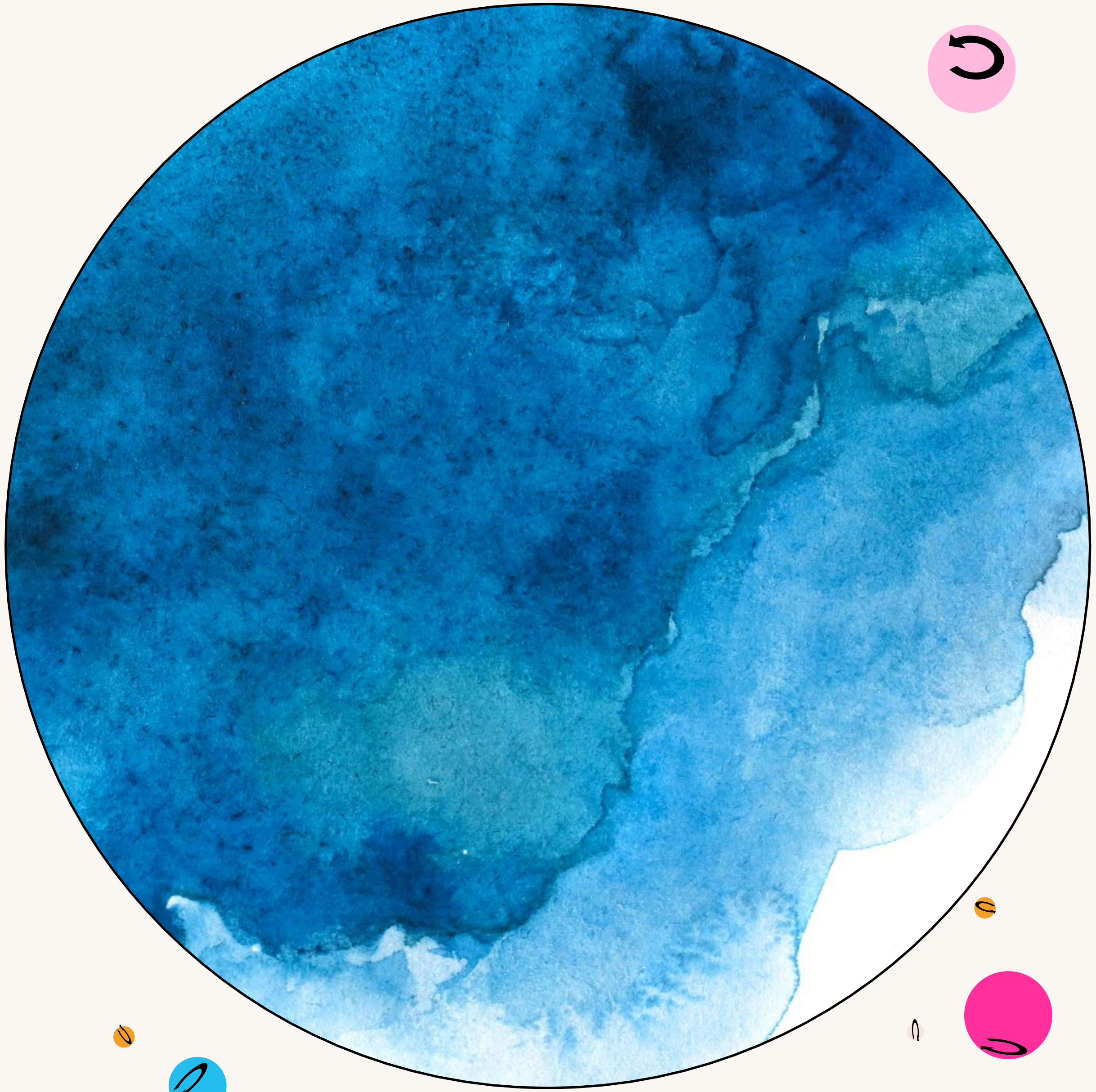
Exact solution for motion of spinning particles near static, spherically symmetric objects



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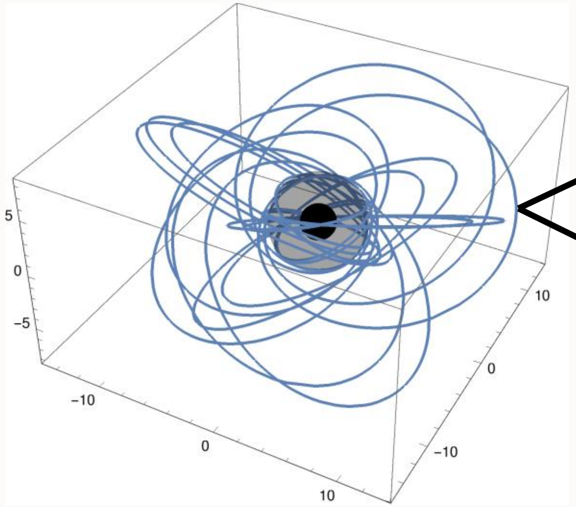
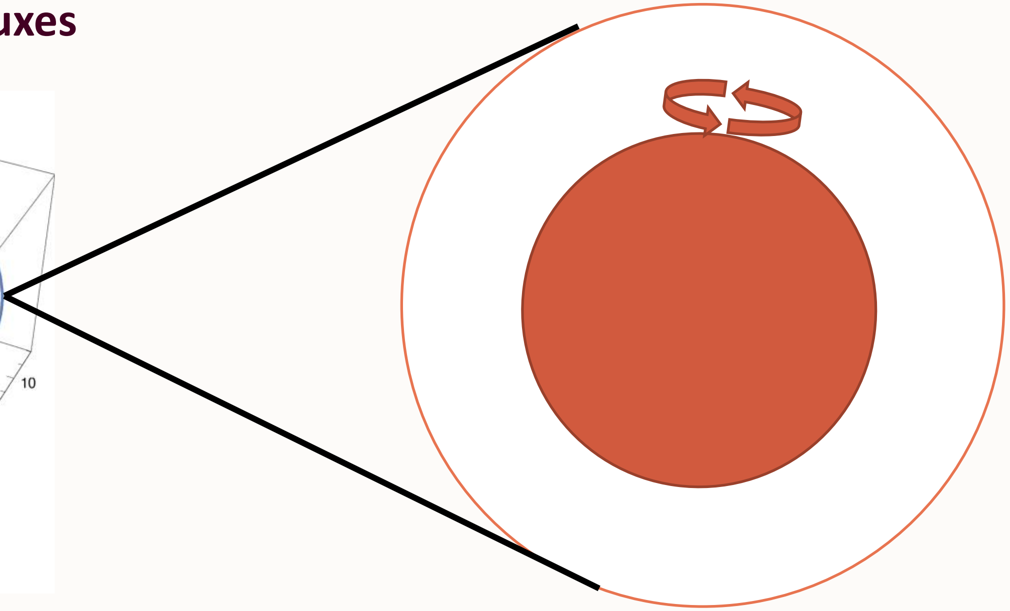
Why care about ”spinning particle”?

- In compact object binaries, spin-orbital coupling appears as a conservative effect at linear order in mass ratio (assuming Kerr bound for black holes, mass-shedding limits for neutron stars)
- Also 1.5PN in a post-Newtonian expansion, 0.5PN earlier than non-trivial gravitational self-force (not obtainable from $m \to \mu$ rescaling and Schwarzschild geodesics)
- That places it confidently into the **1PA iteration** of large-mass ratio waveforms.
    - Need to solve for **motion perturbed by spin (linear order)**
    - Need to solve for **correction to fluxes**





Context

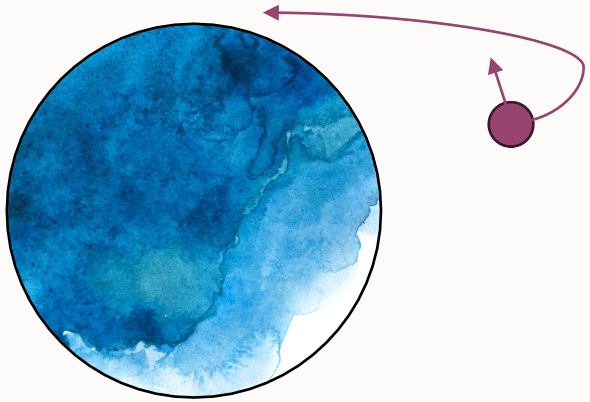
- 2018 (VW, Steinhoff, Lukes-Gerakopoulos 1808.06582): Hamiltonians and canonical coordinates for spinning particles
- 2019 (VW, 1903.03651): Hamilton-Jacobi equation, separability in Kerr, constants of motion $\mathcal{E}_{so}, \mathcal{L}_{so}, \mathcal{K}_{so}, s_{\parallel}$, solution depends on adapted Marck tetrad and its **connection**

$$\frac{\mathrm{d}r}{\mathrm{d}\lambda} = \pm\Delta\sqrt{{w'_r}^2 - e_{0r} e^{\kappa}_{C;r} e_{\kappa B} \tilde{s}^{CD}} \,,$$

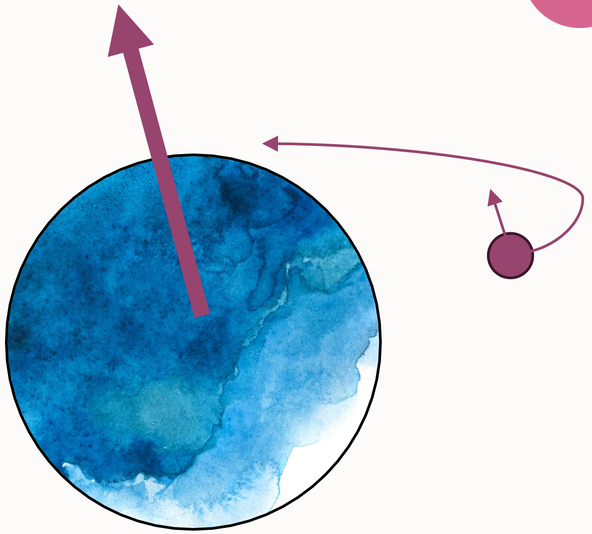
$$\frac{\mathrm{d}\vartheta}{\mathrm{d}\lambda} = \pm\sqrt{{w'_\vartheta}^2 - e_{0\vartheta} e^{\kappa}_{C;\vartheta} e_{\kappa B} \tilde{s}^{CD}} \,,$$

Annoying to deal with (ask about connection terms and watch Gabriel's expression)

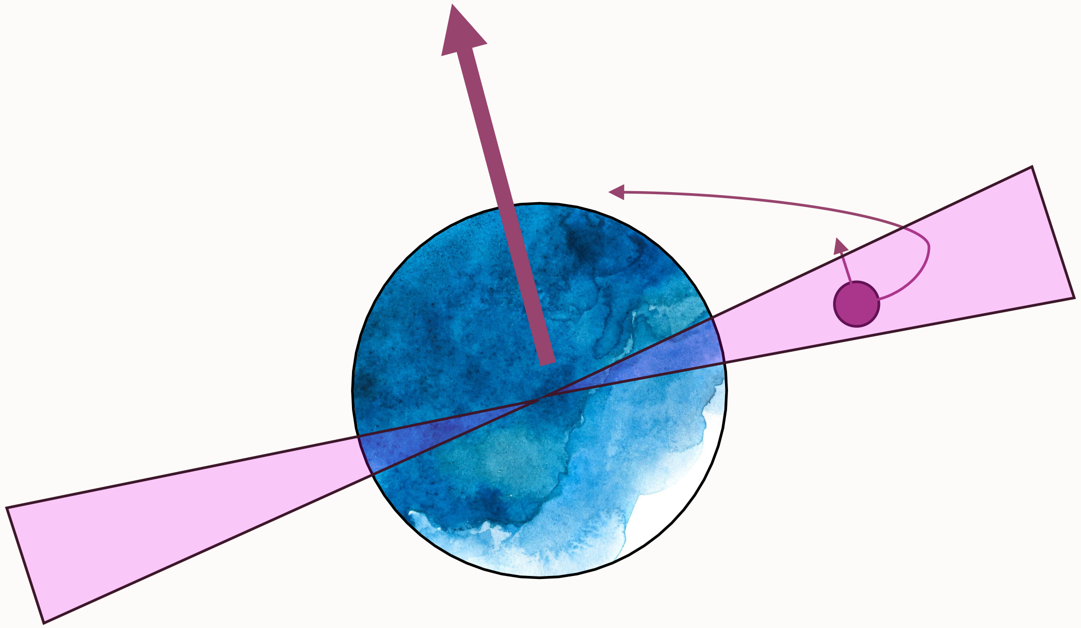
Sanity checks in equatorial plane, independent methods in special limits (Schwarzschild), contact with
Drummond & Hughes (2201.13334, 2201.13335),
Skoupý & Lukes-Gerakopoulos (2102.04819)
Saijo+ (1998),…



Symmetries



$$\vec{J} = \text{"}\vec{L} + \vec{S}\text{"}$$



Aligned coordinates:
$\delta\theta = O(S)$

$$s^t = \sqrt{\frac{s^2 - s_\parallel^2}{f(\mathcal{J}^2 + r^2)}} \left(\frac{\mathcal{E}\mathcal{J}\cos\psi}{\sqrt{f}} + \dot{r}\, r \sin\psi \right),$$

$$\ldots \psi + \frac{\mathcal{E} r \sin\psi}{\sqrt{f h (\mathcal{J}^2 + r^2)}} \Bigg),$$

$$\delta\vartheta = \frac{\sqrt{(s^2 - s_\parallel^2)(\mathcal{J}^2 + r^2)}\sin\psi}{\mathcal{J} r}.$$

$$\tilde{T}_r(\chi) = \frac{\mathcal{E}}{\sqrt{(1-\mathcal{E}^2) r_2 (r_1 - r_3)}} \Bigg[-\frac{2M(r_1 - r_3)(r_3 - r_2) - r_3\left(r_1(r_3 - r_2) + r_3(r_2 + r_3)\right) + 2M s_\parallel \mathcal{J}/\mathcal{E}}{r_3 - 2M} F(\chi, k)$$
$$+ (4M + r_1 + r_2 + r_3)(r_2 - r_3)\Pi\left(\chi, \frac{r_1 - r_2}{r_1 - r_3}, k\right) + r_2(r_1 - r_3)E(\chi, k)$$
$$- \frac{(8M^2 - \mathcal{J} s_\parallel/\mathcal{E})2M(r_2 - r_3)}{(r_3 - 2M)(r_2 - 2M)}\Pi\left(\chi, \frac{(r_3 - 2M)(r_1 - r_2)}{(r_2 - 2M)(r_1 - r_3)}, k\right) - \frac{\sqrt{r r_2 (r_1 - r_3)(r_1 - r)(r \ldots}}{\sqrt{r - r_3}}$$

$$\tilde{\Phi}_r(\chi) = \frac{2(\mathcal{J} - s_\parallel \mathcal{E})}{\sqrt{(1-\mathcal{E}^2) r_2 (r_1 - r_3)}} F(\chi, k),$$

$$\tilde{\Psi}_r(\chi) = \frac{\mathcal{E}\mathcal{J}}{\sqrt{(1-\mathcal{E}^2)(r_1 - r_3) r_2}} \Bigg[\frac{2 r_3^2}{\mathcal{J}^2 + r_3^2} F(\chi, k) + \frac{2(r_1 - r_2)(\mathcal{J}^2 + r_2 r_3)}{(\mathcal{J}^2 + r_2^2)(r_1 - r_3)} \mathfrak{Re}\left(\Pi\left(\chi, \frac{i\mathcal{J} + r_3}{i\mathcal{J} + r_2} \frac{r_1 - \ldots}{r_1 - \ldots}\right.\right.$$
$$- \frac{2\mathcal{J}(r_1 - r_2)(r_2 - r_3)}{(\mathcal{J}^2 + r_2^2)(r_1 - r_3)} \mathfrak{Im}\left(\Pi\left(\chi, \frac{i\mathcal{J} + r_3}{i\mathcal{J} + r_2} \frac{r_1 - r_2}{r_1 - r_3}, k\right)\right)\Bigg]$$

$$\Upsilon^t = \frac{\tilde{T}_r(\pi)}{\Upsilon^r} = \frac{\mathcal{E}}{2K(k)} \Bigg[(r_2 - r_3)(r_1 + r_2 + r_3 + 4M)\Pi\left(\frac{r_1 - r \ldots}{r_1 - r \ldots}\right.$$
$$- \frac{2M(r_2 - r_3)(8M^2 - M s_\parallel \mathcal{J}/\mathcal{E})}{(r_2 - 2M)(r_3 - 2M)}\Pi\left(\frac{r_1 - r_2}{r_1 - r_3}\frac{2M - r_3}{2M - r_2}, k\right)$$
$$+ \frac{2M(r_1 - r_3)(r_2 - r_3) + r_3\left(r_3(r_2 + r_3) - r_1(r_2 - r_3)\right) \ldots}{r_3 - 2M}$$

$$\Upsilon^\phi = \frac{\tilde{\Phi}_r(\pi)}{\Upsilon^r} = \mathcal{J} - s_\parallel \mathcal{E},$$

$$\Upsilon^\psi = \frac{\tilde{\Psi}_r(\pi)}{\Upsilon^r} = \frac{\mathcal{E}\mathcal{J} r_3^2}{\mathcal{J}^2 + r_3^2} + \frac{\mathcal{E}\mathcal{J}^2 (r_2 - r_3)}{K(k)(\mathcal{J}^2 + r_2^2)(\mathcal{J}^2 + r_3^2)} \Bigg[\mathcal{J}(r_2 + \ldots$$
$$- (\mathcal{J}^2 - r_2 r_3)\mathfrak{Im}\left(\Pi\left(\frac{i\mathcal{J} + r_3}{i\mathcal{J} + r_2}\frac{r_1 - r_2}{r_1 - r_3}, k\right)\right)\Bigg],$$

$$t(r) - t(r_0) = \pm \int_{r_0}^{r} \frac{\mathrm{d}r'}{\sqrt{\mathcal{R}(r')}} \left[\frac{\mathcal{E}}{f} - \frac{s_\parallel \mathcal{J} f'}{2 f r' \sqrt{f h}} \right],$$

$$\varphi(r) - \varphi(r_0) = \pm \int_{r_0}^{r} \frac{\mathrm{d}r'}{\sqrt{\mathcal{R}(r')}} \left[\frac{\mathcal{J}}{r'^2} - \frac{s_\parallel \mathcal{E}}{r'^2 \sqrt{f h}} \right],$$

$$\psi(r) - \psi(r_0) = \pm \int_{r_0}^{r} \frac{\mathrm{d}r'}{\sqrt{\mathcal{R}(r')}} \left[\frac{\mathcal{E}\mathcal{J}}{\mathcal{J}^2 + r'^2} \right].$$

Metric (why not general -> bosons stars and whatnot):

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + h(r)\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin\theta^2\mathrm{d}\phi^2\right)$$

Killing constants of motion for spinning particle:

$$P^\mu\xi_\mu - \xi_{\rho;\sigma}S^{\rho\sigma}/2$$

Static, spherically symmetric (x,y,z axis rotations):

$$\mathcal{E} = -f\dot{t} + \frac{r^2\sin\theta f' s^\phi\dot{\theta} - s^\theta\dot{\phi}}{2\sqrt{fh}},$$

$$\mathcal{J}_x = -r^2(\sin\phi\,\dot{\theta} + \cos\phi\cos\theta\sin\theta\,\dot{\phi}) + \sqrt{\frac{f}{h}}\Big[\sin\theta\cos\phi\,h(s^t\dot{r} - s^r\dot{t}) + r\sin\phi\sin\theta(s^\phi\dot{t} - s^t\dot{\phi}) + r\cos\phi\cos\theta(s^t\dot{\theta} - s^\theta\dot{t})\Big],$$

$$\mathcal{J}_y = r^2(\cos\phi\,\dot{\theta} - \sin\phi\cos\theta\sin\theta\,\dot{\phi}) + \sqrt{\frac{f}{h}}\Big[\sin\theta\sin\phi\,h(s^t\dot{r} - s^r\dot{t}) + r\cos\phi\sin\theta(s^t\dot{\phi} - s^\phi\dot{t}) + r\sin\phi\cos\theta(s^t\dot{\theta} - s^\theta\dot{t})\Big],$$

$$\mathcal{J}_z = r^2\sin^2\theta\,\dot{\phi} + \sqrt{\frac{f}{h}}\Big[\cos\theta\,h(s^t\dot{r} - s^r\dot{t}) + r\sin\theta(s^\theta\dot{t} - s^t\dot{\theta})\Big].$$

Aligned frame $\vec{z'} \parallel \vec{\mathcal{J}}$: decoupling!!

$$\delta\vartheta = \frac{\sqrt{fh}(s^r\dot{t} - s^t\dot{r})}{r^2\dot{\varphi}} + \mathcal{O}(s^2)\,.$$

$$\dot{r}^2 = \frac{1}{h}\left(-1 + \frac{\mathcal{E}^2}{f} - \frac{\mathcal{J}^2}{r^2}\right) - \frac{s_\parallel \mathcal{E}\mathcal{J}(2f - rf')}{(fh)^{3/2}r^2}\,,$$

$$\dot{t} = \frac{\mathcal{E}}{f} - \frac{s_\parallel \mathcal{J} f'}{2fr\sqrt{fh}}\,,$$

$$\dot{\varphi} = \frac{\mathcal{J}}{r^2} - \frac{s_\parallel \mathcal{E}}{r^2\sqrt{fh}}\,.$$

$$s_\parallel \equiv \frac{l^\mu s_\mu}{\sqrt{l^\nu l_\nu}}\,,\quad \frac{\mathrm{d}s_\parallel}{\mathrm{d}\tau} = 0 + \mathcal{O}(s^2)\,.$$

Spin sector? Parallel transport

Generalization of the tetrad of Marck (1983) for Kerr geodesics to any static, spherically symmetric space-time:

$$e^{\mu}_{(0)} = \left(\mathcal{E}/f, \dot{r}, \mathcal{J}/r^2, 0\right) \ , \qquad \tilde{e}^{\mu}_{(1)} = \left(\frac{\dot{r}r\sqrt{h}}{\sqrt{f(\mathcal{J}^2+r^2)}}, \frac{\mathcal{E}r}{\sqrt{fh(\mathcal{J}^2+r^2)}}, 0, 0\right) \ ,$$

$$e^{\mu}_{(3)} = \left(0, 0, 0, \frac{1}{r}\right) \ , \qquad \tilde{e}^{\mu}_{(2)} = \left(\frac{\mathcal{E}\mathcal{J}}{f\sqrt{\mathcal{J}^2+r^2}}, \frac{\mathcal{J}\dot{r}}{r}, \frac{\sqrt{\mathcal{J}^2+r^2}}{r^2}, 0\right) \ ,$$

$$\dot{\psi} = \frac{\mathcal{E}\mathcal{J}}{\mathcal{J}^2+r^2}$$

$$s^t = \sqrt{\frac{s^2-s_{\parallel}^2}{f(\mathcal{J}^2+r^2)}}\left(\frac{\mathcal{E}\mathcal{J}\cos\psi}{\sqrt{f}} + \dot{r}r\sin\psi\right) \ , \qquad s^{\varphi} = \frac{\sqrt{(s^2-s_{\parallel}^2)(\mathcal{J}^2+r^2)}\cos\psi}{r^2} \qquad \delta\vartheta = \frac{\sqrt{(s^2-s_{\parallel}^2)(\mathcal{J}^2+r^2)}\sin\psi}{\mathcal{J}r}$$

$$s^r = \sqrt{s^2-s_{\parallel}^2}\left(\frac{\mathcal{J}\dot{r}\cos\psi}{r} + \frac{\mathcal{E}r\sin\psi}{\sqrt{fh(\mathcal{J}^2+r^2)}}\right) \qquad s^{\vartheta} = \frac{s_{\parallel}}{r} \ .$$

Taster: radial motion in Schwarzschild

$$\lambda(r)-\lambda(r_2)=\int_{r_2}^{r}\frac{\mathrm{d}r'}{\sqrt{R(r)}} = \frac{2F(\chi,k)}{\sqrt{(1-\mathcal{E}^2)(r_1-r_3)r_2}}\,,$$

$$R(r)\equiv(1-\mathcal{E}^2)(r_1-r)(r-r_2)(r-r_3)r\,,$$

$$r_{1,2}=\frac{p}{1\mp e}$$

$$r_3=\frac{2Mp}{p-4M}-\frac{2s_\parallel\sqrt{Mp\left[p^2-4Mp+4M^2(1-e^2)\right]}}{(p-4M)^2}$$

$$\sin\chi\equiv\sqrt{\frac{(r_1-r_3)(r-r_2)}{(r_1-r_2)(r-r_3)}}\,,$$

$$k^2=\frac{(r_1-r_2)r_3}{(r_1-r_3)r_2}\,,$$

$$\mathcal{E}^2=\frac{(p-2M)^2-4M^2e^2}{p\left[p-M(3+e^2)\right]} - s_\parallel\frac{(e^2-1)^2M\sqrt{Mp\left[p^2-4Mp-4M^2(e^2-1)\right]}}{p^2\left[p-M(3+e^2)\right]^2}$$

ISCO position in closed form:

$$p_c=(6+2e)M-2s_\parallel\sqrt{\frac{2(1+e)}{3+e}}$$

Inverting and t-motion (cf. van de Meent, 1906.05090)

$$r(\lambda) = \frac{r_3(r_1 - r_2)\mathrm{sn}^2\left(\frac{K(k)}{\pi}q^r, k\right) + r_2(r_1 - r_3)}{(r_1 - r_2)\mathrm{sn}^2\left(\frac{K(k)}{\pi}q^r, k\right) - (r_1 - r_3)},$$

$$q^r \equiv \Upsilon^r \lambda + q_0^r\,,$$

$$\Upsilon^r \equiv \frac{\pi\sqrt{(1-\mathcal{E}^2)(r_1 - r_3)r_2}}{2K(k)}\,,$$

$$t(\lambda) = q^t + \tilde{T}_r\left(\mathrm{am}\left(\frac{q^r}{\pi}K(k), k\right)\right) - \frac{\tilde{T}_r(\pi)}{2\pi}q^r,\quad q^t \equiv \Upsilon^t\lambda + q_0^t,\ \Upsilon^t = \frac{\tilde{T}_r(\pi)}{\Upsilon^r}$$

$$\tilde{T}_r(\chi) = \frac{\mathcal{E}}{\sqrt{(1-\mathcal{E}^2)r_2(r_1 - r_3)}}\left[-\frac{2M(r_1 - r_3)(r_3 - r_2) - r_3\left(r_1(r_3 - r_2) + r_3(r_2 + r_3)\right) + 2Ms_{\|}\mathcal{J}/\mathcal{E}}{r_3 - 2M}F(\chi, k)\right.$$
$$+ (4M + r_1 + r_2 + r_3)(r_2 - r_3)\Pi\left(\chi, \frac{r_1 - r_2}{r_1 - r_3}, k\right) + r_2(r_1 - r_3)E(\chi, k)$$
$$\left. - \frac{(8M^2 - \mathcal{J}s_{\|}/\mathcal{E})2M(r_2 - r_3)}{(r_3 - 2M)(r_2 - 2M)}\Pi\left(\chi, \frac{(r_3 - 2M)(r_1 - r_2)}{(r_2 - 2M)(r_1 - r_3)}, k\right) - \frac{\sqrt{rr_2(r_1 - r_3)(r_1 - r)(r - r_2)}}{\sqrt{r - r_3}}\right],$$

Conclusions

- Spinning particle motion in spherically symmetric, static space-times fully solved by quadrature
- ISCO, energy, angular momentum shifts, turning points, all depend only on $s_{\parallel}$
- $s_{\parallel}$=0 means geodesic with precessing plane!
- For Schwarzschild full closed-form solution in terms of angle variables q^r, q^t, q^ϕ, q^ψ
- Lookout on arxiv for Witzany & Piovano 2307.XXXXX