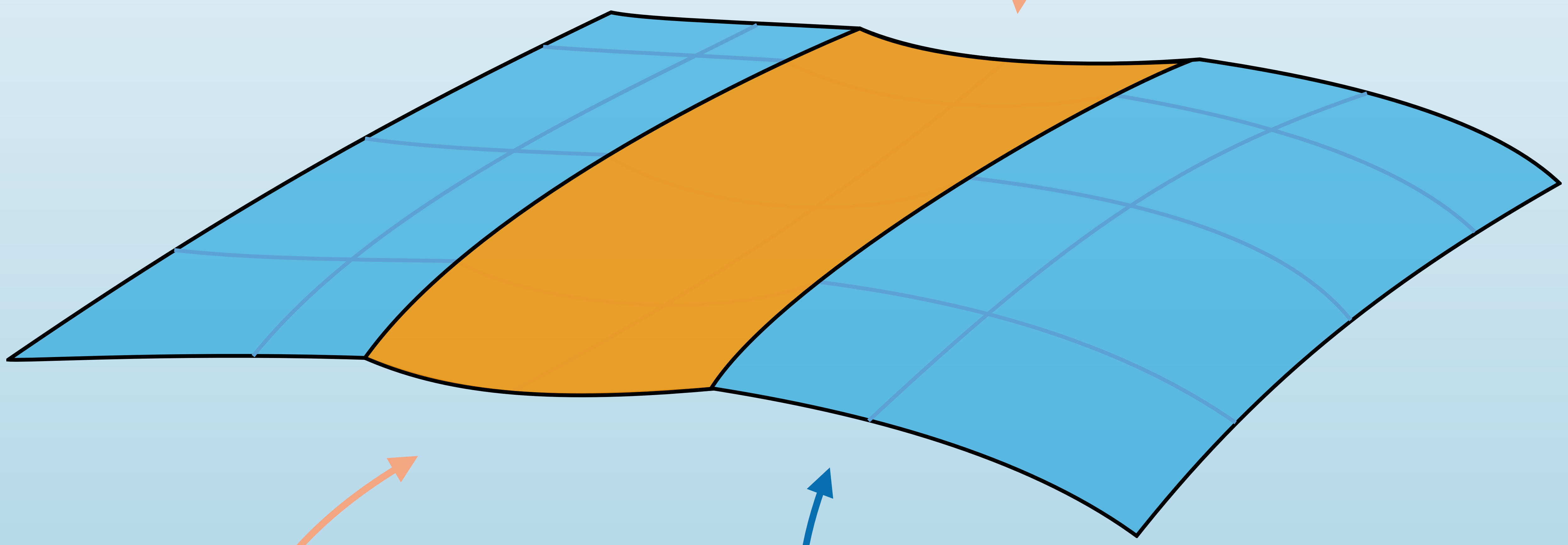
$$S^{\rm eff}_{\ell mn}(r) \longrightarrow \psi^{\mathcal{R}}_{\ell mn}(r)$$

$$\omega_{mn} := m\Omega_\varphi + n\Omega_r, \qquad m,n\in\mathbb{Z}$$

Scalar self-force (SSF)

$$\Box_{\ell mn}\psi^{\mathcal{R}}_{\ell mn}(r) = J_{\ell mn}(r) - \Box_{\ell mn}\psi^{\mathcal{P}}_{\ell mn}(r) := S^{\text{eff}}_{\ell mn}(r)$$

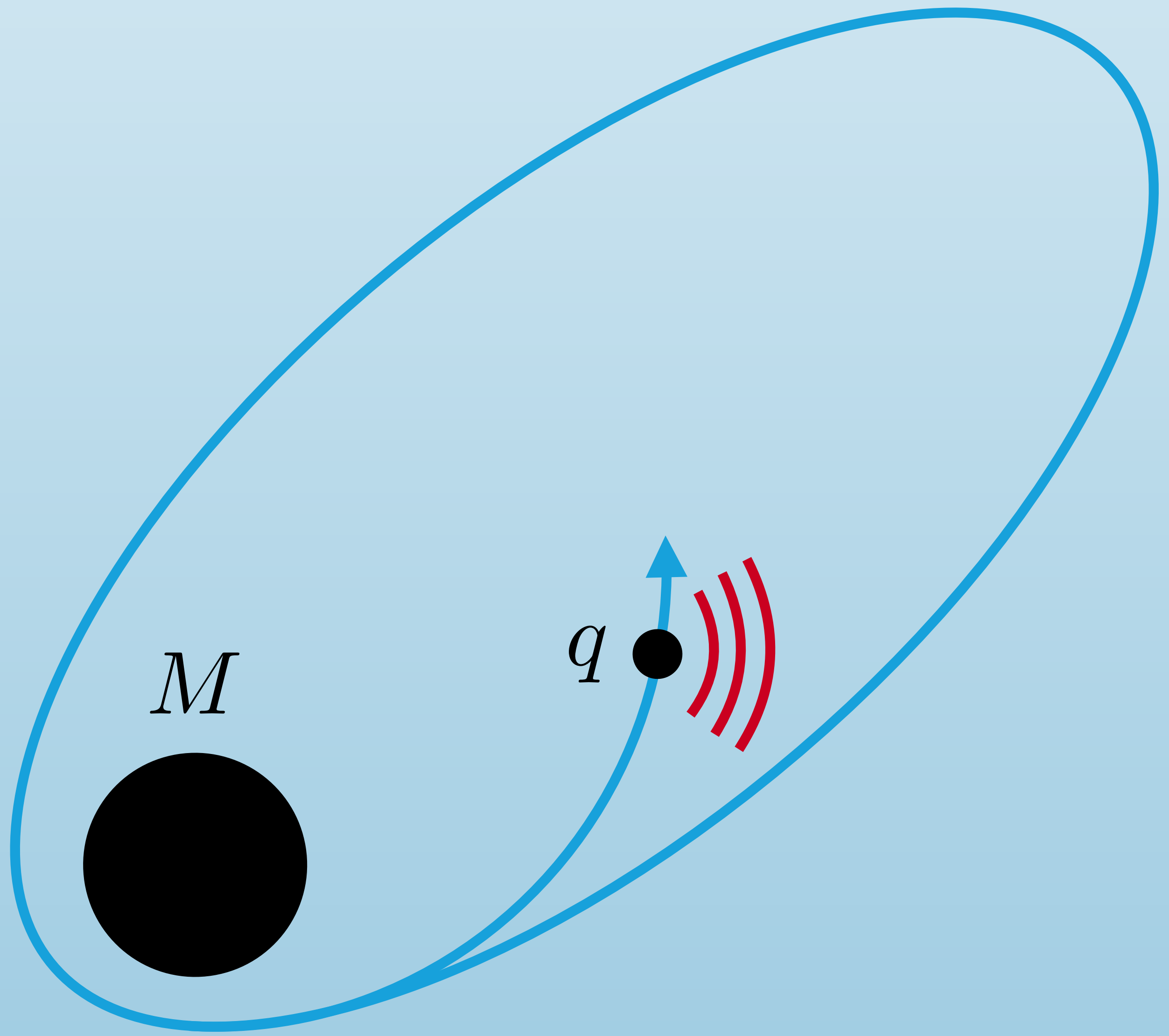
Residual Field



Worldtube

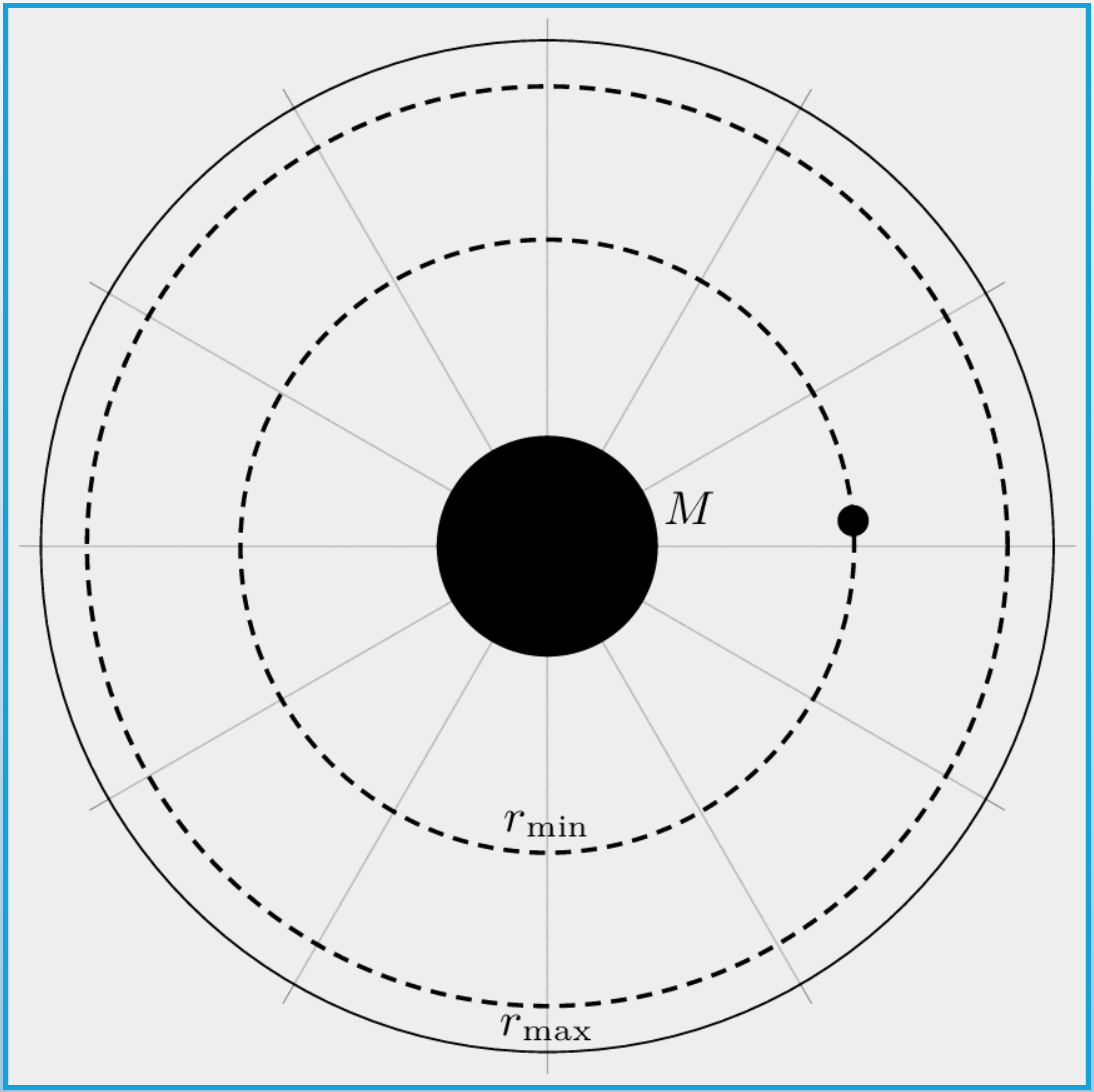
$$\Box_{\ell mn}\psi_{\ell mn}(r) = 0$$

Retarded Field

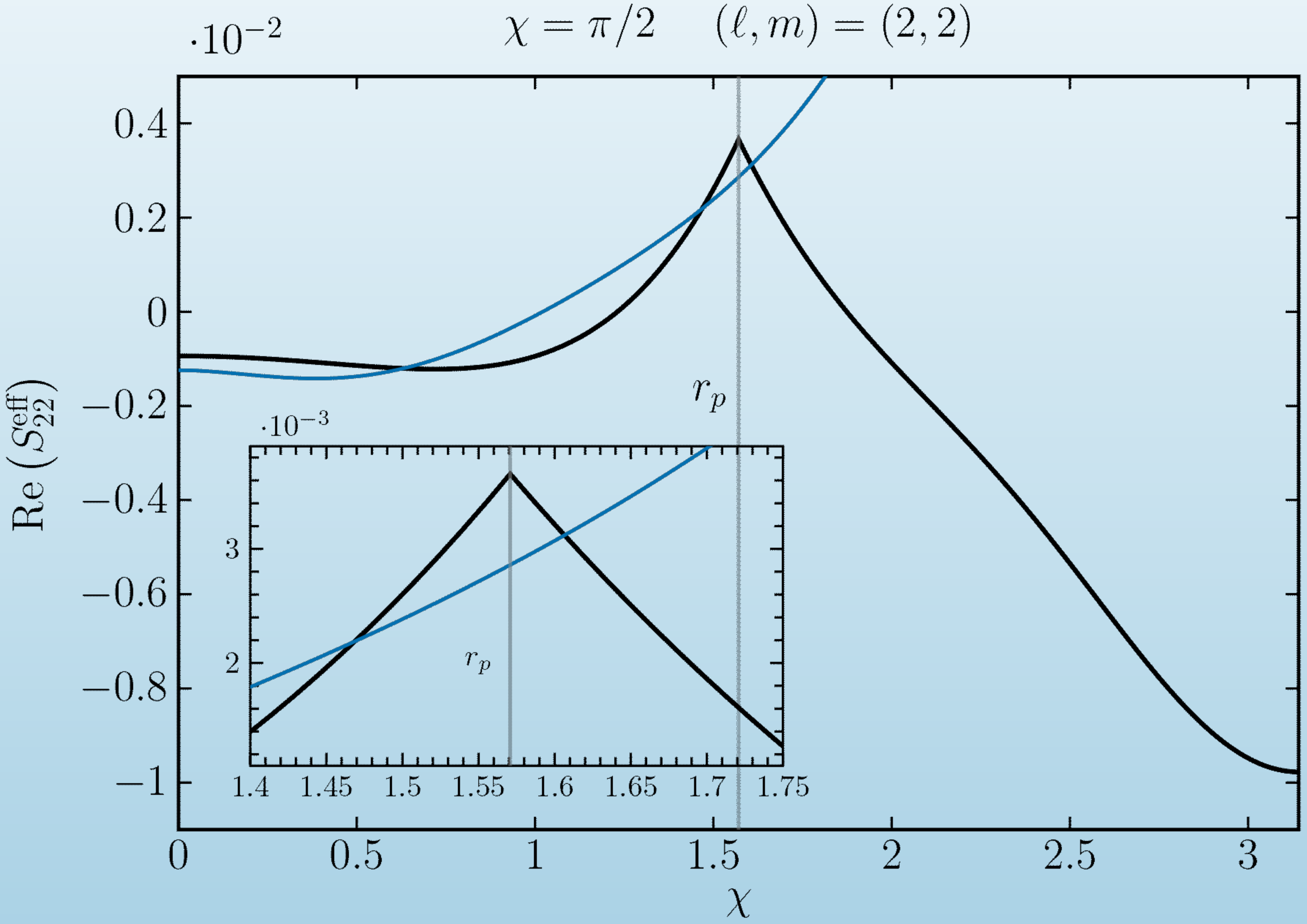


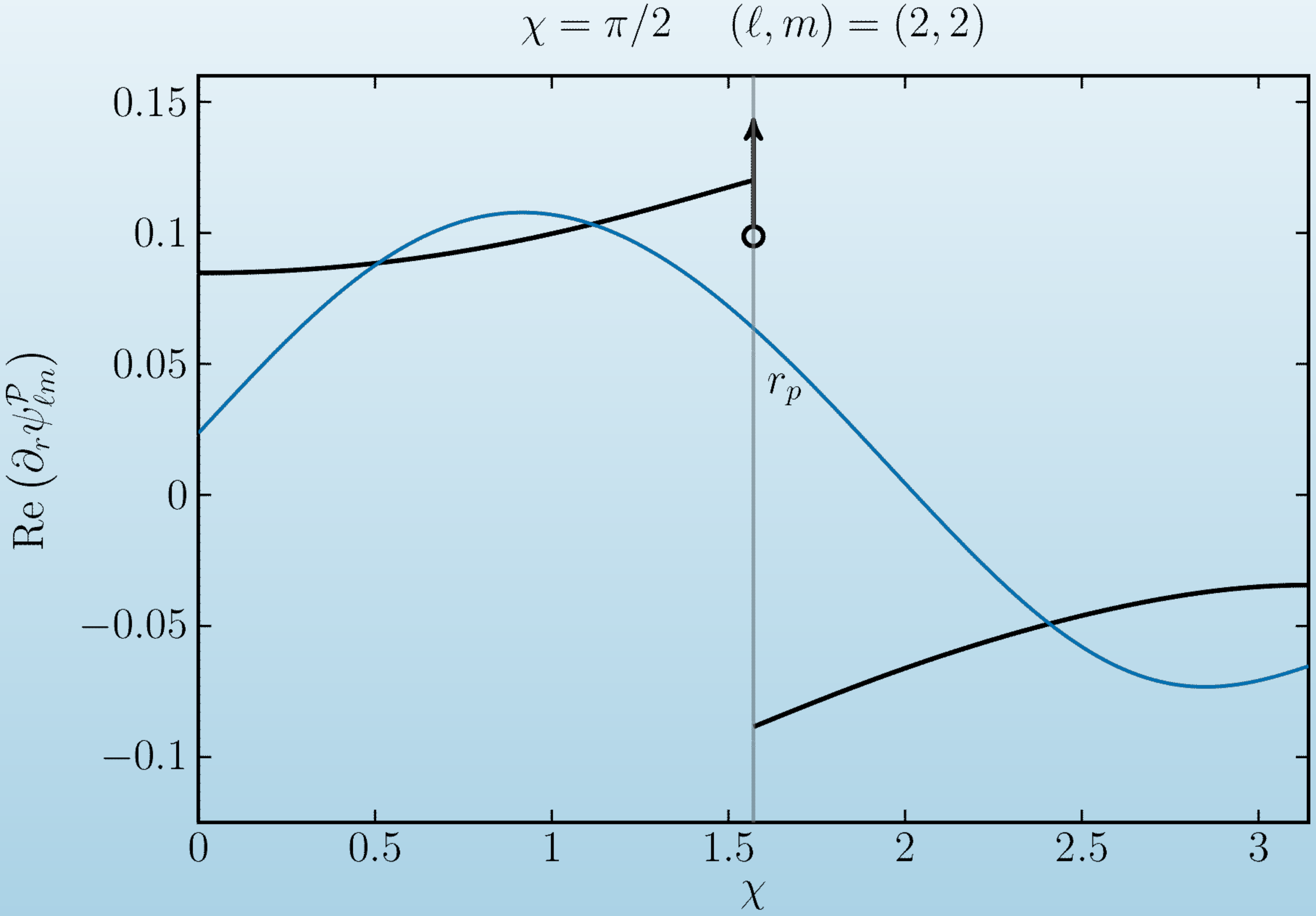
Schwarzschild Background

The challenge of eccentric orbits



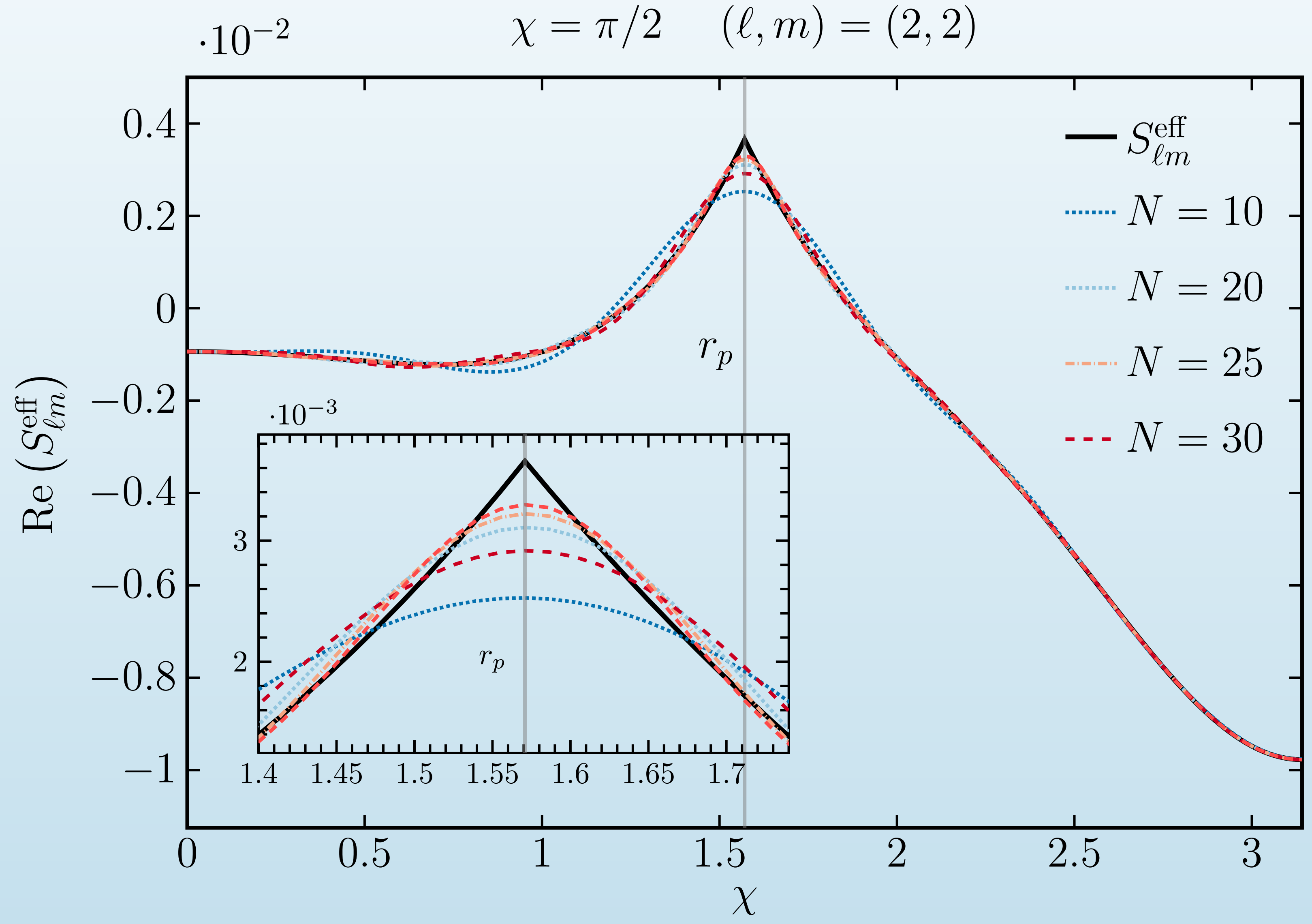
Gibbs Phenomenon

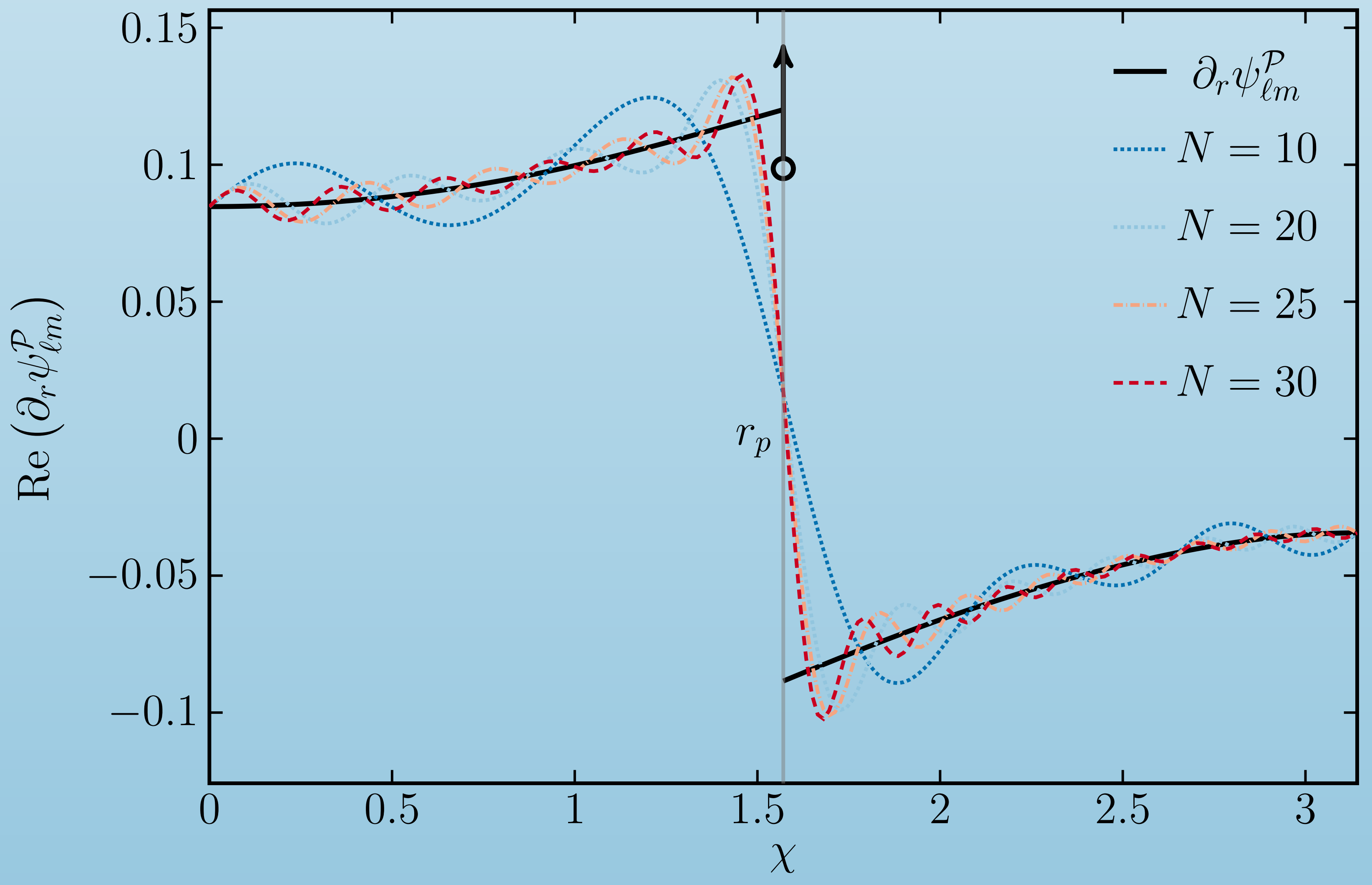


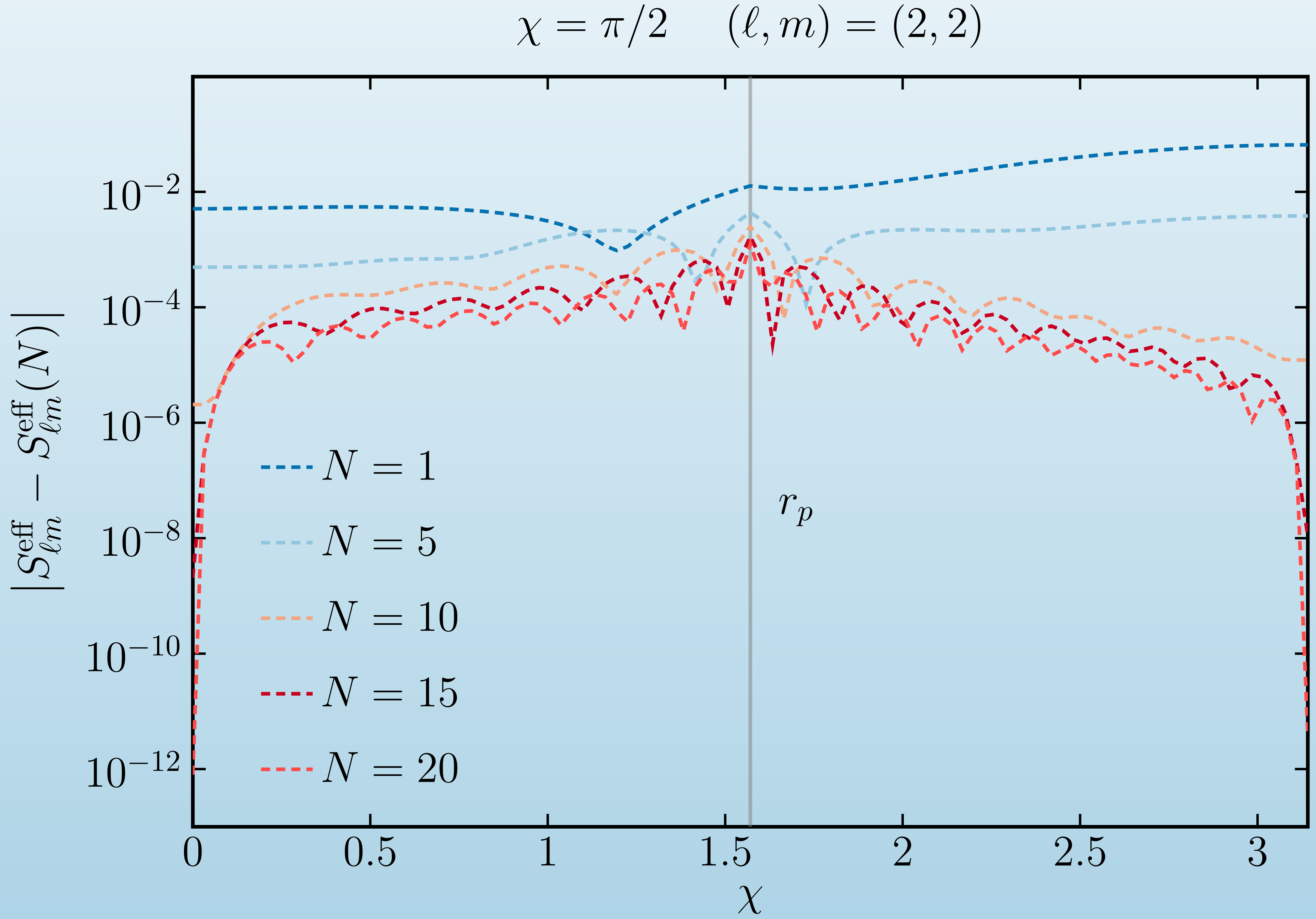


$$S^{\mathrm{eff}}_{\ell mn}(r) = \frac{1}{T_r}\int_0^{T_r} S^{\mathrm{eff}}_{\ell m}(t,r)e^{-i\omega_{mn}t}\,dt \quad \longrightarrow \quad S^{\mathrm{eff}}_{\ell m}(t,r) = \lim_{N\to\infty}\sum_{n=-N}^{N} S^{\mathrm{eff}}_{\ell mn}(r)e^{-i\omega_{mn}t}$$

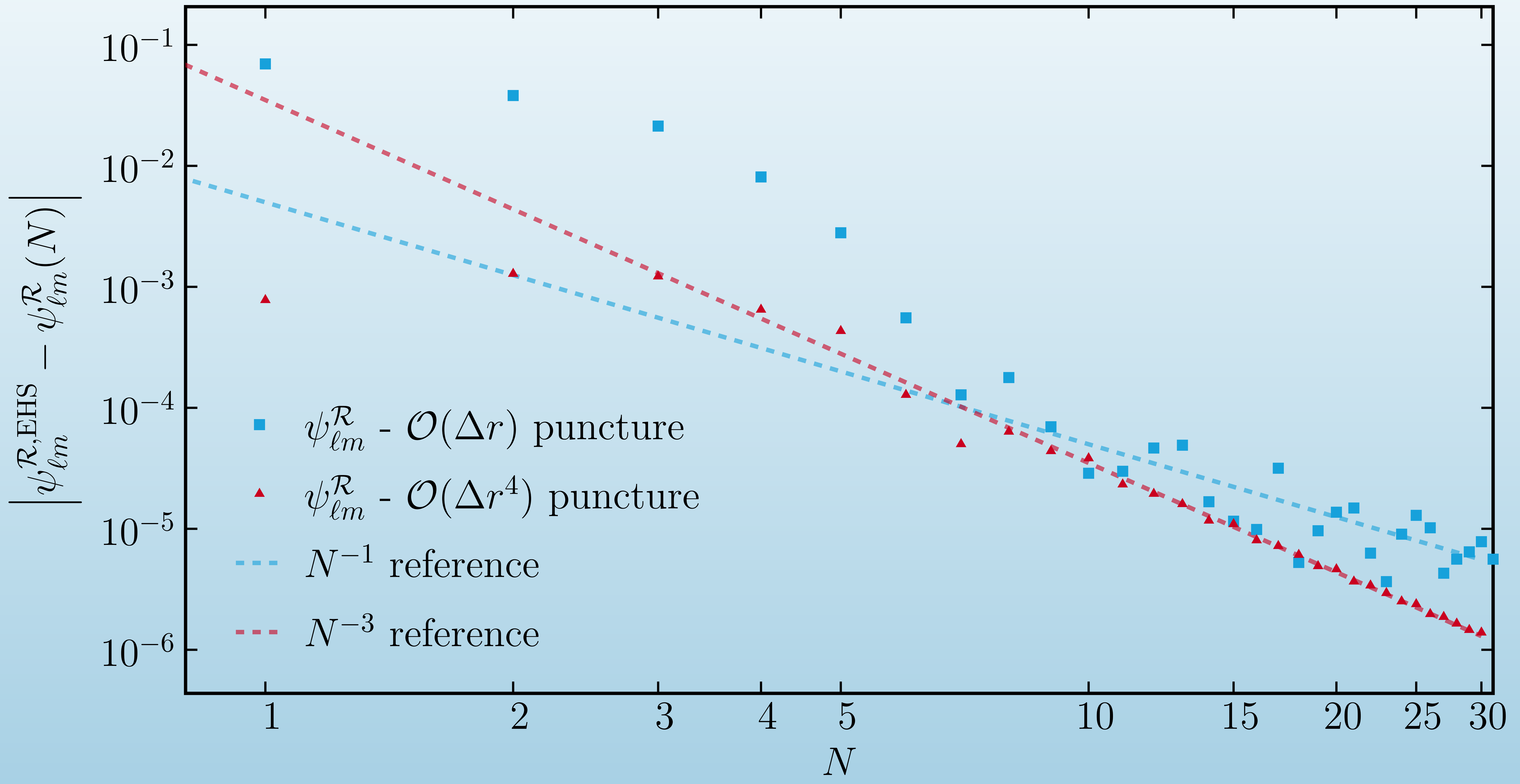
Gibbs Phenomenon







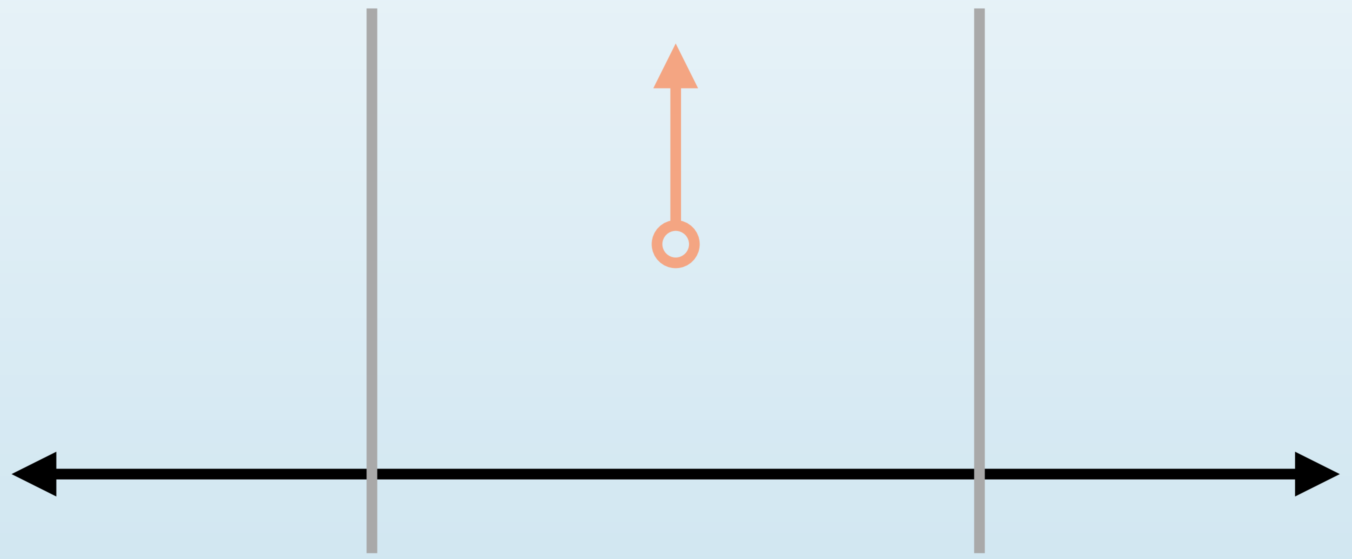
Fourier reconstruction of the residual field



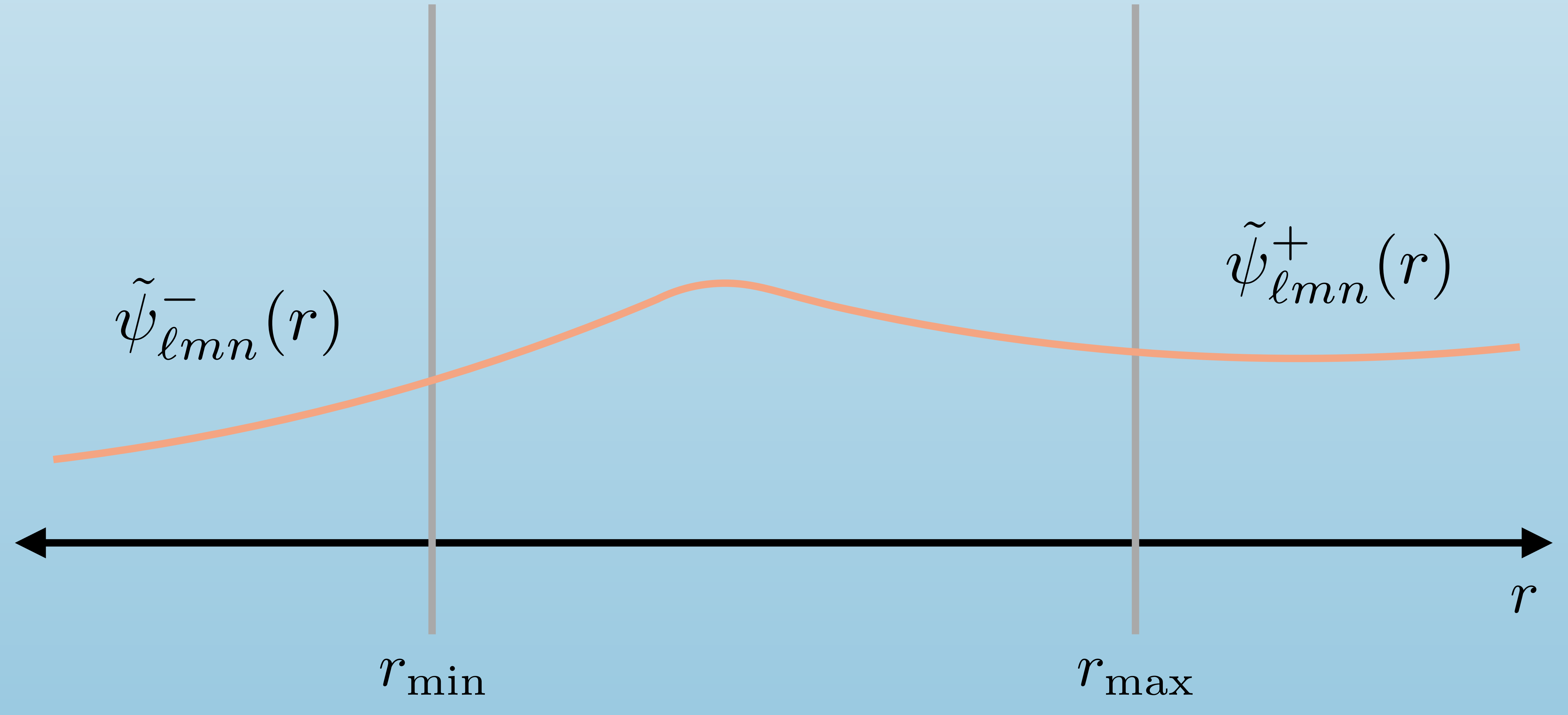
$$\psi^{\mathcal{R}}_{\ell m}(t,r) = \lim_{N\to\infty} \sum_{n=-N}^{N} \psi^{\mathcal{R}}_{\ell mn}(r) e^{-i\omega_{mn}t}$$

Extended Homogeneous Solutions (EHS)

Distributional Source



Fourier Domain



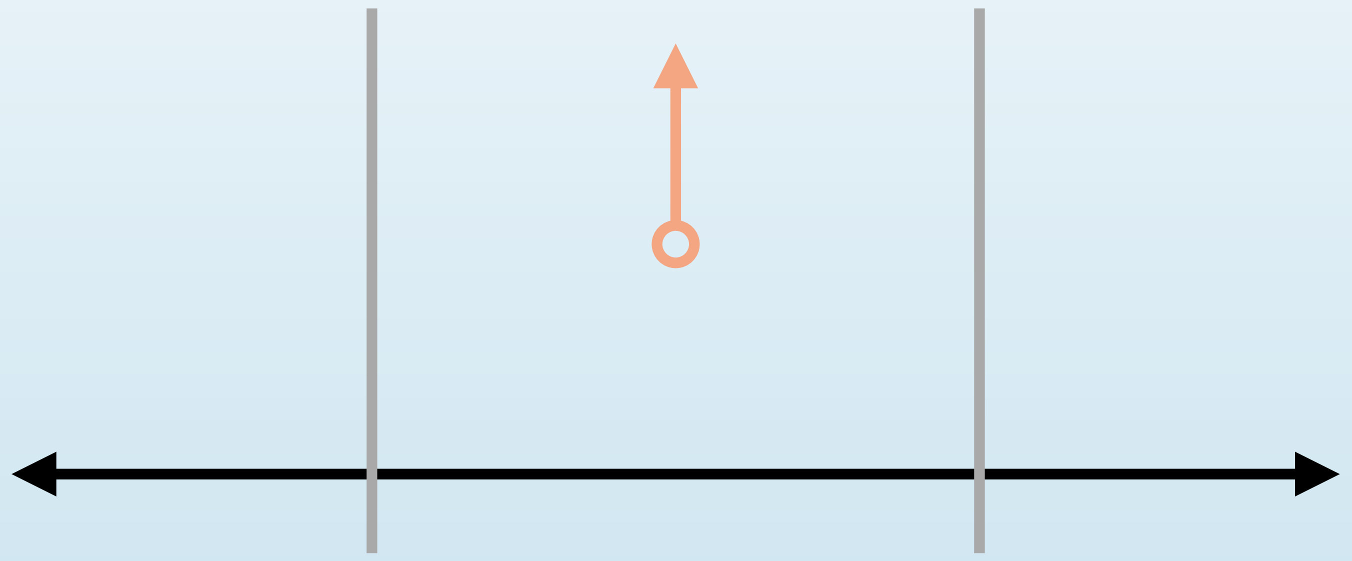
$$\tilde{\psi}^{\pm}_{\ell mn}(r) := C^{\pm}_{\ell mn}\psi^{\infty/h}_{\ell mn}(r)$$

$$\tilde{\psi}^{\pm}_{\ell m}(t,r) := \sum_{n=-\infty}^{\infty} \tilde{\psi}^{\pm}_{\ell mn}(r)e^{-i\omega_{mn}t}$$

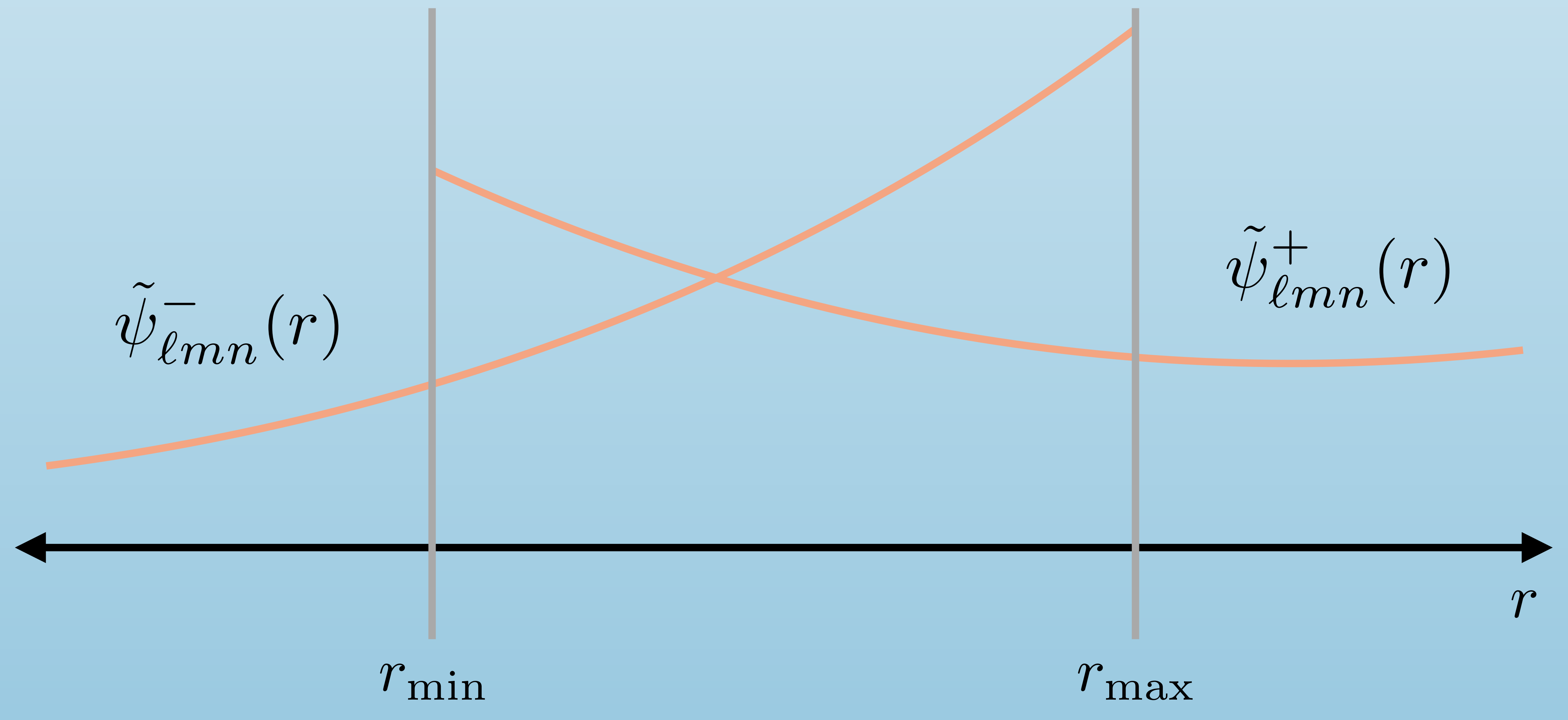
$$\psi^{\mathrm{EHS}}_{\ell m}(t,r) := \tilde{\psi}^{+}_{\ell m}(t,r)\Theta[r - r_p(t)] + \tilde{\psi}^{-}_{\ell m}(t,r)\Theta[r_p(t) - r]$$

Extended Homogeneous Solutions (EHS)

Distributional Source



Fourier Domain



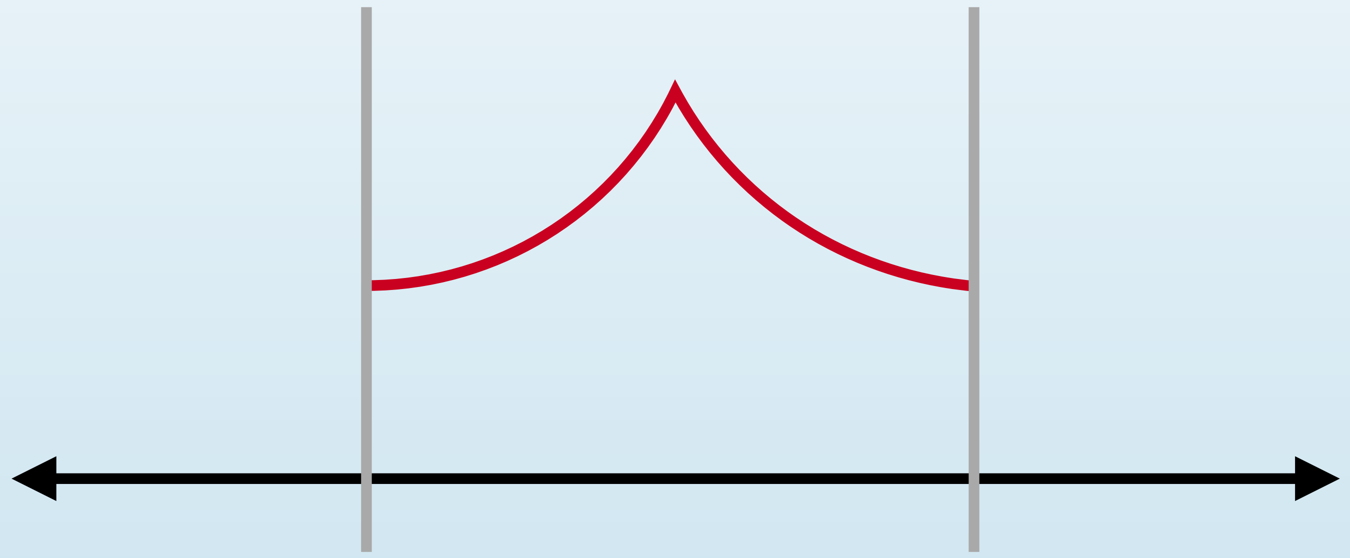
$$\tilde{\psi}^{\pm}_{\ell mn}(r) := C^{\pm}_{\ell mn}\psi^{\infty/h}_{\ell mn}(r)$$

$$\tilde{\psi}^{\pm}_{\ell m}(t,r) := \sum_{n=-\infty}^{\infty} \tilde{\psi}^{\pm}_{\ell mn}(r)e^{-i\omega_{mn}t}$$

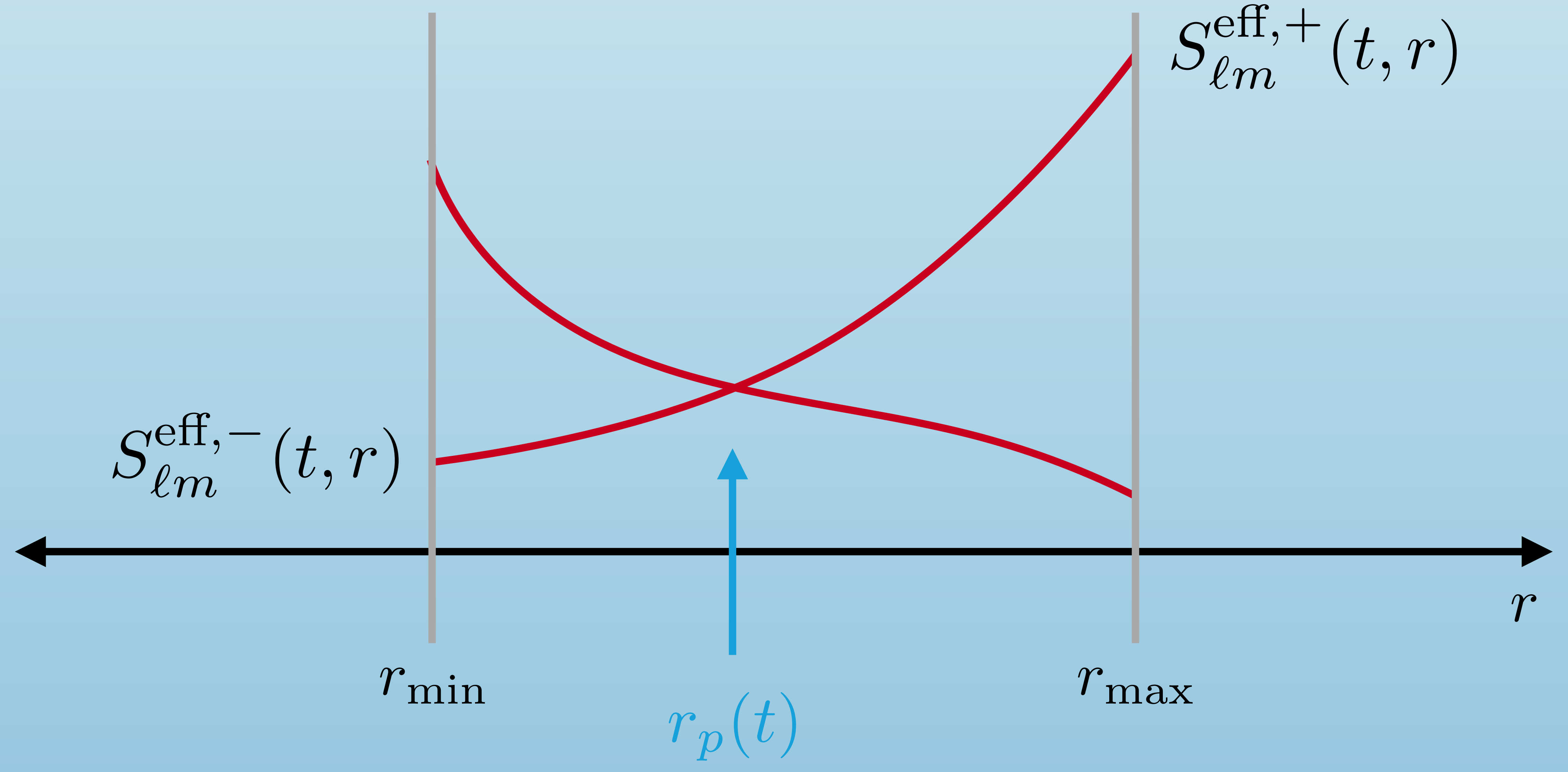
$$\psi^{\text{EHS}}_{\ell m}(t,r) := \tilde{\psi}^{+}_{\ell m}(t,r)\Theta[r - r_p(t)] + \tilde{\psi}^{-}_{\ell m}(t,r)\Theta[r_p(t) - r]$$

Extended Effective Sources (EES)

Effective Source



Time Domain



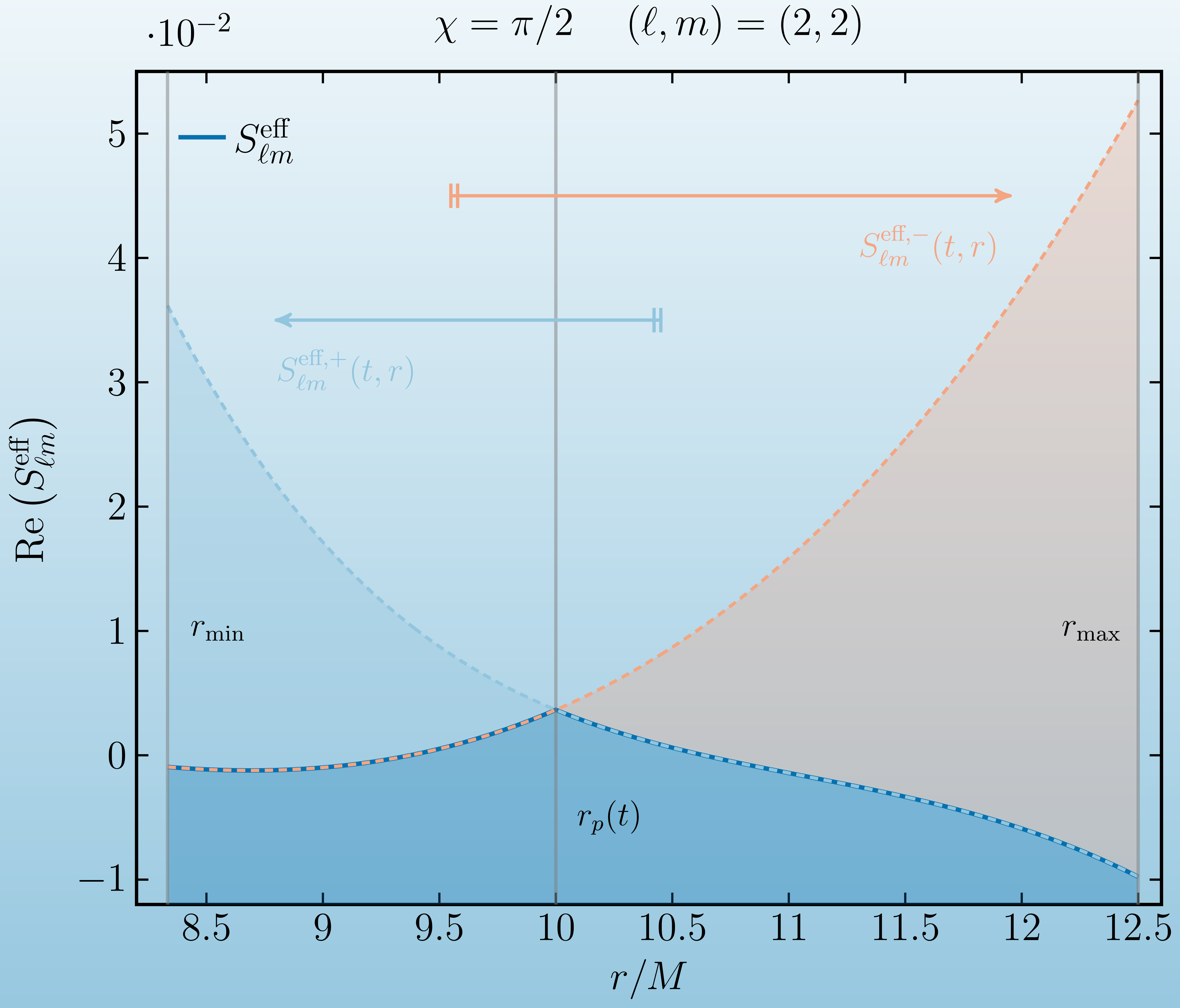
$$S^{\text{eff}}_{\ell m}(t,r) = S^{\text{eff},+}_{\ell m}(t,r)\Theta^{+}(t,r) + S^{\text{eff},-}_{\ell m}(t,r)\Theta^{-}(t,r)$$

$$S^{\text{eff},\pm}_{\ell mn}(r) = \frac{1}{T_r}\int_0^{T_r} S^{\text{eff},\pm}_{\ell m}(t,r)e^{i\omega_{mn}t}dt$$

$$\psi^{\mathcal{R}}_{\ell m}(t,r) = \psi^{\mathcal{R},+}_{\ell m}(t,r)\Theta^{+}(t,r) + \psi^{\mathcal{R},-}_{\ell m}(t,r)\Theta^{-}(t,r)$$

$$\Theta^{\pm}(t,r) = \Theta[\pm(r - r_p(t))]$$

Extended Effective Sources (EES)



Fourier Convolution

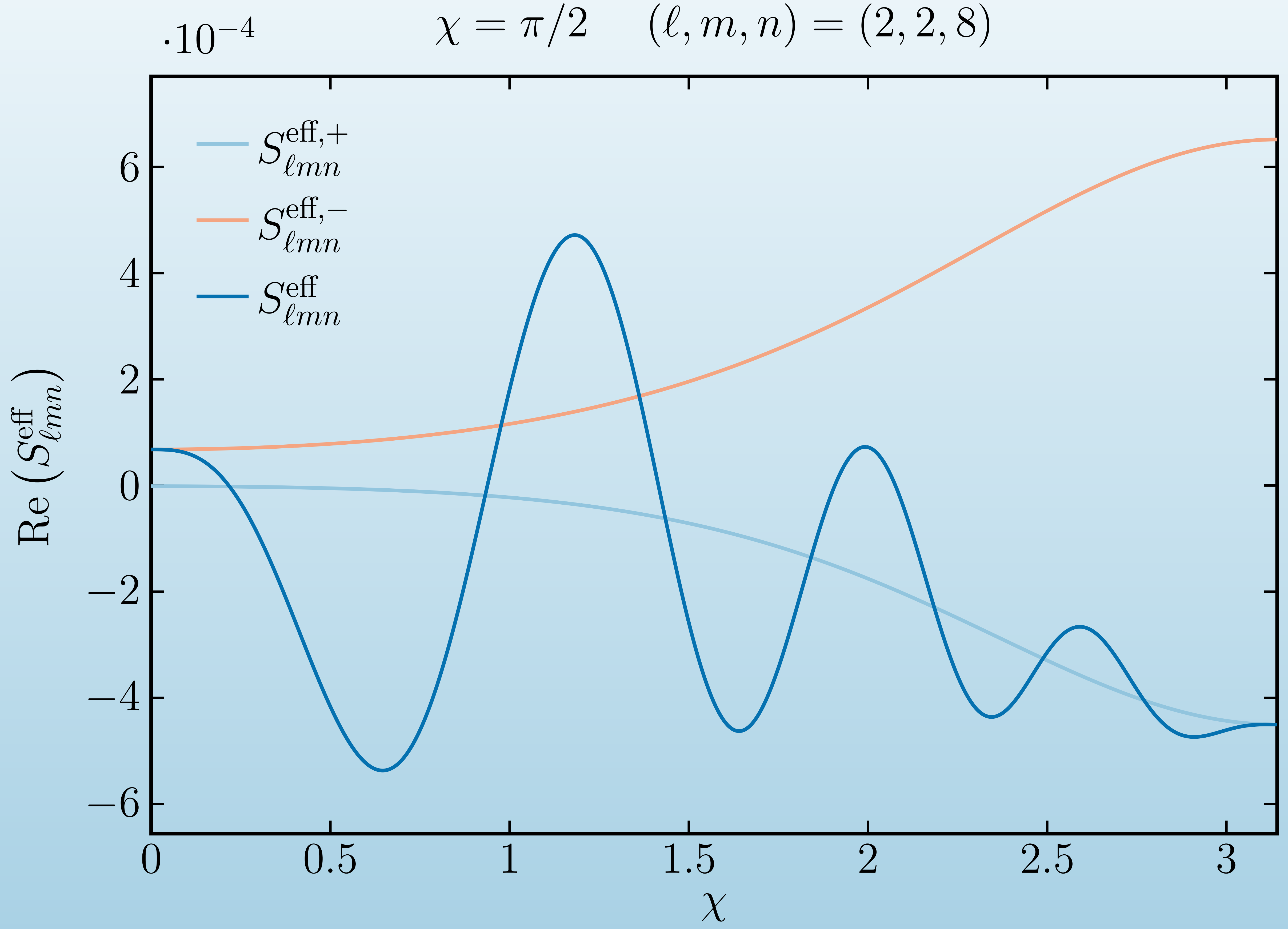
Fourier series of Heaviside function

$$\Theta^{\pm}(t,r) = \sum_{n=-\infty}^{\infty} b_n^{\pm}(r) e^{-in\Omega_r t}$$

$$b_0^{+}(r) = \frac{2t_p(r)}{T_r},$$

$$b_0^{-}(r) = 1 - \frac{2t_p(r)}{T_r},$$

$$b_n^{\pm}(r) = \pm\frac{1}{n\pi}\sin\left(\frac{2n\pi\, t_p(r)}{T_r}\right)$$

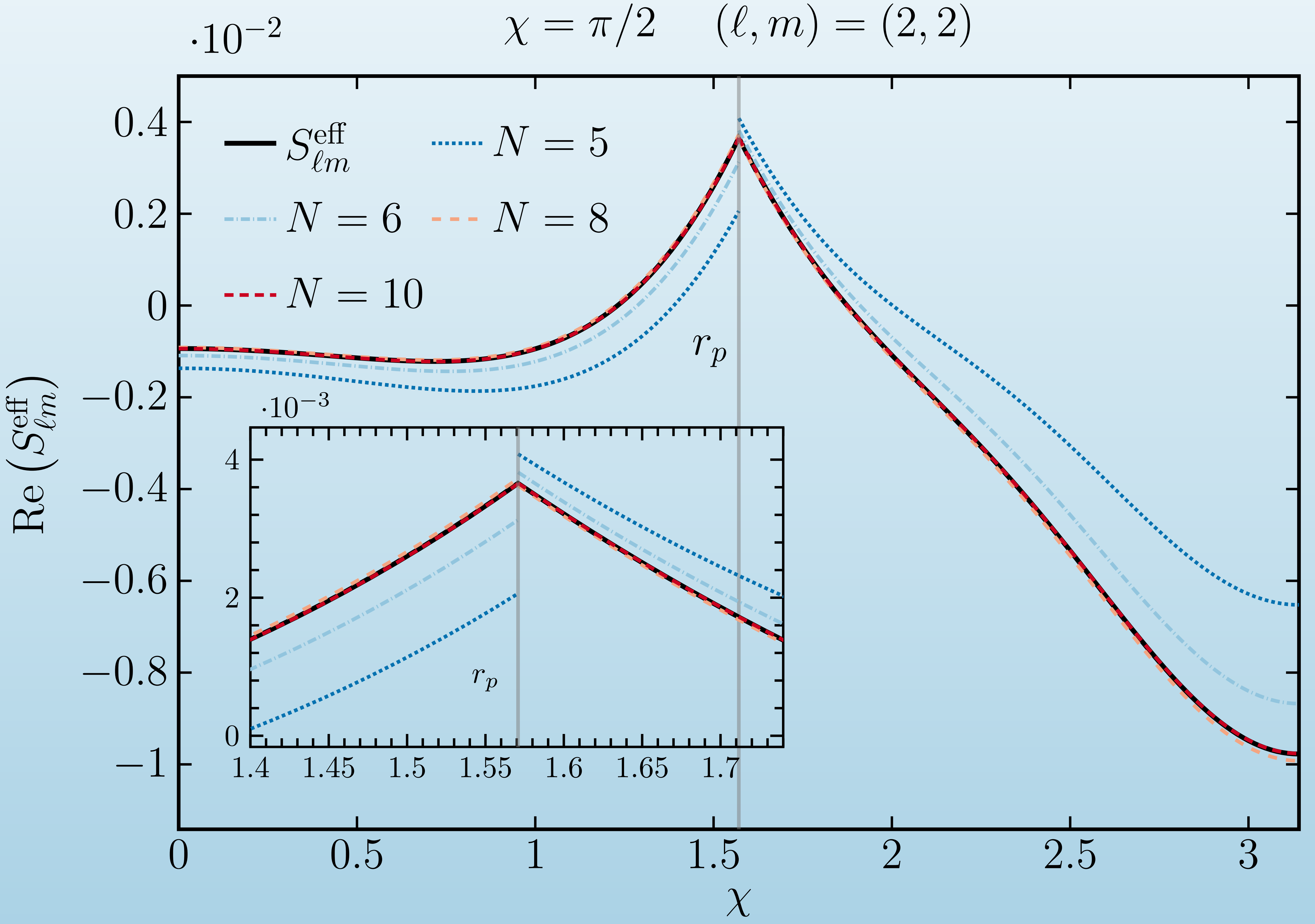


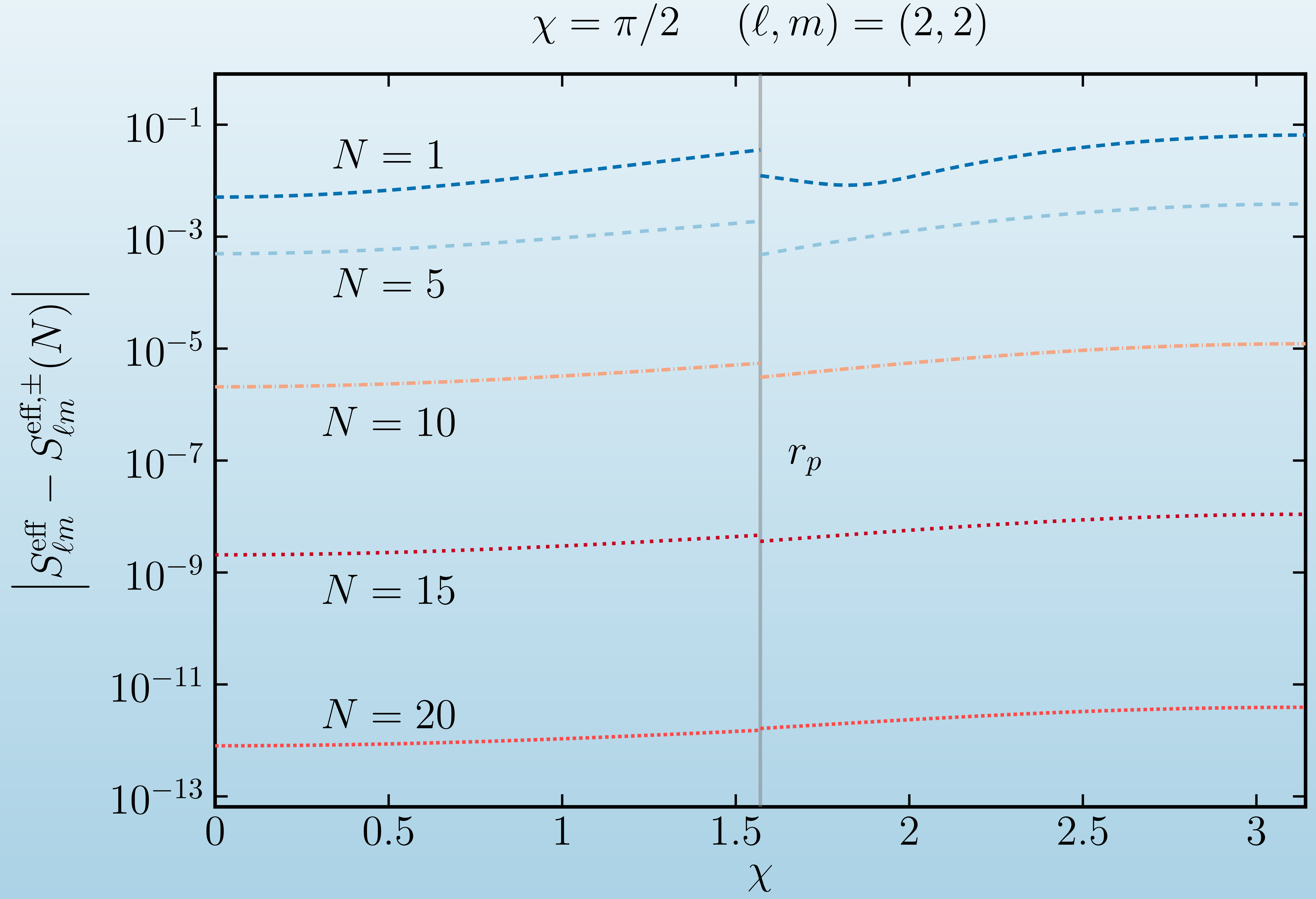
$$r \longrightarrow r_p(\chi)$$

$$S^{\mathrm{eff}}_{\ell mn}(r) = \sum_{n'=-\infty}^{\infty} \left[b^{+}_{n'-n}(r) S^{\mathrm{eff},+}_{\ell mn}(r) + b^{-}_{n'-n}(r) S^{\mathrm{eff},-}_{\ell mn}(r) \right]$$

Ensure spectral convergence

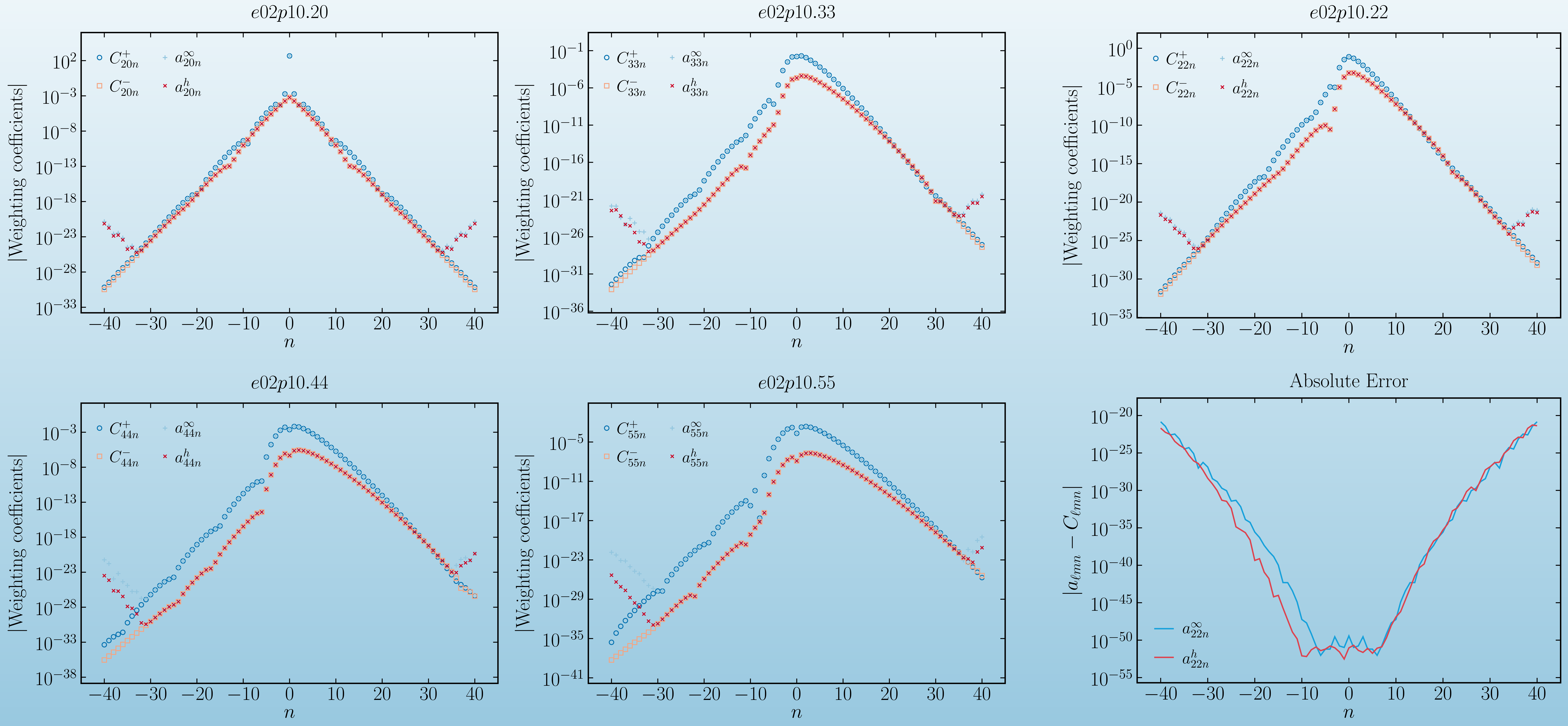
Fourier reconstruction of EES





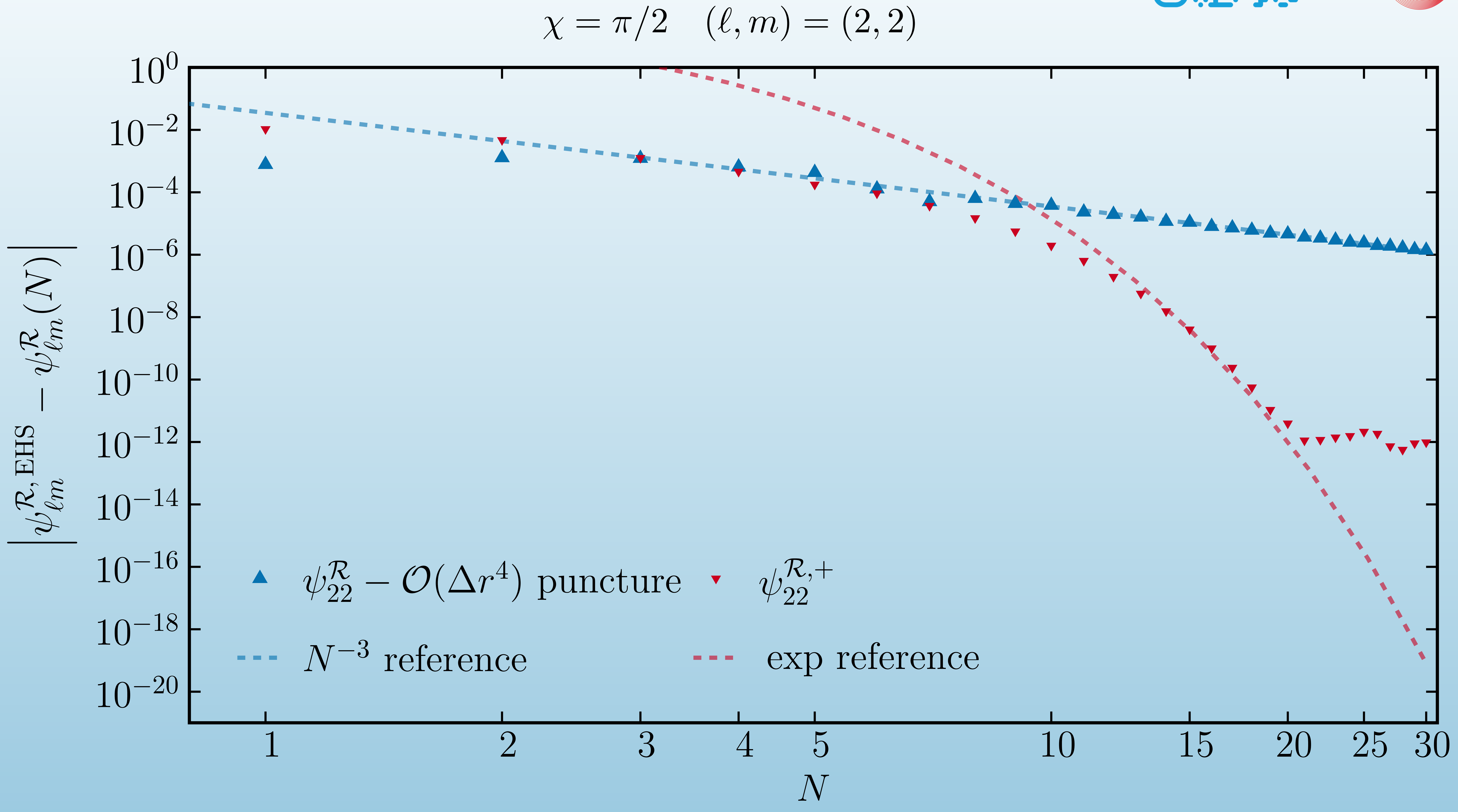
$$S^{\rm eff,\pm}_{\ell mn}(r) = \frac{1}{T_r}\int_0^{T_r} S^{\rm eff,\pm}_{\ell m}(t,r)e^{i\omega_{mn}t}dt \longrightarrow S^{\rm eff,\pm}_{\ell m}(t,r) = \sum_{n=-\infty}^{\infty} S^{\rm eff,\pm}_{\ell mn}(r)e^{-i\omega_{mn}t}$$

Results



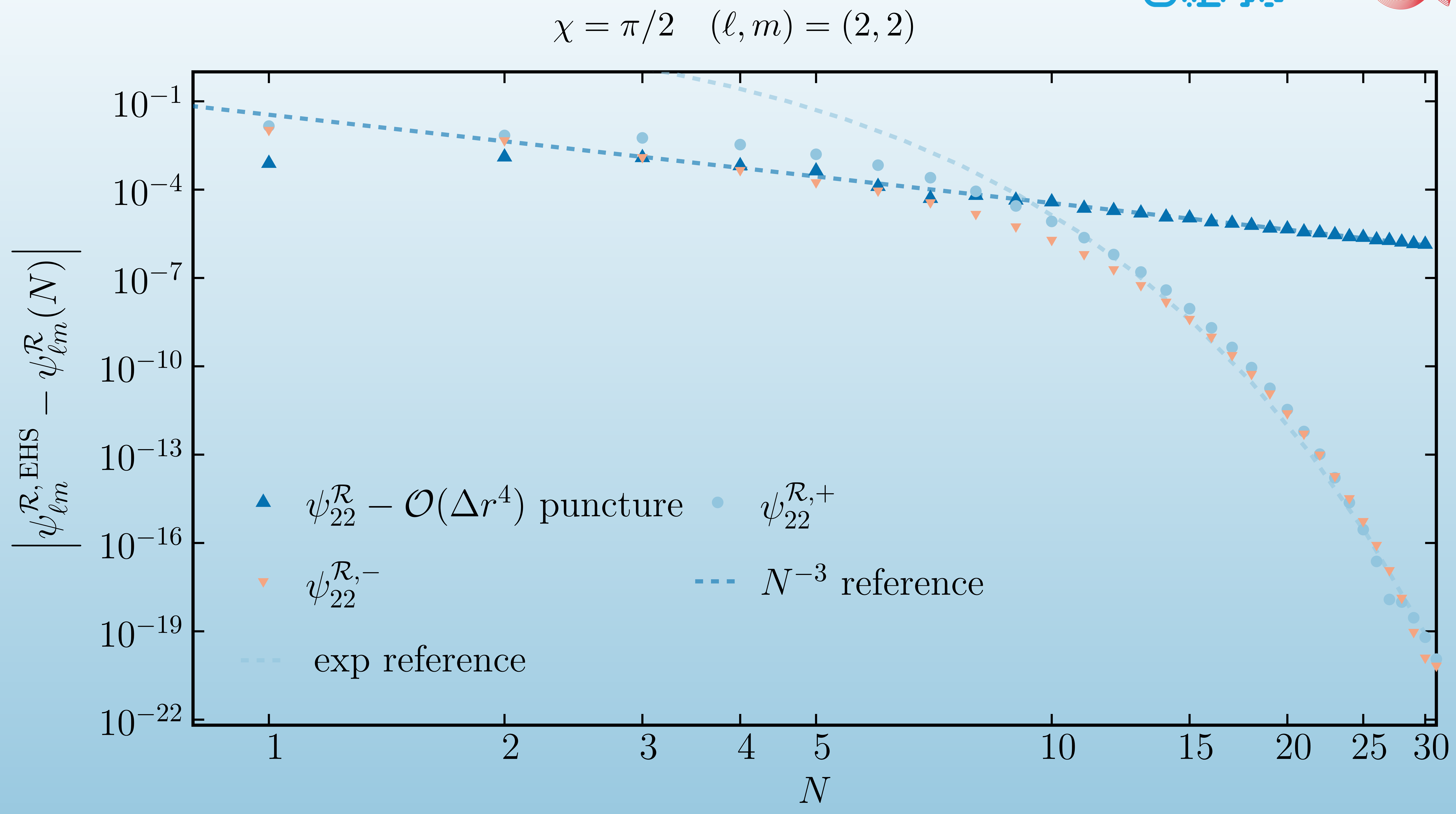
e02p10.20
e02p10.33
e02p10.22
e02p10.44
e02p10.55
Absolute Error
|Weighting coefficients|
n
C^+_{20n} C^-_{20n} a^∞_{20n} a^h_{20n}
C^+_{33n} C^-_{33n} a^∞_{33n} a^h_{33n}
C^+_{22n} C^-_{22n} a^∞_{22n} a^h_{22n}
C^+_{44n} C^-_{44n} a^∞_{44n} a^h_{44n}
C^+_{55n} C^-_{55n} a^∞_{55n} a^h_{55n}
$|a_{\ell mn} - C_{\ell mn}|$
a^∞_{22n}
a^h_{22n}

Results

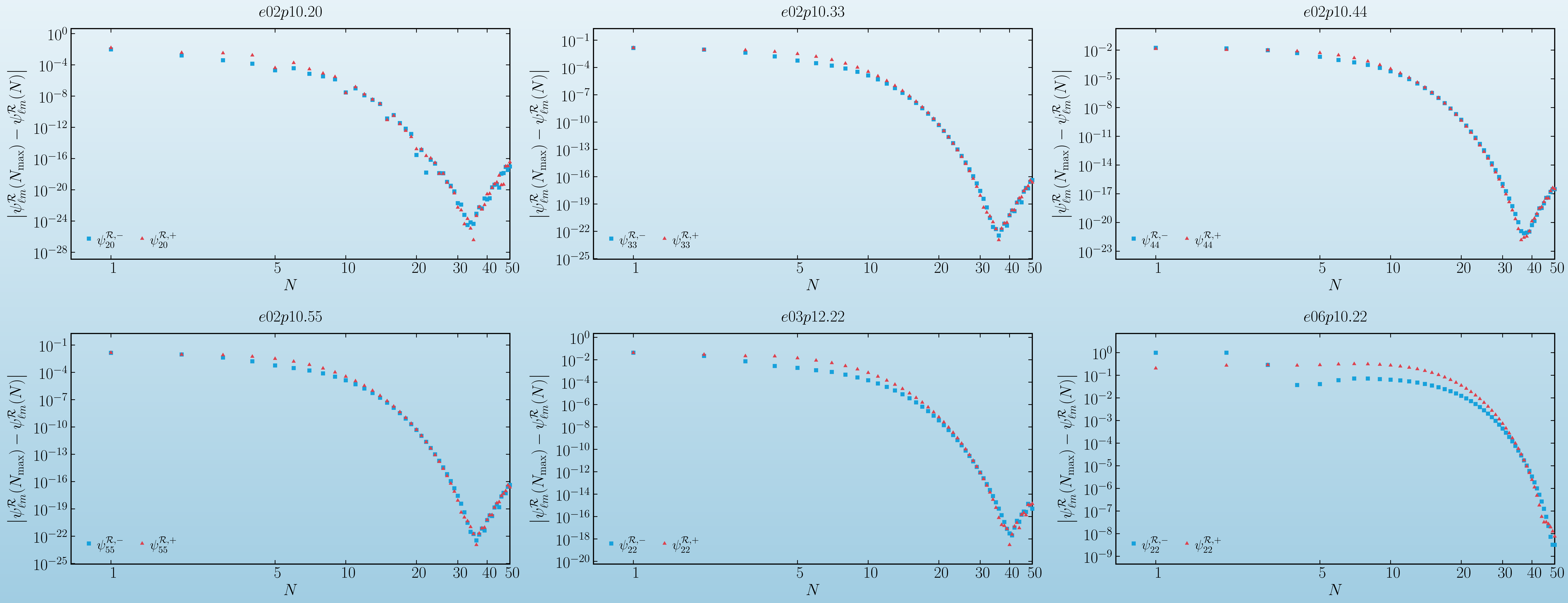


Fourier Transform by numerical integration

Results


$\chi = \pi/2 \quad (\ell, m) = (2, 2)$
$\left|\psi_{\ell m}^{\mathcal{R},\,\mathrm{EHS}} - \psi_{\ell m}^{\mathcal{R}}(N)\right|$
10^{-1}
10^{-4}
10^{-7}
10^{-10}
10^{-13}
10^{-16}
10^{-19}
10^{-22}
1
2
3
4
5
10
15
20
25
30
N
$\psi_{22}^{\mathcal{R}} - \mathcal{O}(\Delta r^4)$ puncture
$\psi_{22}^{\mathcal{R},+}$
$\psi_{22}^{\mathcal{R},-}$
N^{-3} reference
exp reference

Results

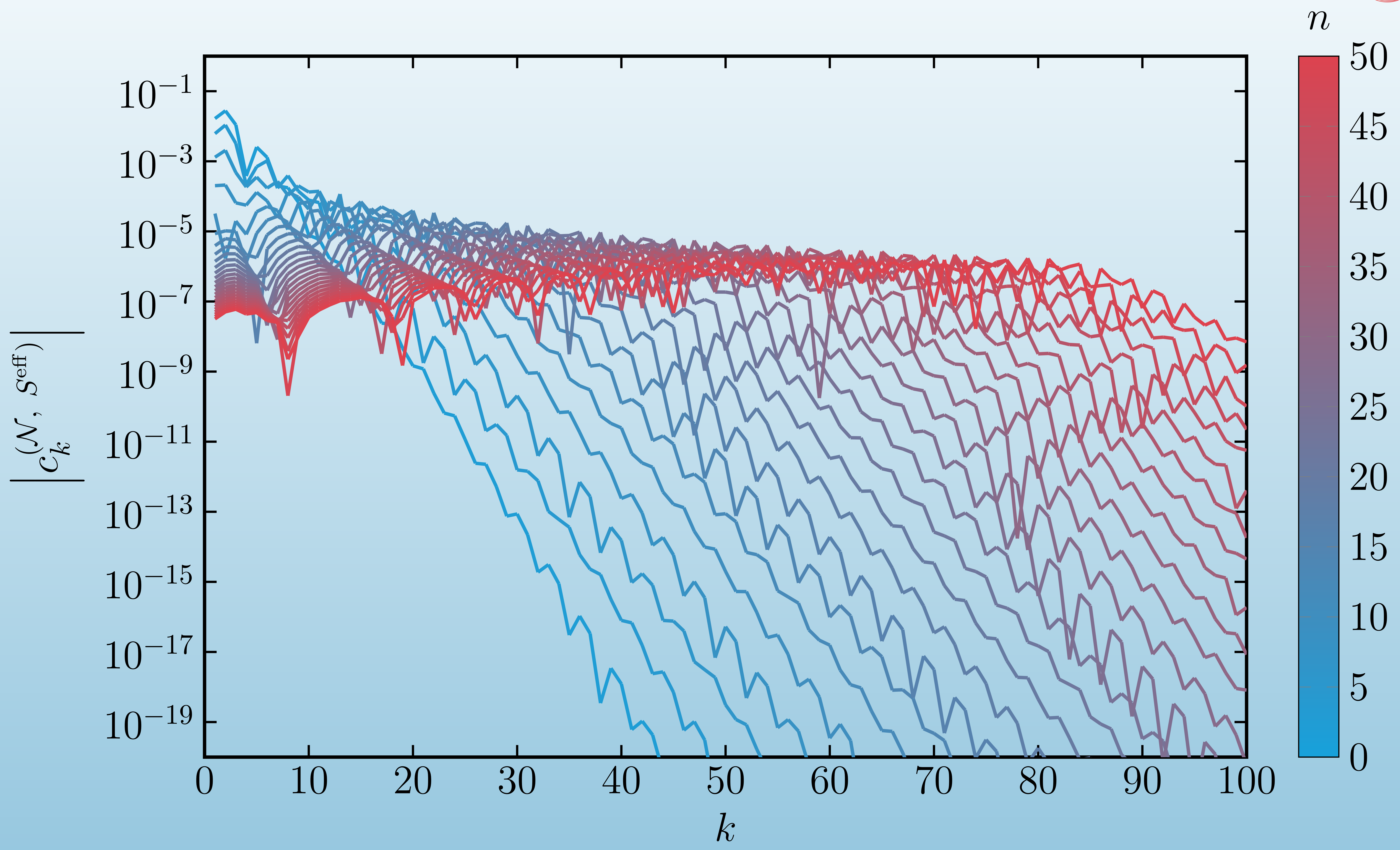


e02p10.20
e02p10.33
e02p10.44
e02p10.55
e03p12.22
e06p10.22
$|\psi^{\mathcal{R}}_{\ell m}(N_{\max}) - \psi^{\mathcal{R}}_{\ell m}(N)|$
N
$\psi^{\mathcal{R},-}_{20}$ $\psi^{\mathcal{R},+}_{20}$
$\psi^{\mathcal{R},-}_{33}$ $\psi^{\mathcal{R},+}_{33}$
$\psi^{\mathcal{R},-}_{44}$ $\psi^{\mathcal{R},+}_{44}$
$\psi^{\mathcal{R},-}_{55}$ $\psi^{\mathcal{R},+}_{55}$
$\psi^{\mathcal{R},-}_{22}$ $\psi^{\mathcal{R},+}_{22}$
$\psi^{\mathcal{R},-}_{22}$ $\psi^{\mathcal{R},+}_{22}$

Conclusion

- EES method overcomes Gibbs Phenomenon and associated slow convergence
- Example calculation computing residual scalar field for a compact source
- Validated results against extended homogeneous solutions
- Apply Worldtube method with spectral methods
- We can now consider eccentric second-order self-force calculations
- Eccentric Kerr…?

Appendix: Chebyshev Interpolation



Appendix: Standard Worldtube Method

We seek a solution of the form

$$\psi^{\mathcal{R}}_{\ell mn}(r) = \begin{cases} a^h_{lmn}\psi^h_{\ell mn}(r), & r \leq r_{\rm min}, \\ b^\infty_{lmn}\psi^\infty_{\ell mn}(r) + b^h_{lmn}\psi^h_{\ell mn}(r) + \psi^{\rm inh}_{\ell mn}(r), & r_{\rm min} < r < r_{\rm max}, \\ a^\infty_{lmn}\psi^\infty_{\ell mn}(r), & r \geq r_{\rm max}. \end{cases}$$

The particular inhomogeneous solution

$$\psi^{\rm inh}_{\ell mn}(r) = C^\infty_{lmn}(r)\psi^\infty_{\ell mn}(r) + C^h_{lmn}(r)\psi^h_{\ell mn}(r)$$

$$C^\infty_{\ell mn}(r) = \int_{r_{\rm min}}^{r} \frac{\psi^h_{\ell mn}(r')S^{\rm eff}_{\ell mn}(r')}{W[\psi^h_{\ell mn}(r'), \psi^\infty_{\ell mn}(r')]}dr'$$

$$C^h_{\ell mn}(r) = \int_{r}^{r_{\rm max}} \frac{\psi^\infty_{\ell mn}(r')S^{\rm eff}_{\ell mn}(r')}{W[\psi^h_{\ell mn}(r'), \psi^\infty_{\ell mn}(r')]}dr'$$

Demanding continuity at the worldtube boundaries

$$a^\infty_{lmn} = \frac{1}{\psi^\infty_{\ell mn}(r_{\rm max})}\{\psi^\infty_{\ell mn}(r_{\rm max})[b^\infty_{lmn} + C^\infty_{lmn}(r_{\rm max})] + b^h_{lmn}\psi^h_{\ell mn}(r_{\rm max}) + \psi^{\mathcal{P}}_{\ell mn}(r_{\rm max})\}$$

$$a^h_{lmn} = \frac{1}{\psi^h_{\ell mn}(r_{\rm min})}\{b^\infty_{lmn}\psi^\infty_{\ell mn}(r_{\rm min}) + \psi^h_{\ell mn}(r_{\rm min})[b^h_{lmn} + C^h_{lmn}(r_{\rm min})] + \psi^{\mathcal{P}}_{\ell mn}(r_{\rm min})\}$$

$$b^\infty = \left.\frac{W[\psi^{\mathcal{P}}_{\ell mn}(r), \psi^h_{\ell mn}(r)]}{W[\psi^h_{\ell mn}(r), \psi^\infty_{\ell mn}(r)]}\right|_{r=r_{\rm min}}$$

$$b^h = \left.\frac{W[\psi^{\mathcal{P}}_{\ell mn}(r), \psi^\infty_{\ell mn}(r)]}{W[\psi^\infty_{\ell mn}(r), \psi^h_{\ell mn}(r)]}\right|_{r=r_{\rm max}}$$

Appendix: EES Worldtube Method

We now seek a solution of the form

$$\psi^{\mathcal{R},-}_{lmn}(r) = \begin{cases} a^h_{lmn}\psi^h_{lmn}(r), & r \le r_{\min} \\ b^{\infty,-}_{lmn}\psi^\infty_{lmn}(r) + b^{h,-}_{lmn}\psi^h_{lmn}(r) + \psi^{\text{inh},-}_{lmn}(r), & r_{\min} < r < r_{\max} \end{cases}$$

$$\psi^{\mathcal{R},+}_{lmn}(r) = \begin{cases} b^{\infty,+}_{lmn}\psi^\infty_{lmn}(r) + b^{h,+}_{lmn}\psi^h_{lmn}(r) + \psi^{\text{inh},+}_{\ell mn}(r), & r_{\min} < r < r_{\max}, \\ a^\infty_{lmn}\psi^\infty_{lmn}(r), & r \ge r_{\max} \end{cases}$$

Particular solution constructed with EES

$$\psi^{\text{inh},\pm}_{\ell mn}(r) = C^{\infty,\pm}_{\ell mn}(r)\psi^\infty_{\ell mn}(r) + C^{h,\pm}_{\ell mn}(r)\psi^h_{\ell mn}(r)$$

$$C^{\infty,\pm}_{\ell mn}(r) = \int_{r_{\min}}^{r} \frac{\psi^h_{\ell mn}(r')S^{\text{eff},\pm}_{\ell mn}(r')}{W[\psi^h_{\ell mn}(r'), \psi^\infty_{\ell mn}(r')]}\,dr'$$

$$C^{h,\pm}_{\ell mn}(r) = \int_{r}^{r_{\max}} \frac{\psi^\infty_{\ell mn}(r')S^{\text{eff},\pm}_{\ell mn}(r')}{W[\psi^h_{\ell mn}(r'), \psi^\infty_{\ell mn}(r')]}\,dr'$$

Weighting coefficients

$$b^{\infty,+}_{\ell mn} = \left.\frac{W[\kappa^{\infty,+}_{\ell mn}(r)\psi^\infty_{\ell mn}(r) - \psi^{\mathcal{P},+}_{\ell mn}(r), \psi^h_{\ell mn}(r)]}{W[\psi^\infty_{\ell mn}(r), \psi^h_{\ell mn}(r)]}\right|_{r=r_{\max}}$$

$$b^{\infty,-}_{\ell mn} = \left.\frac{W[\kappa^{h,-}_{\ell mn}(r)\psi^h_{\ell mn}(r) - \psi^{\mathcal{P},-}_{\ell mn}(r), \psi^\infty_{\ell mn}(r)]}{W[\psi^h_{\ell mn}(r), \psi^\infty_{\ell mn}(r)]}\right|_{r=r_{\min}}$$

$$\kappa^{\infty,+}_{\ell mn}(r) := a^\infty_{\ell mn} - C^{\infty,+}_{\ell mn}(r)$$

$$\kappa^{h,-}_{\ell mn}(r) := a^h_{\ell mn} - C^{h,-}_{\ell mn}(r)$$

$$b^{h,+}_{\ell mn} = \left.\frac{W[\psi^{\mathcal{P},+}_{\ell mn}(r), \psi^\infty_{\ell mn}(r)]}{W[\psi^\infty_{\ell mn}(r), \psi^h_{\ell mn}(r)]}\right|_{r=r_{\max}}$$

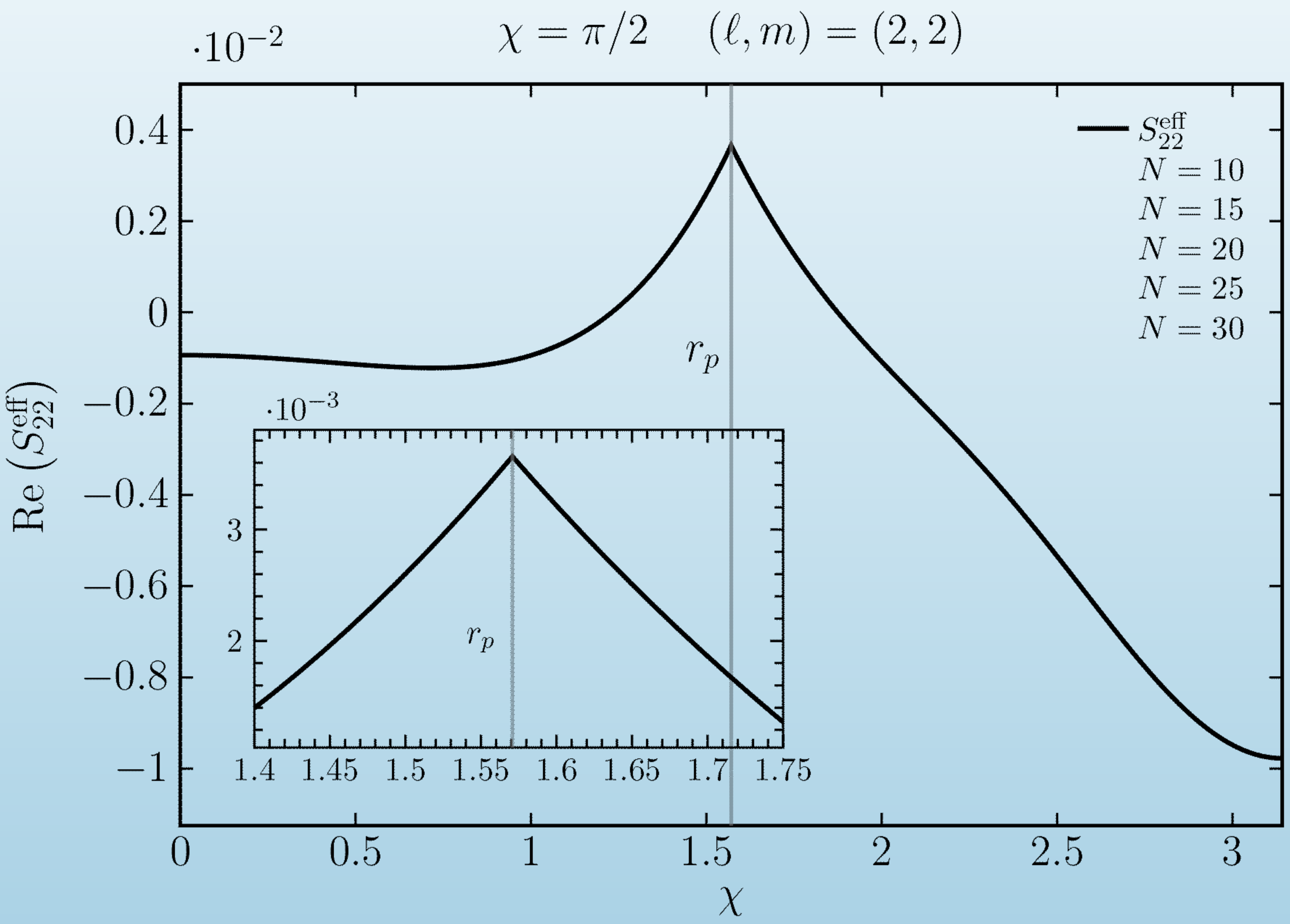
$$b^{h,-}_{\ell mn} = \left.\frac{W[\psi^{\mathcal{P},-}_{\ell mn}(r), \psi^h_{\ell mn}(r)]}{W[\psi^h_{\ell mn}(r), \psi^\infty_{\ell mn}(r)]}\right|_{r=r_{\min}}$$

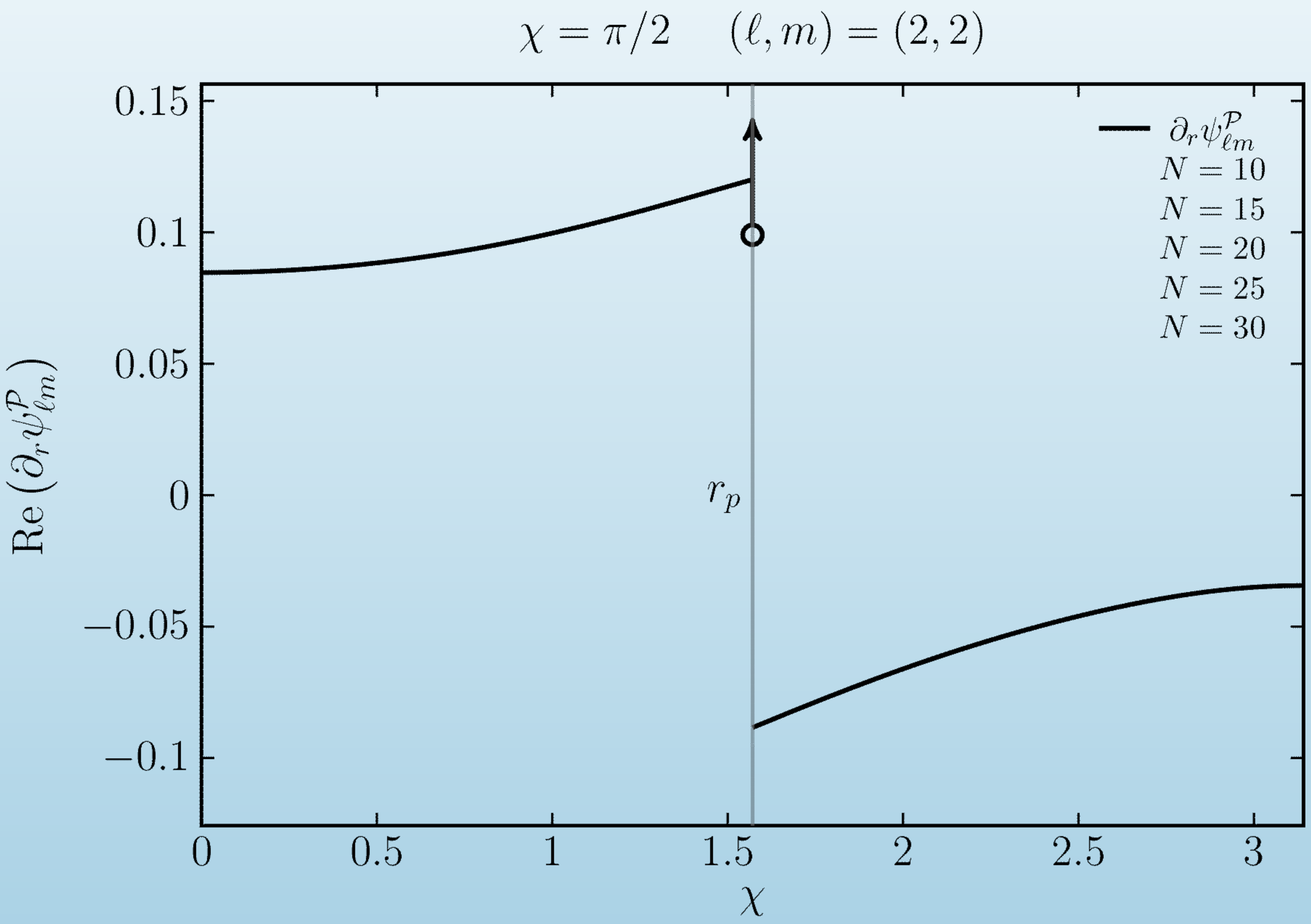
Time domain solution

$$\psi^{\mathcal{R},\pm}_{\ell m}(t,r) = \sum_{n=-\infty}^{\infty} \psi^{\mathcal{R},\pm}_{\ell mn}(r)e^{-i\omega_{mn}t}$$

$$\psi^{\mathcal{R}}_{\ell m}(t,r) = \psi^{\mathcal{R},+}_{\ell m}(t,r)\Theta^+(t,r) + \psi^{\mathcal{R},-}_{\ell m}(t,r)\Theta^-(t,r)$$

Appendix: Gibbs Phenomenon





$$S_{\ell mn}^{\text{eff}}(r) = \frac{1}{T_r}\int_0^{T_r} S_{\ell m}^{\text{eff}}(t,r) e^{-i\omega_{mn}t}\, dt \longrightarrow S_{\ell m}^{\text{eff}}(t,r) = \lim_{N\to\infty}\sum_{n=-N}^{N} S_{\ell mn}^{\text{eff}}(r) e^{-i\omega_{mn}t}$$

Appendix: High Eccentricity