Hyperboloidal 1+1 Teukolsky Time Domain Solver

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Outline

- Brief motivation benefits time symmetric solvers
- Writing 1+1 Teukolsky equation on hyperboloidal Kerr slices
- Symmetric numerical solvers for Teukolsky equation on hyperboloidal slices
- Testing code and results (Price tails, Aretakis Instability, Near extremal behaviour)
- Material covered in talk can be found in below paper.

C. Markakis et al., *Symmetric integration of the 1+1 Teukolsky equation on hyperboloidal foliations of Kerr spacetimes*, Comp. Phys. Comm., accepted [arXiv:2303.08153]

Motivation – Why consider time symmetric?

- Main drawbacks of explicit Runga-Kutta numerical methods typically used as time domain Teukolsky solvers are Courant limit and violate Noether symmetries.
- We explore using symmetric methods which conserve certain Noether charges and are not Courant limited.
- It is expected these features will be of relevance when simulating long-time EMRI’s expected to appear in the LISA band for 2-5 years.

Starting Point - Teukolsky Equation (1)

- 3+1 Teukolsky equation in Boyer-Lindquist coordinates given by

$$\left(\frac{(r^2+a^2)^2}{\Delta}-a^2\sin^2\theta\right)\partial_t^2\psi^{(s)}+\frac{4Mar}{\Delta}\partial_t\partial_\phi\psi^{(s)}+\left(\frac{a^2}{\Delta}-\frac{1}{\sin^2\theta}\right)\partial_\phi^2\psi^{(s)}$$
$$-\Delta^{-s}\partial_r(\Delta^{s+1}\partial_r\psi^{(s)})-\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta\psi^{(s)})+(s^2\cot^2\theta-s)\psi^{(s)}$$
$$-2s\left(\frac{M(r^2-a^2)}{\Delta}-r-\mathrm{i}a\cos\theta\right)\partial_t\psi^{(s)}-2s\left(\frac{a(r-M)}{\Delta}+\frac{\mathrm{i}\cos\theta}{\sin^2\theta}\right)\partial_\phi\psi^{(s)}=0$$

- End goal is to write this equation as a simple linear ODE which can be solved with standard methods

$$\frac{d\mathbf{u}}{d\tau}=\mathbf{L}\mathbf{u}$$

Starting Point - Teukolsky Equation (2)

- To convert to 1+1, expand the field in spherical harmonics

$$\psi^{(s)}(t,r,\theta,\phi) = (r\Delta)^s \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \varphi_{\ell m}^{(s)}(t,r)\, {}_sY_{\ell m}(\theta,\phi_\star)$$

$$\phi_\star = \phi + \int dr \frac{a}{\Delta} = \phi + \frac{a}{r_+ - r_-} \ln\left(\frac{r - r_+}{r - r_-}\right)$$

- And introduce tortoise radial coordinate

$$x = \int dr \frac{r^2 + a^2}{\Delta} = r - r_+ + \frac{2M}{r_+} \ln \frac{r - r_+}{r_+ - r_-} - \frac{2M}{r_-} \ln \frac{r - r_-}{r_+ - r_-}$$

Starting Point - 1+1 Teukolsky Equation

- Results in 1+1 Teukolsky equation

$$
\begin{aligned}
&(\partial_t^2 - \partial_x^2)\varphi_{\ell m}^{(s)} + T(r)\partial_t \varphi_{\ell m}^{(s)} + X(r)\partial_x \varphi_{\ell m}^{(s)} + W(r)\varphi_{\ell m}^{(s)} \\
&- K(r)[a^2\partial_t^2(C_{++}^{\ell}\varphi_{\ell+2,m}^{(s)} + C_{+}^{\ell}\varphi_{\ell+1,m}^{(s)} + C_0^{\ell m}\varphi_{\ell m}^{(s)} + C_{-}^{\ell}\varphi_{\ell-1,m}^{(s)} + C_{--}^{\ell}\varphi_{\ell-2,m}^{(s)}) \\
&- 8\mathrm{i}as\partial_t(c_{+}^{\ell}\varphi_{\ell+1,m}^{(s)} + c_0^{\ell}\varphi_{\ell,m}^{(s)} + c_{-}^{\ell}\varphi_{\ell-1,m}^{(s)})] = 0
\end{aligned}
$$

Barack & Giudice, Time-domain metric reconstruction for self-force applications, Phys. Rev. D 95, 104033 (2017)

Hyperboloidal Transformation – (1)

- We scale our tortoise coordinates to make the equation dimensionless

$$t_\star = \frac{t}{4M}, \quad r_\star = \frac{x}{4M}$$

- Introduce radial coordinate σ defined by

$$\sigma = \frac{2M}{r}$$

- We convert to hyperboloidal coordinates (τ, σ) using height function h and compactification function g as follows

$$t_\star = \tau - h(\sigma)$$
$$r_\star = g(\sigma)$$

Hyperboloidal Transformation (2)

- Compactification function g then follows from subbing our scaling and radial coordinate σ into the definition of the tortoise radial coordinate.

$$g(\sigma) = \int \frac{1+\sigma^2\chi^2}{2\sigma^2(\sigma(1-\sigma\chi^2)-1)}$$
$$= \frac{1}{2}\left[\frac{1}{\sigma} + \frac{1}{1-\kappa^2}\ln\left(\frac{1+\kappa^2}{\sigma}-1\right) - \frac{\kappa^2}{1-\kappa^2}\ln\left(\frac{1+\kappa^2}{\sigma}-\kappa^2\right)\right]$$

- We determine the minimal gauge height function by integrating outgoing null geodesics asymptotically and requiring that the level sets of the time coordinate become null surfaces near null infinity

$$h(\sigma) = g(\sigma) - \frac{1}{\sigma} + \ln\sigma + \sigma + \mathcal{O}(\sigma^2)$$

1+1 Teukolsky equation on hyperboloidal Kerr slices

- Performing coordinate transformation and imposing minimal gauge we obtain the remarkable simple hyperboloidal 1+1D Teukolsky equation:

$$
\begin{aligned}
&(1+\sigma)(1-\chi^2\sigma)\partial_\tau^2\varphi_\ell^{(s)} + (s(\sigma-1)-\chi^2\sigma(1+3\sigma)+2\sigma+\mathrm{i}m\chi(1+2\sigma))\partial_\tau\varphi_\ell^{(s)} \\
&-(1-\sigma^2(2-\chi^2(1+2\sigma)))\partial_\tau\partial_\sigma\varphi_\ell^{(s)} + \sigma(s(\sigma-2)-2+\sigma(3+2\mathrm{i}m\chi-4\chi^2\sigma))\partial_\sigma\varphi_\ell^{(s)} \\
&-\sigma^2(1-\sigma+\chi^2\sigma^2)\partial_\sigma^2\varphi_\ell^{(s)} + [\ell(\ell-s)(\ell+1+s)+(1+s+2\mathrm{i}m\chi)\sigma-2\chi^2\sigma^2]\varphi_\ell^{(s)} \\
&-\chi^2\partial_\tau^2(C_{--}^\ell\varphi_{\ell-2}^{(s)} + C_-^\ell\varphi_{\ell-1}^{(s)} + C_0^\ell\varphi_\ell^{(s)} + C_+^\ell\varphi_{\ell+1}^{(s)} + C_{++}^\ell\varphi_{\ell+2}^{(s)}) \\
&+\mathrm{i}s\chi\partial_\tau(c_-^\ell\varphi_{\ell-1}^{(s)} + c_0^\ell{\varphi_\ell}^{(s)} + c_+^\ell\varphi_{\ell+1}^{(s)}) = 0
\end{aligned}
$$

Matrix Form – First Order Time, Second Space

- Introducing the auxiliary variable π can write equation in matrix form as first order in time and second order in space

$$\sum_{k=\ell_{\min}}^{\ell_{\max}} \begin{pmatrix} \delta_{\ell k} & 0 \\ 0 & \mathrm{A}_{\ell k} \end{pmatrix} \partial_\tau \begin{pmatrix} \varphi_k^{(s)} \\ \pi_k^{(s)} \end{pmatrix} = \sum_{k=\ell_{\min}}^{\ell_{\max}} \begin{pmatrix} 0 & \delta_{\ell k} \\ \Delta_{\ell k}\partial_\sigma + \mathrm{E}_{\ell k}\partial_\sigma^2 + \mathrm{Z}_{\ell k} & \mathrm{B}_{\ell k} + \Gamma_{\ell k}\partial_\sigma \end{pmatrix} \begin{pmatrix} \varphi_k^{(s)} \\ \pi_k^{(s)} \end{pmatrix}$$

Method of Lines (1)

- Discretizing in space (using finite difference or pseudo-spectral grid)

$$\sum_{k=\ell_{\min}}^{\ell_{\max}} \sum_{j=0}^{N} \begin{pmatrix} \delta_{\ell k}^{ij} & 0 \\ 0 & \mathrm{A}_{\ell k}^{ij} \end{pmatrix} \frac{d}{d\tau} \begin{pmatrix} \varphi_k^j \\ \pi_k^j \end{pmatrix} = \sum_{k=\ell_{\min}}^{\ell_{\max}} \sum_{j=0}^{N} \begin{pmatrix} 0 & \delta_{\ell k}^{ij} \\ \mathrm{M}_{\ell k}^{ij} & \mathrm{N}_{\ell k}^{ij} \end{pmatrix} \begin{pmatrix} \varphi_k^j \\ \pi_k^j \end{pmatrix}$$

- With 4-dimensional arrays given by

$$\mathrm{A}_{\ell k}^{ij} := \delta_{ij} \mathrm{A}_{\ell k}(\sigma_i)$$

$$\mathrm{M}_{\ell k}^{ij} := \Delta_{\ell k}(\sigma_i) D_{ij} + \mathrm{E}_{\ell k}(\sigma_i) D_{ij}^{(2)} + \delta_{ij} \mathrm{Z}_{\ell k}(\sigma_i)$$

$$\mathrm{N}_{\ell k}^{ij} := \mathrm{B}_{\ell k}(\sigma_i) \delta_{ij} + \Gamma_{\ell k}(\sigma_i) D_{ij}$$

- And rectangular matrix

$$U_{i\ell}(\tau) := u_\ell(\tau, \sigma_i) = \begin{pmatrix} \varphi_\ell(\tau, \sigma_i) \\ \pi_\ell(\tau, \sigma_{i-N+1}) \end{pmatrix} = \begin{pmatrix} \varphi_\ell^i(\tau) \\ \pi_\ell^{i-N+1}(\tau) \end{pmatrix}, \quad i = 0, 1, \ldots, 2N+2$$

Method of Lines (2)

- Rectangular matrix can be flattened to column vector and rank 4 arrays matricized

$$\mathbf{U} = \begin{pmatrix} U_{00} & U_{10} \\ U_{01} & U_{11} \\ U_{02} & U_{12} \\ U_{03} & U_{13} \\ U_{04} & U_{14} \\ U_{05} & U_{15} \end{pmatrix} \rightarrow \mathbf{u} = \begin{pmatrix} U_{00} \\ U_{01} \\ U_{02} \\ U_{10} \\ U_{11} \\ U_{12} \\ U_{03} \\ U_{04} \\ U_{05} \\ U_{13} \\ U_{14} \\ U_{15} \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} \begin{pmatrix} A^{00}_{00} & A^{00}_{01} \\ A^{00}_{10} & A^{00}_{11} \end{pmatrix} & \begin{pmatrix} A^{01}_{00} & A^{01}_{01} \\ A^{01}_{10} & A^{01}_{11} \end{pmatrix} & \begin{pmatrix} A^{02}_{00} & A^{02}_{01} \\ A^{02}_{10} & A^{02}_{11} \end{pmatrix} \\ \begin{pmatrix} A^{10}_{00} & A^{10}_{01} \\ A^{10}_{10} & A^{10}_{11} \end{pmatrix} & \begin{pmatrix} A^{11}_{00} & A^{11}_{01} \\ A^{11}_{10} & A^{11}_{11} \end{pmatrix} & \begin{pmatrix} A^{12}_{00} & A^{12}_{01} \\ A^{12}_{10} & A^{12}_{11} \end{pmatrix} \\ \begin{pmatrix} A^{20}_{00} & A^{20}_{01} \\ A^{20}_{10} & A^{20}_{11} \end{pmatrix} & \begin{pmatrix} A^{21}_{00} & A^{21}_{01} \\ A^{21}_{10} & A^{21}_{11} \end{pmatrix} & \begin{pmatrix} A^{22}_{00} & A^{22}_{01} \\ A^{22}_{10} & A^{22}_{11} \end{pmatrix} \end{pmatrix} \rightarrow$$

$$\mathbb{A} = \begin{pmatrix} A^{00}_{00} & A^{00}_{01} & A^{01}_{00} & A^{01}_{01} & A^{02}_{00} & A^{02}_{01} \\ A^{00}_{10} & A^{00}_{11} & A^{01}_{10} & A^{01}_{11} & A^{02}_{10} & A^{02}_{11} \\ A^{10}_{00} & A^{10}_{01} & A^{11}_{00} & A^{11}_{01} & A^{12}_{00} & A^{12}_{01} \\ A^{10}_{10} & A^{10}_{11} & A^{11}_{10} & A^{11}_{11} & A^{12}_{10} & A^{12}_{11} \\ A^{20}_{00} & A^{20}_{01} & A^{21}_{00} & A^{21}_{01} & A^{22}_{00} & A^{22}_{01} \\ A^{20}_{10} & A^{20}_{11} & A^{21}_{10} & A^{21}_{11} & A^{22}_{10} & A^{22}_{11} \end{pmatrix}$$

Final System of ODEs

- We can now write the system in the form where $\boldsymbol{I}, \boldsymbol{A}, \mathrm{M}, \mathrm{N}$ are square matrices and u is a column vector

$$\begin{pmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{A} \end{pmatrix} \frac{d\mathbf{u}}{d\tau} = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{M} & \mathbb{N} \end{pmatrix} \mathbf{u}$$

- Because the matrix $\mathbf{A}$ is now square and well-conditioned, we can typically compute an inverse for it, which now allows us to write the final system of ODEs to solve in the form

$$\frac{d\mathbf{u}}{d\tau} = \mathbf{L}\mathbf{u} \qquad \mathbf{L} = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{A}^{-1}\mathbb{M} & \mathbb{A}^{-1}\mathbb{N} \end{pmatrix}$$

Solving Numerically

- We can exactly go from $\mathbf{U}(t_n)$ to $\mathbf{U}(t_{n+1})$ using

$$\mathbf{U}(t_{n+1}) = e^{\mathbf{L}\Delta\mathrm{t}}\mathbf{U}(t_n)$$

- If we then approximate $e^{\mathbf{L}\Delta\mathrm{t}}$ with a one-point Taylor expansion, we recover Runga-Kutta methods

$$e^{\mathbf{L}\,\Delta\mathrm{t}} \approx \mathbf{I} + \mathbf{L}\,\Delta\mathrm{t} + \frac{1}{2!}(\mathbf{L}\,\Delta\mathrm{t})^2 + \ldots + \frac{1}{n!}(\mathbf{L}\,\Delta\mathrm{t})^n$$

- While a two-point Taylor expansion up to the second term, equivalent to the Hermite integration method or a 2nd order Pade expansion, yields the time-symmetric approximant

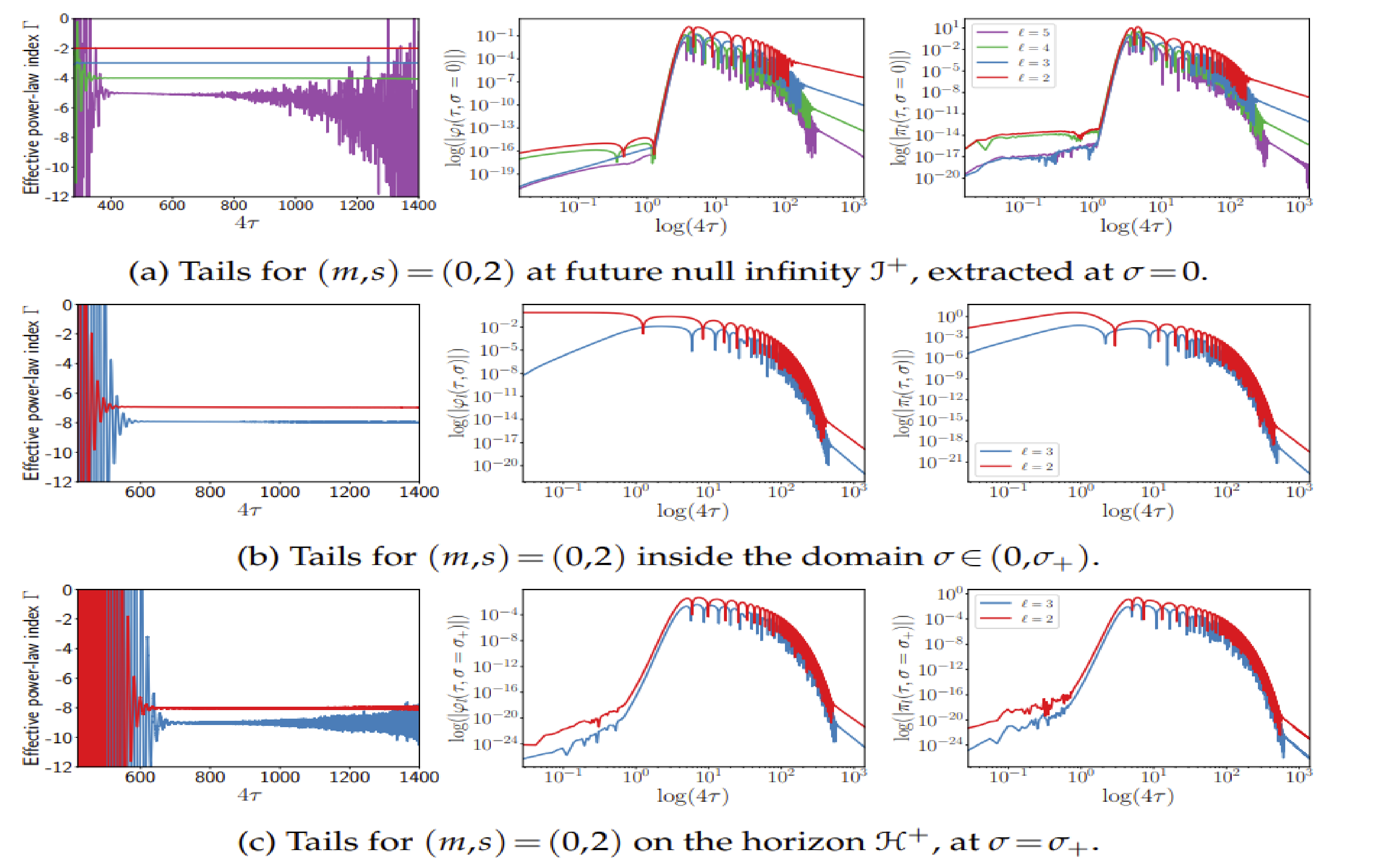
$$e^{\mathbf{L}\,\Delta\mathrm{t}} \approx \left(\mathbf{I} - \frac{\mathbf{L}\,\Delta\mathrm{t}}{2} + \frac{1}{12}(\mathbf{L}\,\Delta\mathrm{t})^2\right)^{-1}\left(\mathbf{I} + \frac{\mathbf{L}\,\Delta\mathrm{t}}{2} + \frac{1}{12}(\mathbf{L}\,\Delta\mathrm{t})^2\right)$$

Can calculate Price Tails until field decays below machine precision

- Price tails are a good check to see that our code can compute accurately until late times.

- Tails are calculated with the below formula

$$\Gamma_\ell = \tau \frac{\Re(\varphi_\ell)\,\Re(\pi_\ell) + \Im(\varphi_\ell)\,\Im(\pi_\ell)}{\Re(\varphi_\ell)^2 + \Im(\varphi_\ell)^2}$$



(a) Tails for $(m,s) = (0,2)$ at future null infinity $\mathcal{I}^+$, extracted at $\sigma = 0$.

(b) Tails for $(m,s) = (0,2)$ inside the domain $\sigma \in (0, \sigma_+)$.

(c) Tails for $(m,s) = (0,2)$ on the horizon $\mathcal{H}^+$, at $\sigma = \sigma_+$.

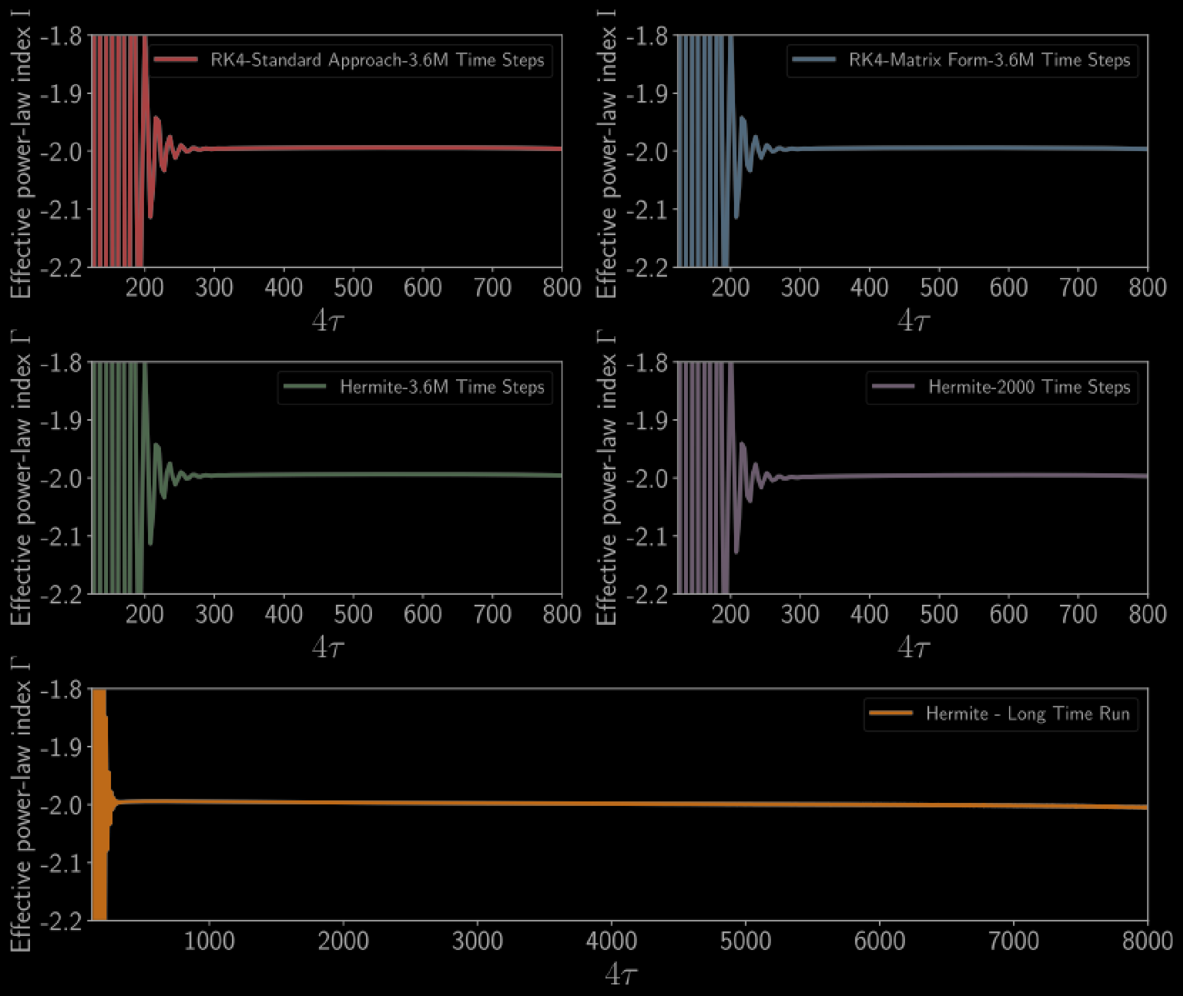
Comparison of Runga-Kutta & Hermite Solvers

- Compare Runga-Kutta and Hermite for computing Price tail for first mode for (m,s)=(0,2). (Run 1-4)

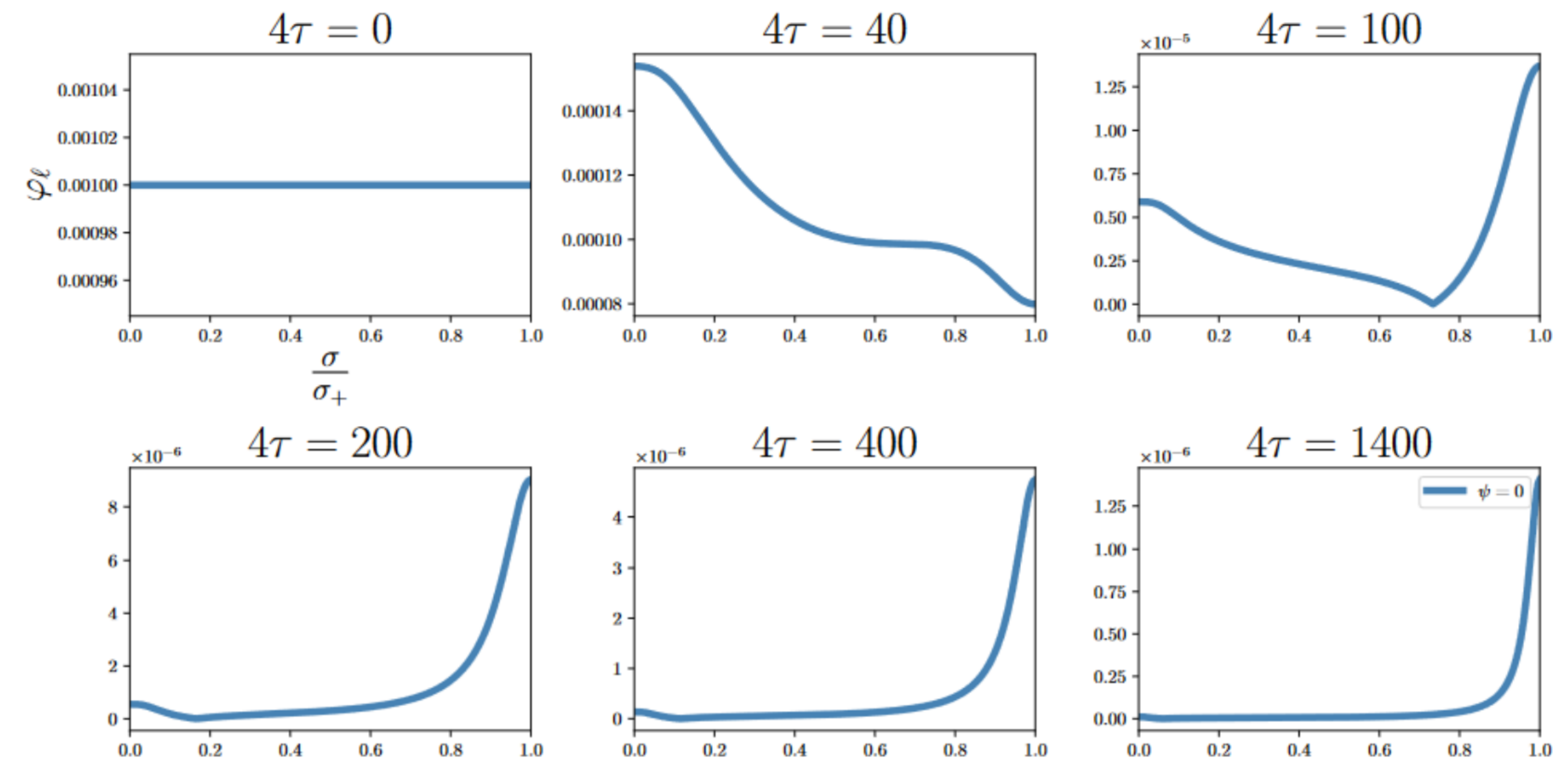
- Additionally tested accuracy of Hermite approach for long time run (Run 5).

- Comparison each run:

	Numerical Method	4τ	Spatial Grid Size	Numer Time Steps	Run Time - Seconds
Run 1	Runga-Kutta Standard	800	220	3,600,000	1758
Run 2	Runga-Kutta Matrix Form	800	220	3,600,000	883
Run 3	Hermite	800	220	3,600,000	259
Run 4	Hermite	800	220	2,000	0.15
Run 5	Hermite	8,000	770	200,000	399



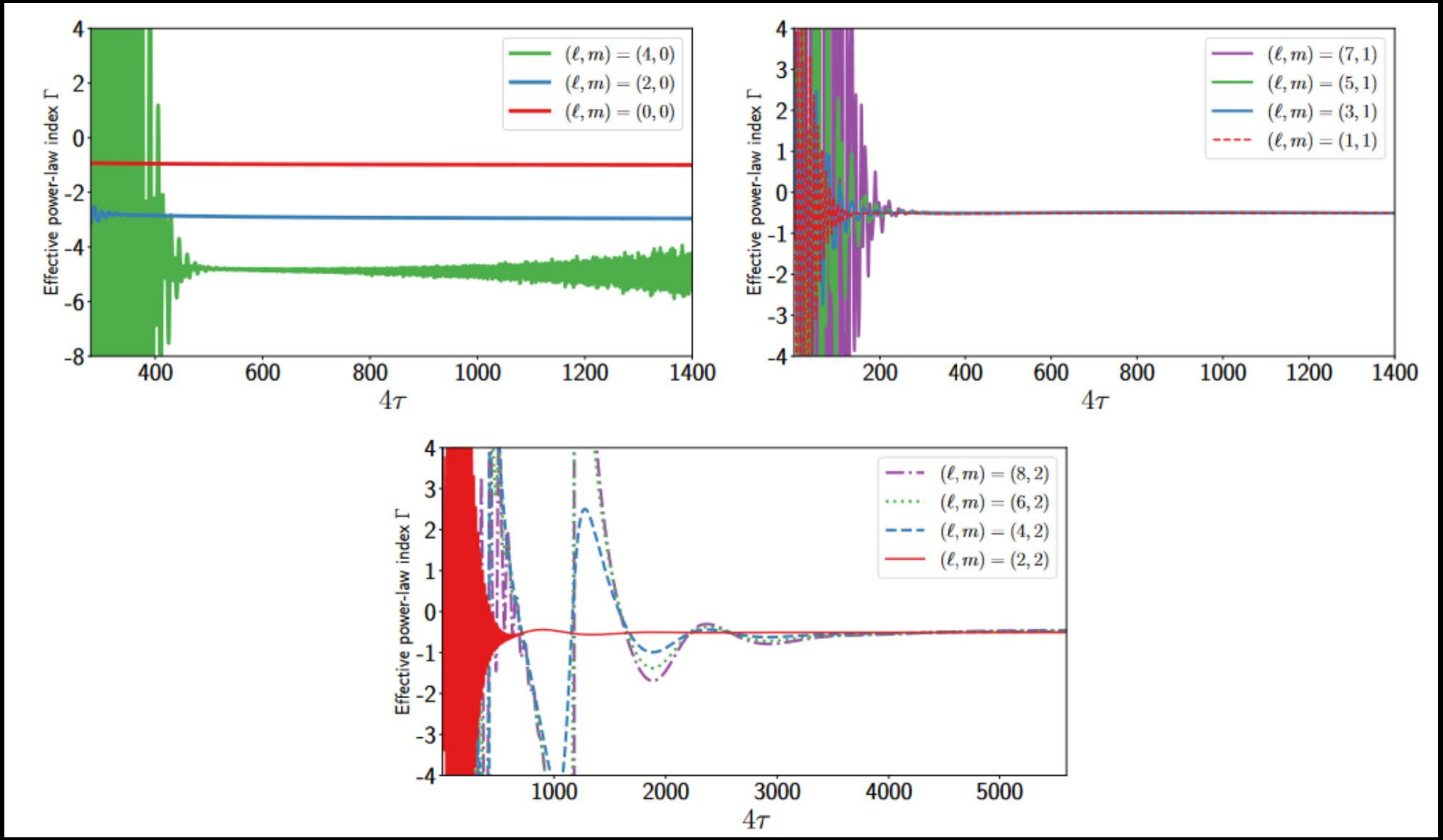
Aretakis' horizon instability (extremal *a*=*M* Kerr)



Sharp gradients form near the horizon.

Non-degenerate energy of ϕ_{lm} generically **concentrates** on the horizon at $\phi_{lm} \to \infty$!

Aretakis' horizon instability (extremal *a*=*M* Kerr)



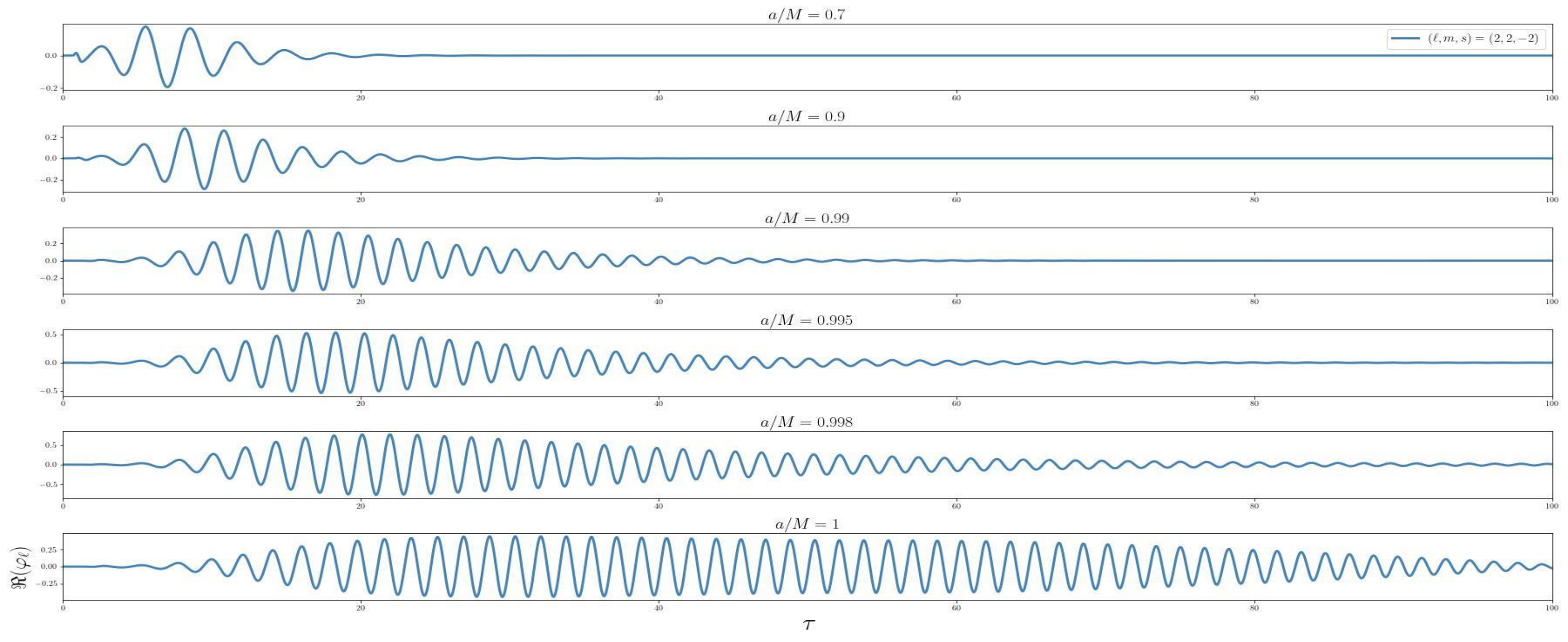
$$\psi_{lm} \sim \tau^{-1}, \;\; m = 0$$

$$\psi_{lm} \sim \tau^{-1/2}, \;\; m \neq 0$$

Ringing of near-extremal $|a|\to M$ Kerr at Scri+

- When Kerr parameter is near extremality decay of the field is much slower: $\psi_m \sim \tau^{-1}\sum_l e^{i(m\omega_+\tau+\sqrt{-a_{lm}}\ln\tau)}$
- Further work is to investigate observational signature in LIGO O4 data

$$\text{vs. } \psi_m \sim \tau^{-2-m}, \ |a| < M$$



N. A. Andersson & K. Glampedakis, *A superradiance resonance cavity outside rapidly rotating black holes*, PRL 84 4537-4540, 2000
K. Glampedakis & N. A. Andersson, *Late-time dynamics of rapidly rotating black holes*, Phys.Rev. D64, 104021, 2001

Summary

- We have written the 1+1D Teukolsky equation in a convenient form in hyperboloidal coordinates
- Converted the system into the form of a linear ODE which easily solvable
- Computed Price tails, calculated Aretakis instability and near extremal behaviour.
- Next steps incorporate particle & consider second order.