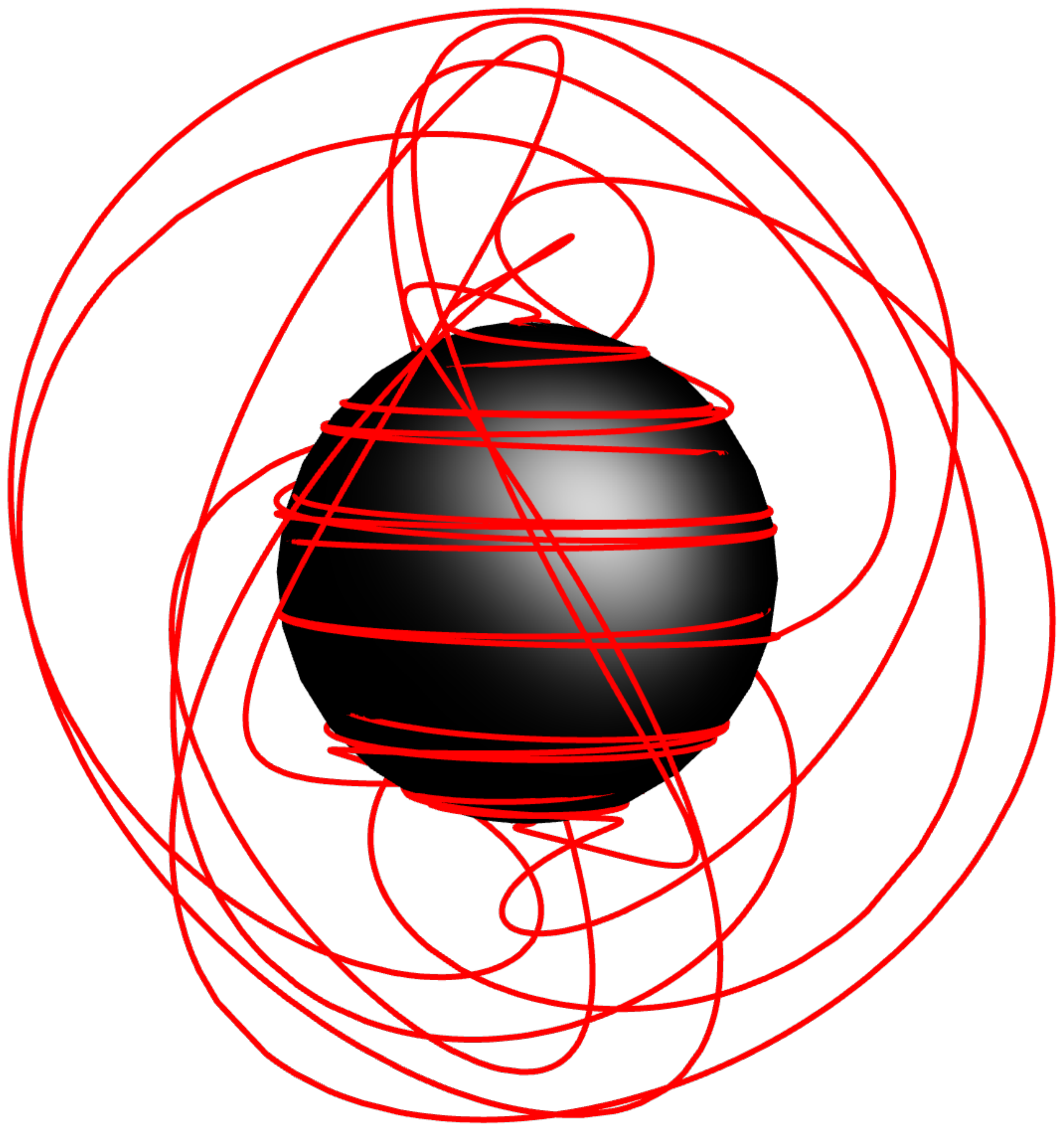
Analytic Solutions to Plunging Geodesics in Kerr



Conor Dyson

in collaboration with Maarten van de Meent [arxiv:2302.03704]

strong. niels bohr institute

KØBENHAVNS UNIVERSITET

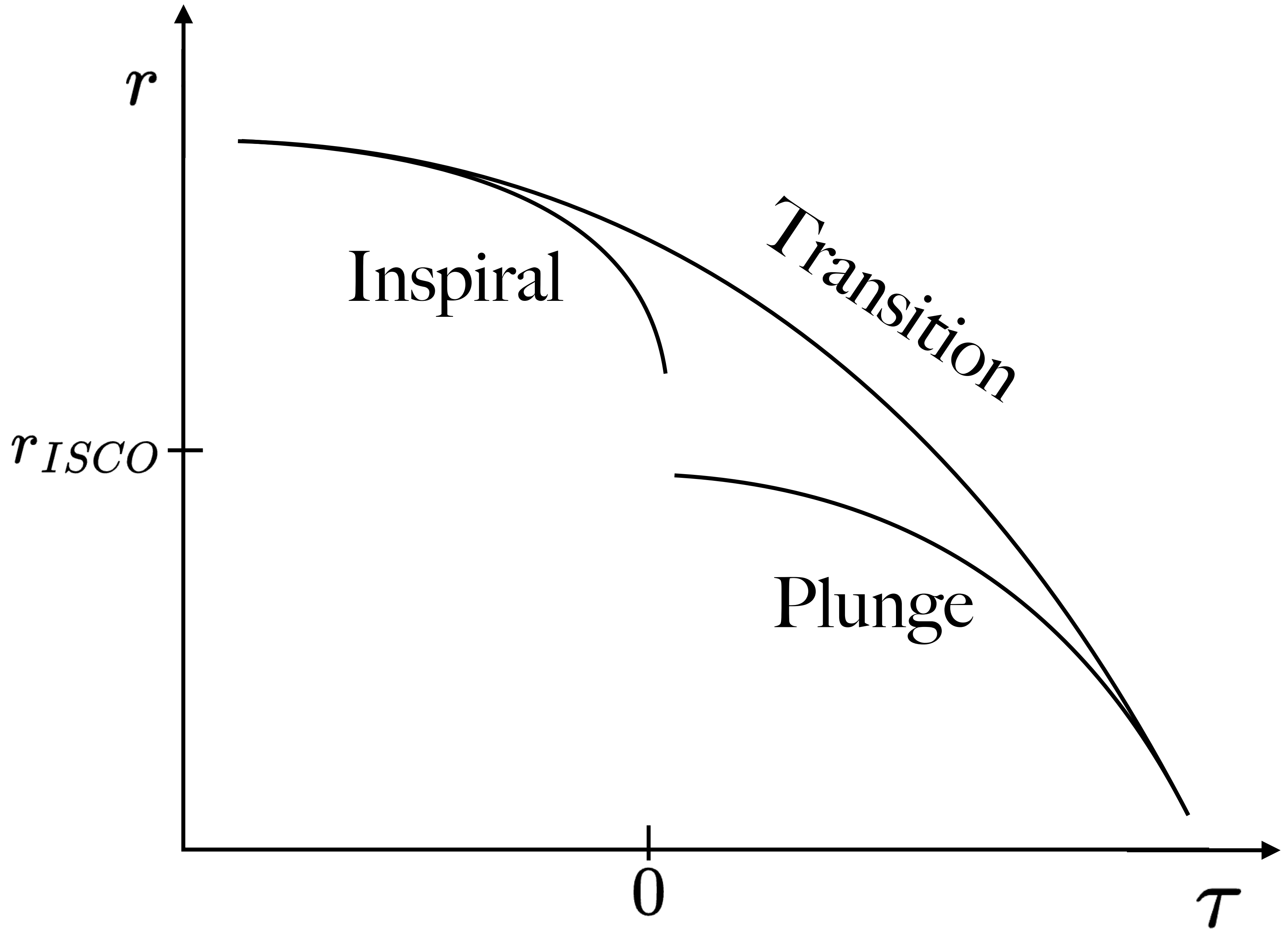
Danmarks Grundforskningsfond Danish National Research Foundation

VILLUM FONDEN

erc European Research Council Established by the European Commission

The Plunge in Self-Force

- Geodesic motion is the 0th order motion in the self-force expansion
- Transition to Plunge smoothly connects inspiral to a plunging geodesic
- For IMRI/EMRIs plunging geodesic sources the ringdown



Adapted from [Ori, Thorn: arXiv:gr-qc/0003032]

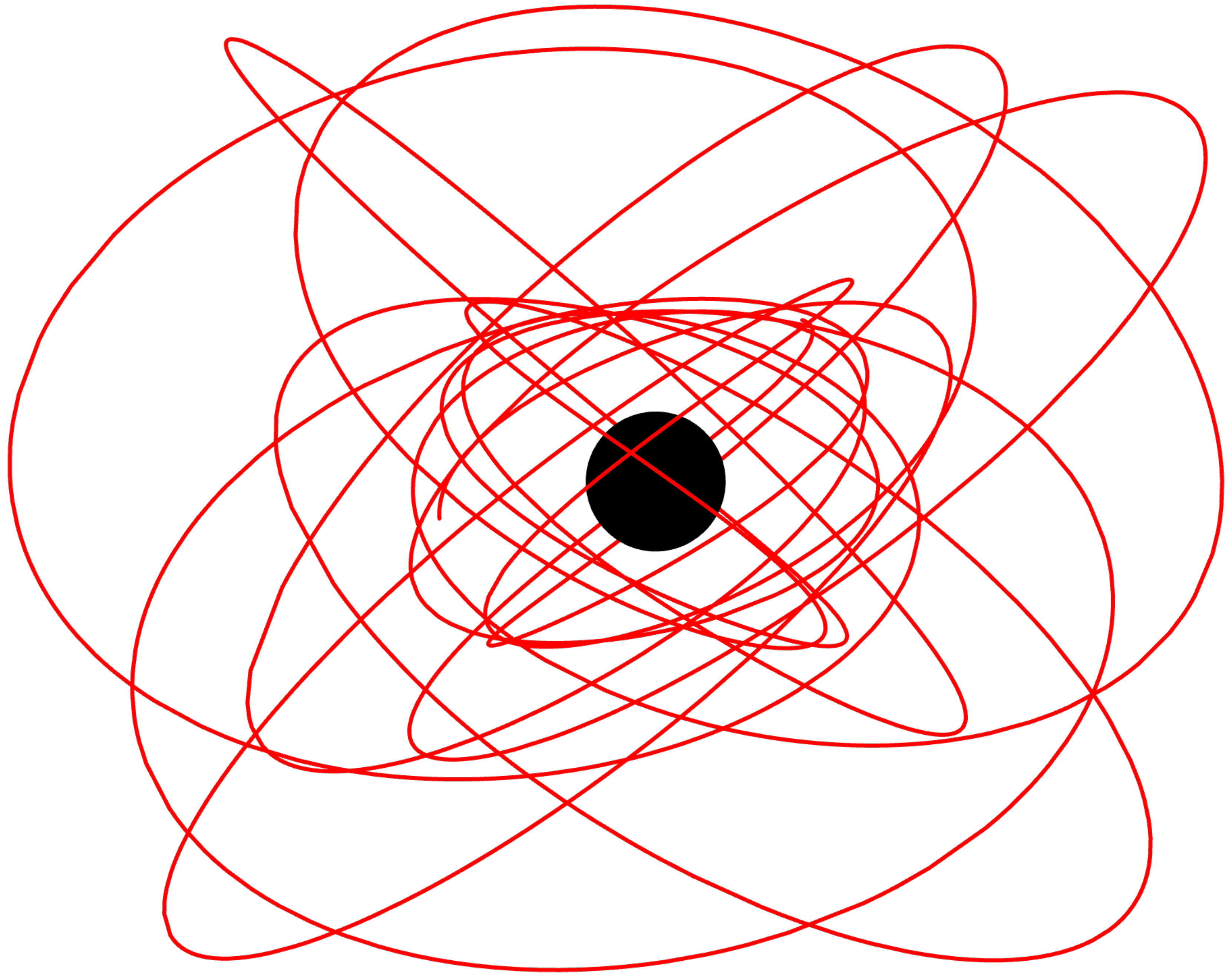
Previous Work on Analytic Solutions

- Extensive history of solutions to special cases, Chandrasekhar, Wilkins etc…

- Introduction of Mino time, $d\tau = \Sigma d\lambda$
  [Mino: 03020]

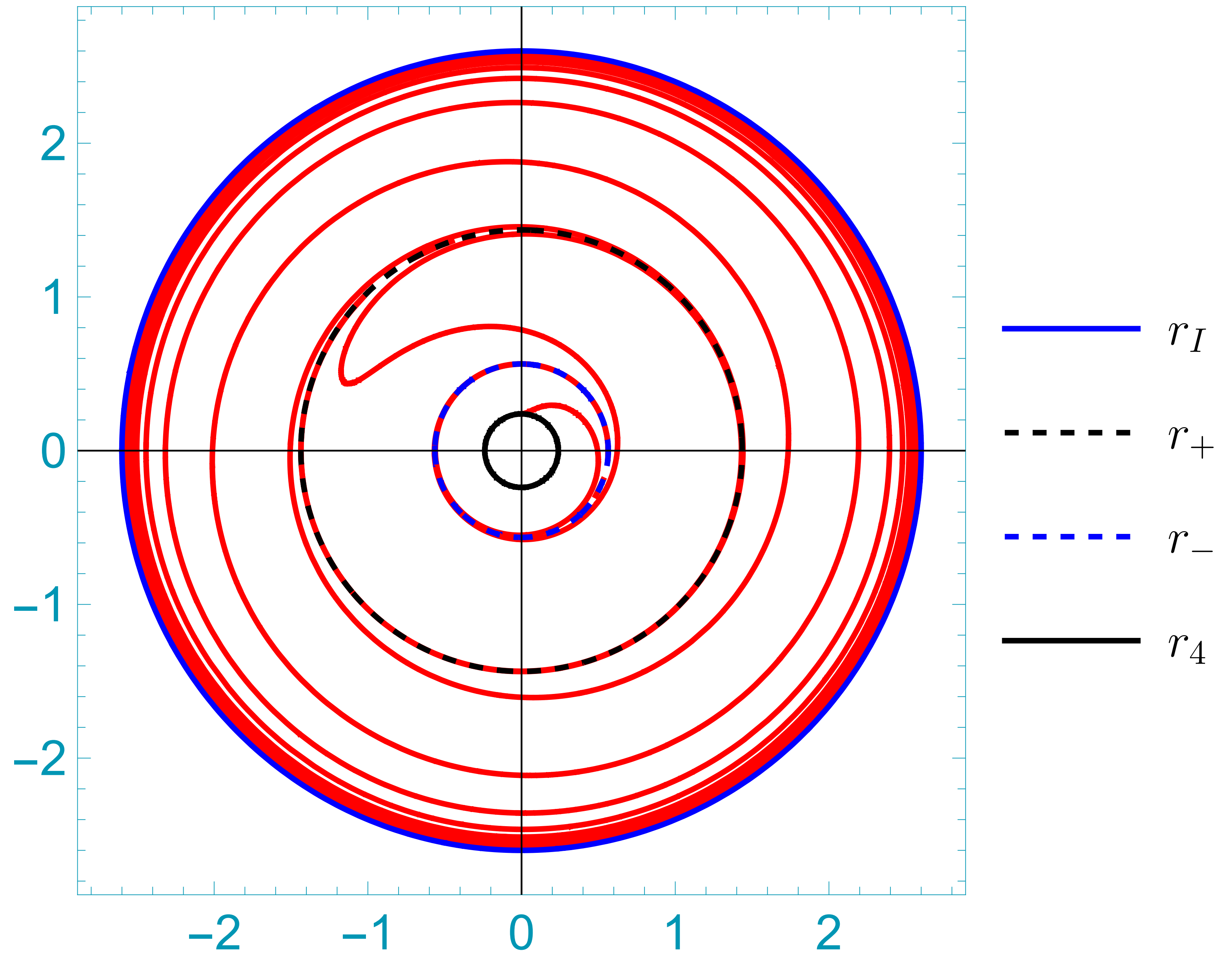
- Analytic solutions to generic bound geodesics
  [Fujita, Hikida: 0906]

- And much more…



Aims of the Work

- Plunge following the approach of Fujita and Hikida
- Easily related back to the action angle parameterisation
- Efficiently implement in the Black Hole Perturbation Toolkit



Bound Plunges

- Write geodesic equations in terms of conserved quantities $(\mathscr{E}, \mathscr{L}, Q)$

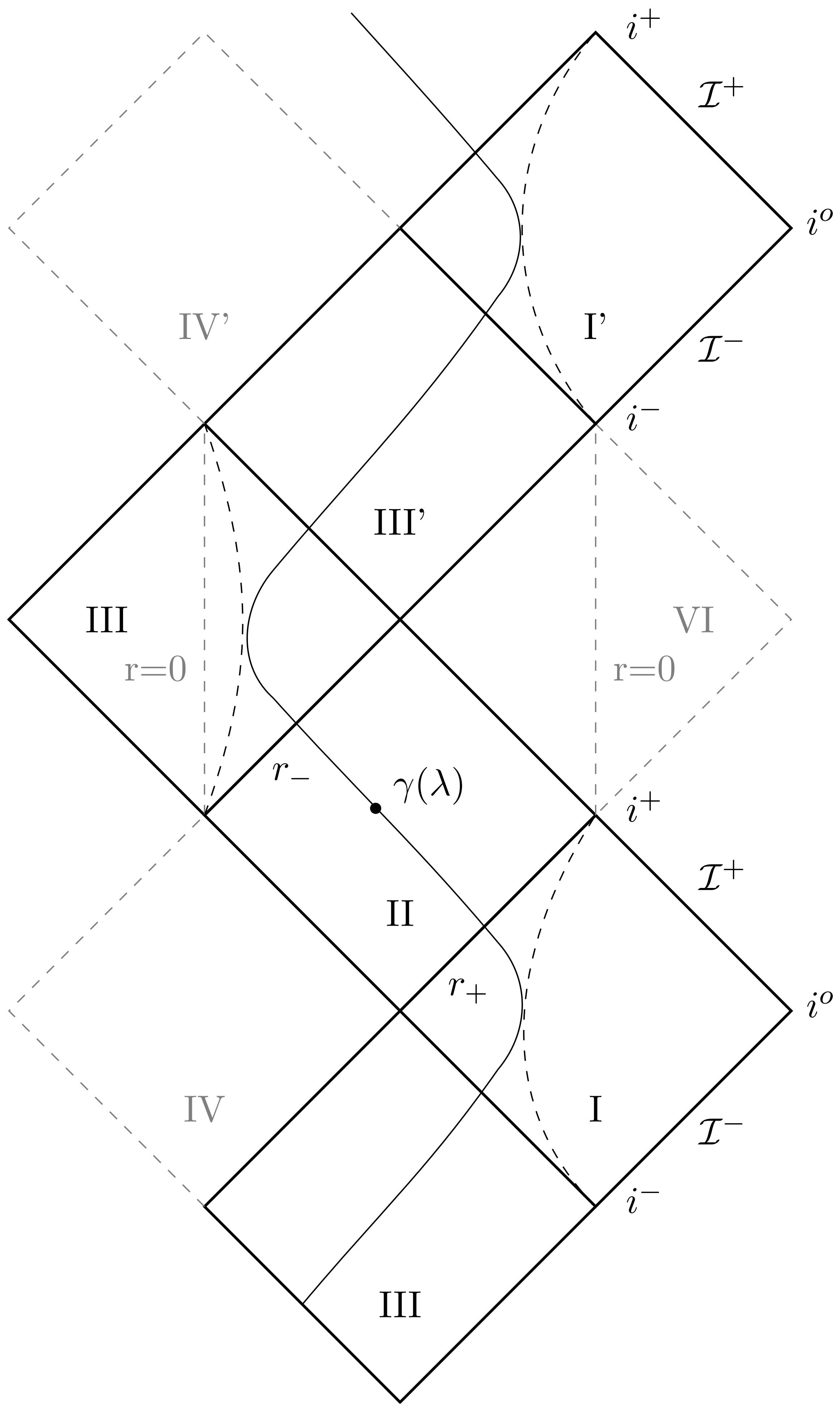
$$\left(\frac{dr}{d\lambda}\right)^2 = R(r), \quad \text{- fourth order poly in } r$$

$$\left(\frac{dz}{d\lambda}\right)^2 = Z(z^2), \quad \text{- second order order } z^2$$

$$\frac{dt}{d\lambda} = T_r(r) + T_z(z),$$

$$\frac{d\phi}{d\lambda} = \Phi_r(r) + \Phi_z(z)\,.$$

- Restrict to $\mathscr{E} < 1$, Bound Geodesics



Elliptic Integrals

- Need to solve new sets of radial elliptic integrals

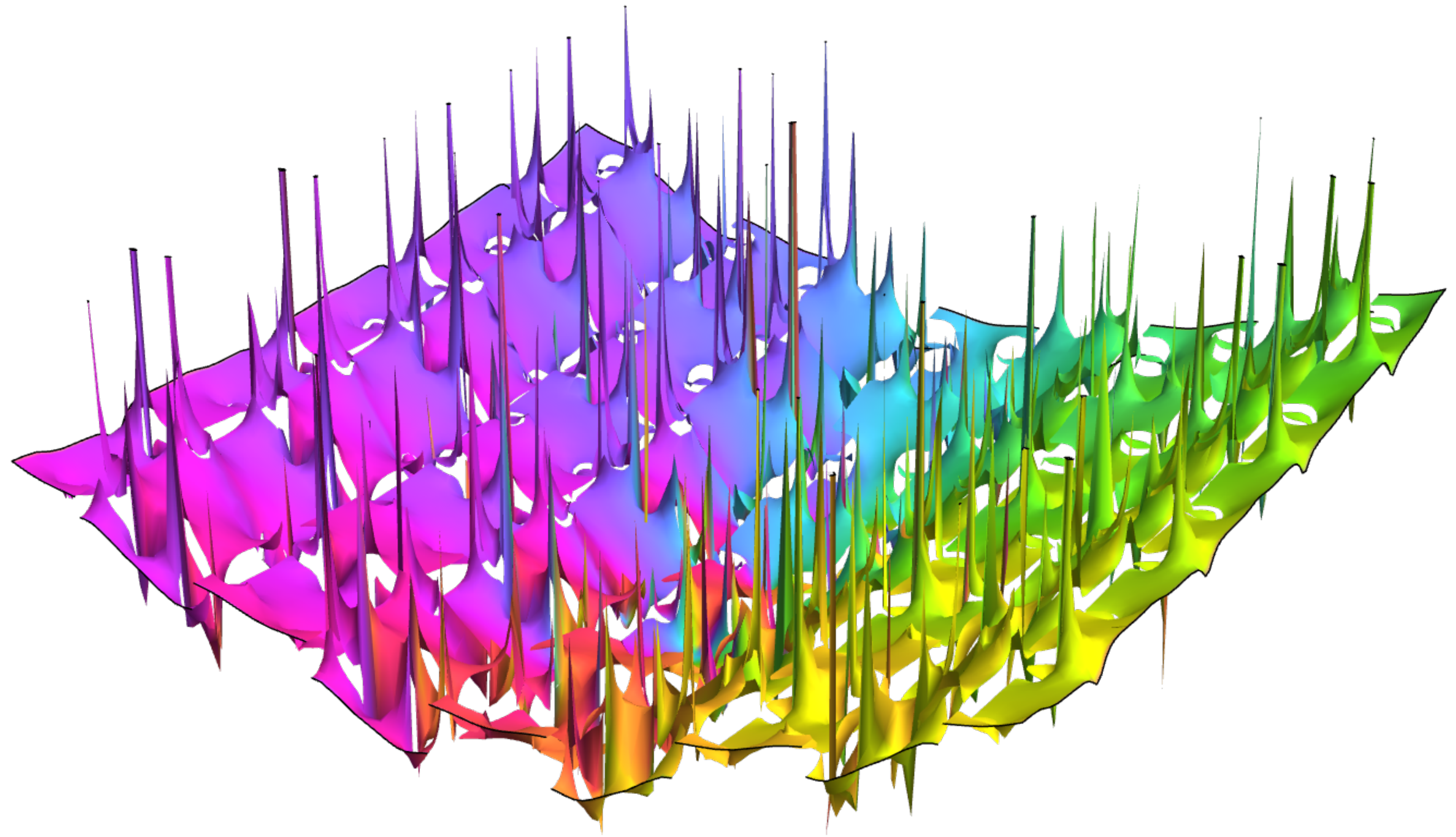
- With $P_{0,1}$(x) rational polynomials in x

- y^2 is a polynomial in x with order three or four

$$\mathcal{I} = \int_a^b \frac{f_0(x) + f_1(x)y}{g_0(x) + g_1(x)y} dx = \int_a^b P_0(x) + \int_a^b \frac{P_1(x)}{y} dx$$

[Labahn and Mutrie: 0197]

Reducing the Problem



- Reduce $P_1(x)$ to sum over rational polynomials with simple poles

- One can then decomposed into four fundamental integrals

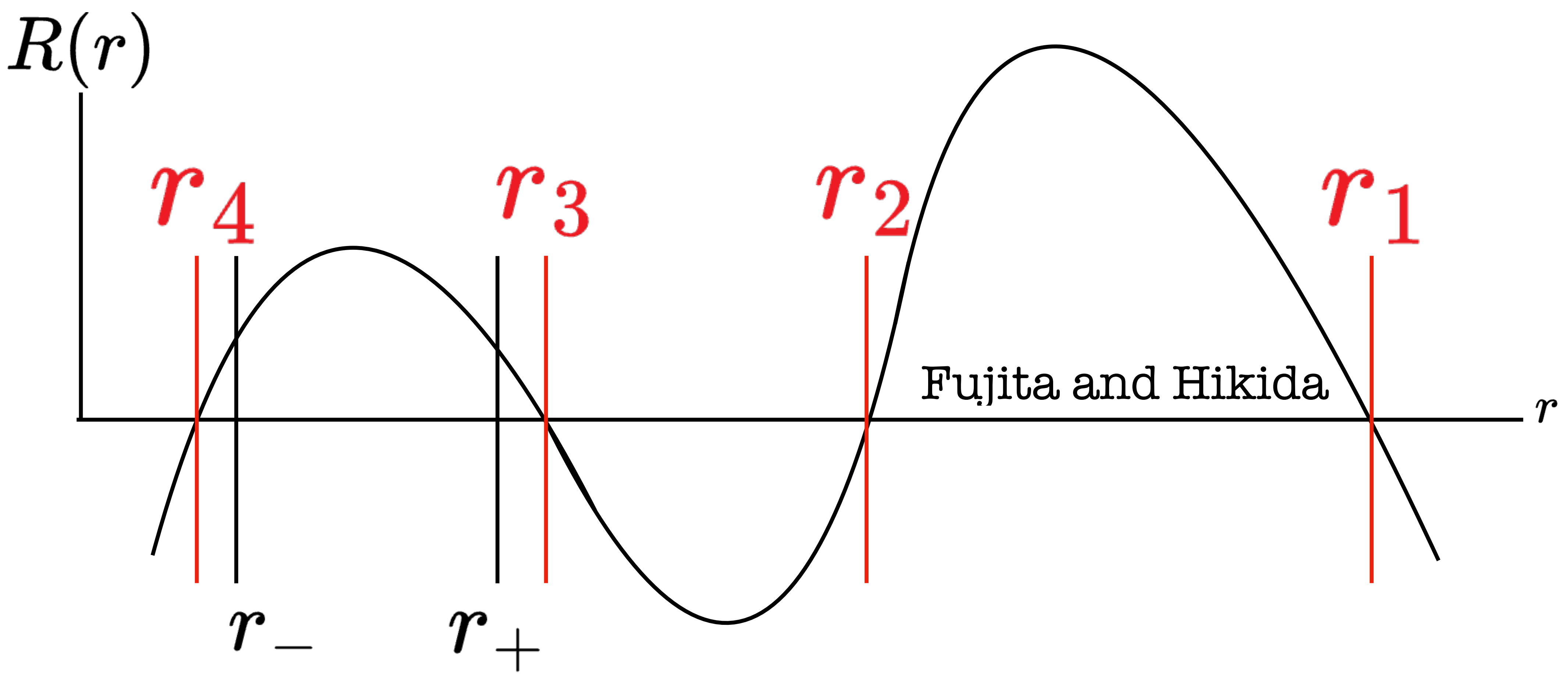
$$\lambda = -\int \frac{dr}{\sqrt{R(r)}}, \quad \mathcal{I}_r = \int \frac{r dr}{\sqrt{R(r)}},$$

$$\mathcal{I}_{r^2} = \int \frac{r^2 dr}{\sqrt{R(r)}} \text{ and } \mathcal{I}_{r_\pm} = \int \frac{dr}{(r - r_\pm)\sqrt{R(r)}}$$

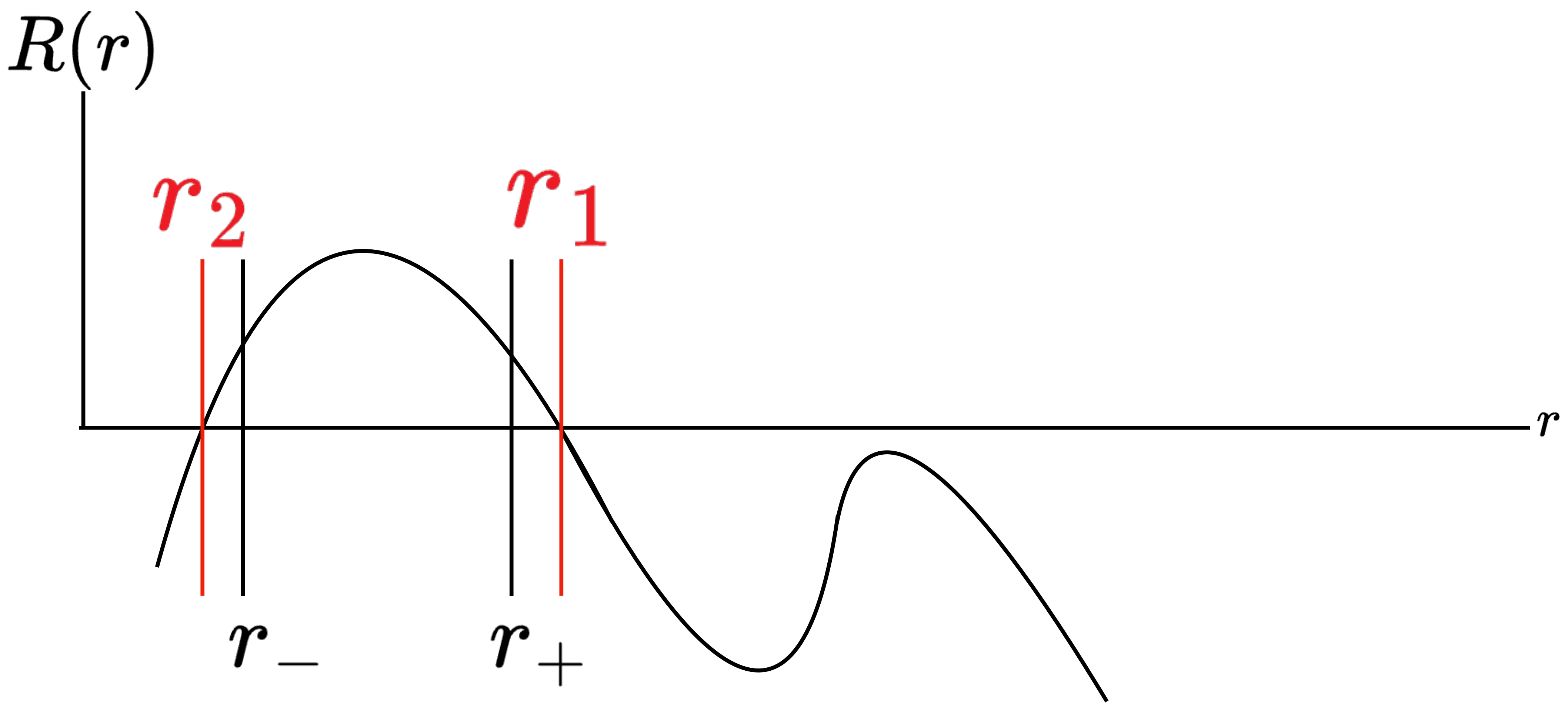
Bound Plunges: Two Cases

- Geodesic type given by root structure of radial potential, $\frac{dr}{d\lambda} = \pm\sqrt{R(r)}$.
- Roots correspond of turning points of the trajectories

Four real roots



Two real and two complex roots



Complex Roots: Solving the Integrals

- Four real roots, $R(r) = A(r - r_1)(r - r_2)(r - r_3)(r - r_4)$

- Two complex two real roots, $R(r) = B(r - r_1)(r - r_2)\left(r^2 - 2\Re(r_3)r + \Re(r_3) - \Im(r_4)\ \right)$

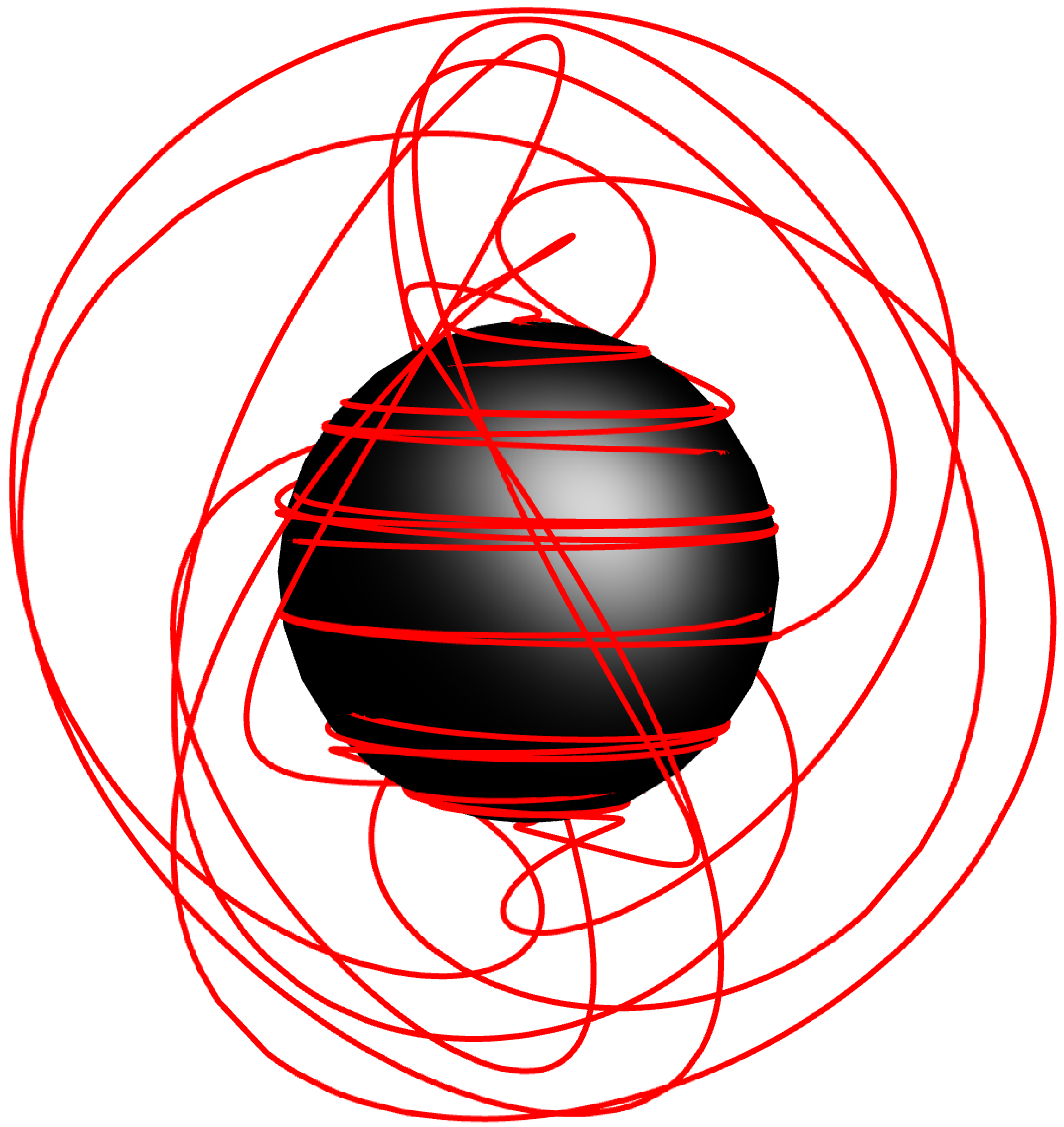
- Make manifestly real substitution $r(y)$ such that $$\frac{dr}{\sqrt{R(r)}} \to C\frac{dy}{\sqrt{(1 - k_r^2 y^2)(1 - y^2)}}$$

- Allows one to solve explicitly for the four fundamental integrals

- Analytical continue everything!

The Plunging Kerr Solutions

- Solutions in terms of Elementary and Elliptic functions
- All function manifestly real parameterised by - $(r_1, r_2, \Re(r_3), \Im(r_4))$
- Immediately obtain Mino frequencies - $\Upsilon_r = \Upsilon_r(\mathscr{E}, r_1, r_2, \Re(r_3), \Im(r_4))$.
- Can be easily recast in action angles parameterisation
- Implemented in the Black Hole Perturbation Toolkit

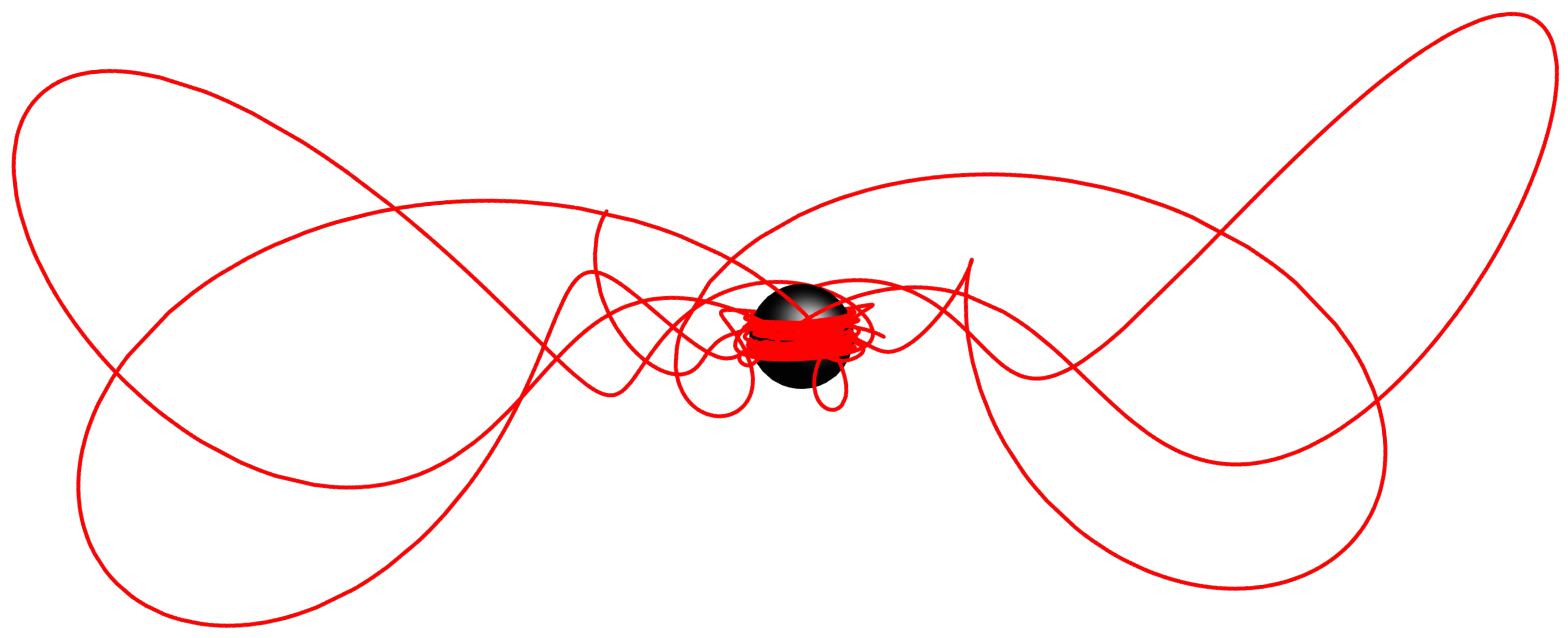


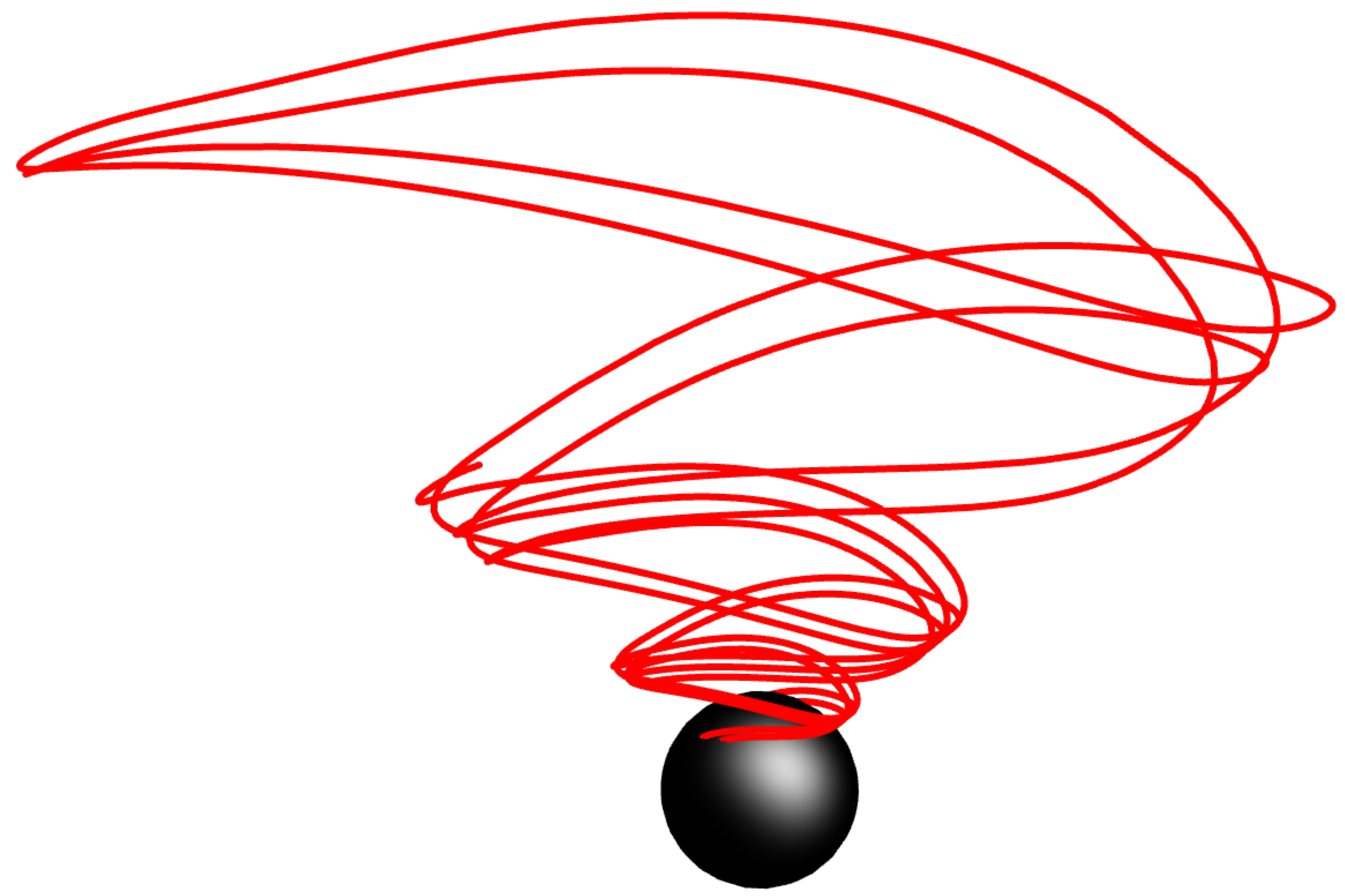
Bonus: Approach Works Generally

- Recent work exploring phenomenology of dyonically charged particles in Kerr-Newman. [Dyson and Pereñiguez: arXiv:2306.15751]

$$u^a \nabla_a u_b = \frac{1}{m}\left(eF_{ba} - g \star F_{ba}\right) u^a \,.$$

- Same procedure can be used generically to find geodesics solutions analytically
- Solve for generic Bounded and Plunging geodesics in Dyonic Kerr-Newman





Outcome

- Solutions in terms of Elementary and Elliptic functions
- Can be easily recast in action angles parameterisation
- Implemented in the Black Hole Perturbation Toolkit