Progress toward inspiral-merger-ringdown waveforms at 1PA order

Adam Pound
with Geoffrey Compere, Leanne Durkan, Lorenzo Kuchler, Niels Warburton, Barry Wardell

26th Capra Meeting - Niels Bohr Institute

3 July 2023

UNIVERSITY OF Southampton   STAG RESEARCH CENTRE   THE ROYAL SOCIETY

First-post-adiabatic (1PA) waveforms

- *Why?* Capra's longstanding goal: subradian waveform accuracy
- *What?* Solve Einstein equations through second order in mass ratio (ε) using multiscale expansion

$$\Rightarrow h = \sum_{k^r, k^\theta, k^\phi} \left[\varepsilon h^{(1)}_{k^A}(\mathcal{J}^A) + \varepsilon^2 h^{(2)}_{k^A}(\mathcal{J}^A)\right] e^{-ik^A\varphi_A}$$

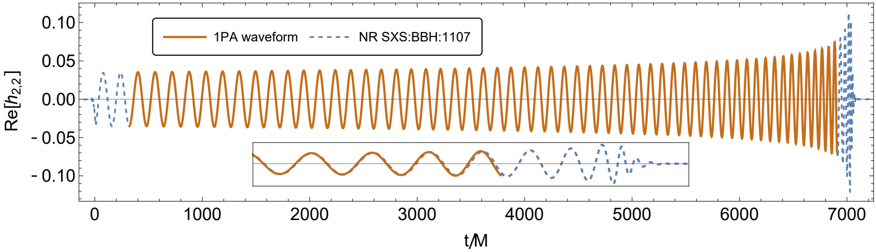
  - orbital phases: $\dfrac{d\varphi_A}{dt} = \Omega_A(\mathcal{J}^B)$
  - slowly evolving parameters:

$$\frac{d\mathcal{J}^A}{dt} = \varepsilon\big[F^A_{(0)}(\mathcal{J}^B) + \varepsilon F^A_{(1)}(\mathcal{J}^B)\big]$$

1PA waveforms for quasicircular, nonspinning binaries

[Wardell, AP, Warburton, Miller, Durkan, and Le Tiec 2021]

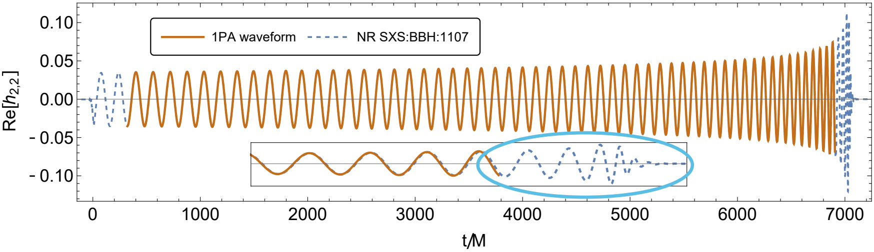
Mass ratio $\varepsilon = 1/10$



1PA waveforms for quasicircular, nonspinning binaries

[Wardell, AP, Warburton, Miller, Durkan, and Le Tiec 2021]

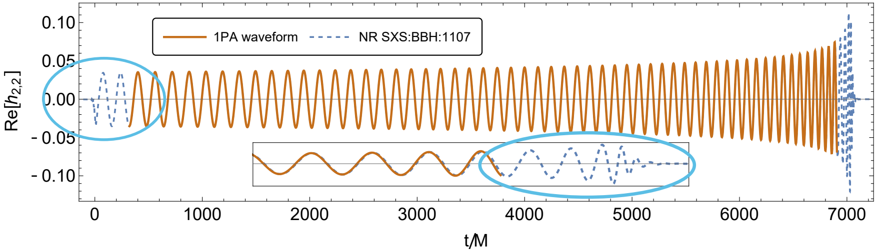
Mass ratio $\varepsilon = 1/10$



1PA waveforms for quasicircular, nonspinning binaries

[Wardell, AP, Warburton, Miller, Durkan, and Le Tiec 2021]

Mass ratio $\varepsilon = 1/10$



Multiscale expansion for quasicircular orbits in Schwarzschild [Miller & AP 2020]

- parameters: $\mathcal{J}_A = (\Omega, M_{BH}, J_{BH})$, $J_{BH} \sim \varepsilon$
- expansions:

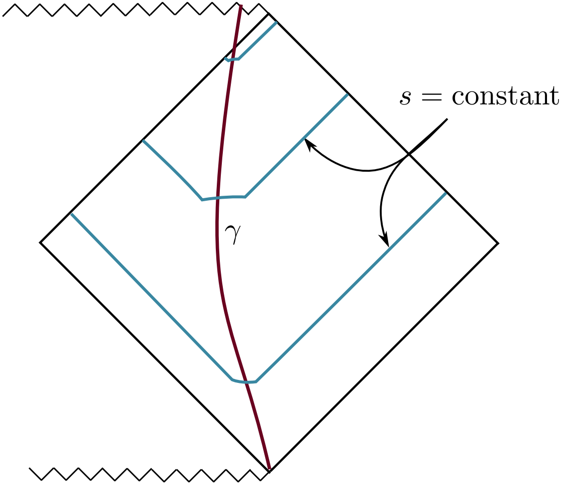
$$h^{(n)}_{\mu\nu} = \sum_{i\ell m} h^{(n)}_{i\ell m}(\mathcal{J}_A, r) e^{-im\phi_p} Y^{i\ell m}_{\mu\nu}$$

$$\frac{d\phi_p}{ds} = \Omega$$

$$\frac{d\Omega}{ds} = \nu \left[F^{(0)}_\Omega(\Omega) + \nu F^{(1)}_\Omega(\mathcal{J}_A) \right]$$

- in Einstein equations, $\frac{\partial}{\partial s} = \Omega \frac{\partial}{\partial \phi_p} + \frac{d\mathcal{J}_A}{ds} \frac{\partial}{\partial \mathcal{J}_A} = -im\Omega + O(\varepsilon)$
- solve field equations for amplitudes $h^{(n)}_{i\ell m}$. Precompute and store $\Rightarrow$ fast waveform generation.

Choice of foliation



Early inspiral

- PN expansion: $x := (M_{\rm tot}\Omega)^{2/3} \ll 1$
- evolution:

$$\frac{d\phi_p}{dt} = \Omega$$
$$\frac{d\Omega}{dt} = \nu x^{11/2}\left[f_\Omega^{(0)}(\nu) + x f_\Omega^{(1)}(\nu) + x^{3/2} f_\Omega^{(1.5)}(\nu) + \ldots + O(x^5)\right]$$

- $h = \nu x \sum_{n\geq 0}\sum_{\ell m} x^{n/2} H_{\ell m}^{(n)}(\nu) e^{-im\phi_p}\,{}_{-2}Y_{\ell m}$

Transition to plunge

- evolution on timescale $\sim 1/(\varepsilon^{1/5}\Omega)$ on frequency interval $(\Omega - \Omega_{\rm isco}) \sim \varepsilon^{2/5}$
- parameters: $\mathcal{J}_A = \{\Delta\tilde{\Omega}, M_{BH}, J_{BH}\}$, $\Delta\tilde{\Omega} := \frac{\Omega - \Omega_{\rm isco}}{\varepsilon^{2/5}}$
- evolution:

$$\frac{d\phi_p}{dt} = \Omega_{\rm isco} + \varepsilon^{2/5}\Delta\tilde{\Omega}$$
$$\frac{d\Delta\tilde{\Omega}}{dt} = \varepsilon^{1/5}\left[F^{(0)}_{\Delta\tilde{\Omega}}(\Delta\tilde{\Omega}) + \varepsilon^{1/5}F^{(1)}_{\Delta\tilde{\Omega}}(\Delta\tilde{\Omega}) + O(\varepsilon^{2/5})\right]$$

- $h_{\mu\nu} = \varepsilon \sum_{n\geq 0}\sum_{i\ell m} \varepsilon^{n/5} j^{(n)}_{i\ell m}(\mathcal{J}_A, r) e^{-im\phi_p} Y^{i\ell m}_{\mu\nu}$

Plunge

- evolution on short time scale $\sim 1/(\varepsilon^0\Omega)$
- parameters: $\mathcal{J}_A = \{\Omega, M_{BH}, J_{BH}\}$
- evolution:

$$\frac{d\phi_p}{dt} = \Omega$$
$$\frac{d\Omega}{dt} = F_\Omega^{(0)}(\Omega) + \varepsilon F_\Omega^{(1)}(\Omega) + O(\varepsilon^2)$$

- $h_{\mu\nu} = \sum_{i\ell m} \left[\varepsilon h_{i\ell m}^{(1)}(\mathcal{J}^A, r) + \varepsilon^2 h_{i\ell m}^{(2)}(\mathcal{J}^A, r) + O(\varepsilon^3)\right] e^{-im\phi_p} Y_{\mu\nu}^{i\ell m}$

Composite expansions

- Example: GSF & PN

$$\frac{d\Omega}{dt} = \nu F_{\Omega}^{(0)}(\Omega) + O(\nu^2)$$
$$\frac{d\Omega}{dt} = \nu x^{11/2} f_{\Omega}^{(0)}(\nu) + O(x^{13/2})$$

- Composite:

$$\frac{d\Omega}{dt} = \nu \left[F_{\Omega}^{(0)}(\Omega) + x^{11/2} f_{\Omega}^{(0)}(\nu) - x^{11/2} f_{\Omega}^{(0)}(0) + O(\nu)\right]$$

Continued-fraction resummation

$$M_{\text{Bondi}}(s) = E_{\text{bind}}(\mathcal{J}_A(s)) + m + M_{BH}(s)$$

$$\Rightarrow -\mathcal{F}_\infty = \frac{\partial E_{\text{bind}}}{\partial \Omega}\dot{\Omega} + \frac{\partial E_{\text{bind}}}{\partial M_{\text{BH}}}\dot{M}_{BH} + \frac{\partial E_{\text{bind}}}{\partial J_{BH}}\dot{J}_{\text{BH}} + \dot{M}_{BH}$$

Continued-fraction resummation

$$M_{\text{Bondi}}(s) = E_{\text{bind}}(\mathcal{J}_A(s)) + m + M_{BH}(s)$$

$$\Rightarrow -\mathcal{F}_\infty = \frac{\partial E_{\text{bind}}}{\partial \Omega}\dot{\Omega} + \frac{\partial E_{\text{bind}}}{\partial M_{\text{BH}}}\cancel{\dot{M}_{BH}} + \frac{\partial E_{\text{bind}}}{\partial J_{BH}}\cancel{\dot{J}_{\text{BH}}} + \overbrace{\dot{M}_{BH}}^{\mathcal{F}_{\mathcal{H}}}$$

Continued-fraction resummation

$$M_{\rm Bondi}(s) = E_{\rm bind}(\mathcal{J}_A(s)) + m + M_{BH}(s)$$

$$\Rightarrow -\mathcal{F}_\infty = \frac{\partial E_{\rm bind}}{\partial \Omega}\dot{\Omega} + \frac{\partial E_{\rm bind}}{\partial M_{\rm BH}}\cancel{\dot{M}_{BH}} + \frac{\partial E_{\rm bind}}{\partial J_{BH}}\cancel{\dot{J}_{\rm BH}} + \overbrace{\dot{M}_{BH}}^{\mathcal{F}_{\mathcal{H}}}$$

$$\Rightarrow \dot{\Omega} = \frac{-\mathcal{F}}{\partial E_{\rm bind}/\partial\Omega}$$

Continued-fraction resummation

$$M_{\text{Bondi}}(s) = E_{\text{bind}}(\mathcal{J}_A(s)) + m + M_{BH}(s)$$

$$\Rightarrow -\mathcal{F}_\infty = \frac{\partial E_{\text{bind}}}{\partial \Omega}\dot{\Omega} + \frac{\partial E_{\text{bind}}}{\partial M_{\text{BH}}}\cancel{\dot{M}_{BH}} + \frac{\partial E_{\text{bind}}}{\partial J_{BH}}\cancel{\dot{J}_{\text{BH}}} + \overbrace{\dot{M}_{BH}}^{\mathcal{F}_{\mathcal{H}}}$$

$$\Rightarrow \dot{\Omega} = \frac{-\mathcal{F}}{\partial E_{\text{bind}}/\partial\Omega}$$

$$= \frac{\nu^2\mathcal{F}^{(1)}(\Omega) + \nu^3\mathcal{F}^{(2)}(\Omega) + \dots}{\nu\partial E^{(0)}_{\text{bind}}/\partial\Omega + \nu^2\partial E^{(1)}_{\text{bind}}/\partial\Omega + \dots}$$

Continued-fraction resummation

$$M_{\rm Bondi}(s) = E_{\rm bind}(\mathcal{J}_A(s)) + m + M_{BH}(s)$$

$$\Rightarrow -\mathcal{F}_\infty = \frac{\partial E_{\rm bind}}{\partial \Omega}\dot\Omega + \frac{\partial E_{\rm bind}}{\partial M_{\rm BH}}\cancel{\dot M_{BH}} + \frac{\partial E_{\rm bind}}{\partial J_{BH}}\cancel{\dot J_{\rm BH}} + \overbrace{\dot M_{BH}}^{\mathcal{F}_{\mathcal{H}}}$$

$$\Rightarrow \dot\Omega = \frac{-\mathcal{F}}{\partial E_{\rm bind}/\partial\Omega}$$

$$= \frac{\nu^2\mathcal{F}^{(1)}(\Omega) + \nu^3[\mathcal{F}_R^{(2)}(\Omega) + \dot\Omega_0\mathcal{F}_S^{(2)}(\Omega)] + \ldots}{\nu\partial E^{(0)}_{\rm bind}/\partial\Omega + \nu^2\partial E^{(1)}_{\rm bind}/\partial\Omega + \ldots}$$

Continued-fraction resummation

$$M_{\text{Bondi}}(s) = E_{\text{bind}}(\mathcal{J}_A(s)) + m + M_{BH}(s)$$

$$\Rightarrow -\mathcal{F}_\infty = \frac{\partial E_{\text{bind}}}{\partial \Omega}\dot{\Omega} + \frac{\partial E_{\text{bind}}}{\partial M_{\text{BH}}}\cancel{\dot{M}_{BH}} + \frac{\partial E_{\text{bind}}}{\partial J_{BH}}\cancel{\dot{J}_{\text{BH}}} + \overbrace{\dot{M}_{BH}}^{\mathcal{F}_{\mathcal{H}}}$$

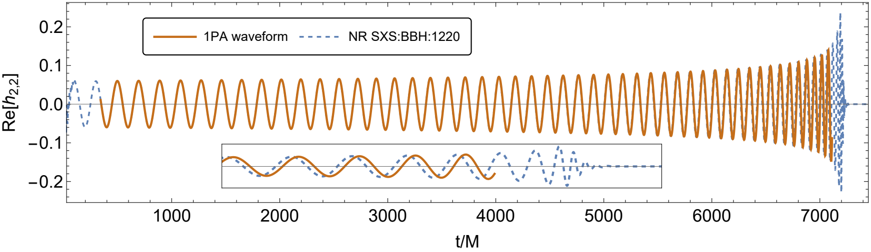
$$\Rightarrow \dot{\Omega} = \frac{-\mathcal{F}}{\partial E_{\text{bind}}/\partial \Omega}$$

$$= \frac{\nu^2 \mathcal{F}^{(1)}(\Omega) + \nu^3[\mathcal{F}_R^{(2)}(\Omega) + \dot{\Omega}_0 \mathcal{F}_S^{(2)}(\Omega)] + \ldots}{\nu \partial E_{\text{bind}}^{(0)}/\partial \Omega + \nu^2 \partial E_{\text{bind}}^{(1)}/\partial \Omega + \ldots}$$

$$\Rightarrow \dot{\Omega} = \frac{\nu^2 \mathcal{F}^{(1)}(\Omega) + \nu^3 \left[\mathcal{F}_R^{(2)}(\Omega) + \frac{\mathcal{F}^{(1)}(\Omega)}{\partial E_{\text{bind}}^{(0)}/\partial\Omega + \nu \partial E_{\text{bind}}^{(1)}/\partial\Omega + \ldots}\mathcal{F}_S^{(2)}(\Omega)\right] + \ldots}{\nu \partial E_{\text{bind}}^{(0)}/\partial \Omega + \nu^2 \partial E_{\text{bind}}^{(1)}/\partial \Omega + \ldots}$$

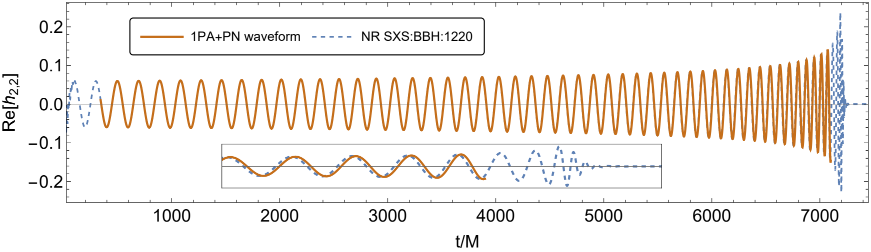
1PA waveforms + PN information

Mass ratio $\varepsilon = 1/4$

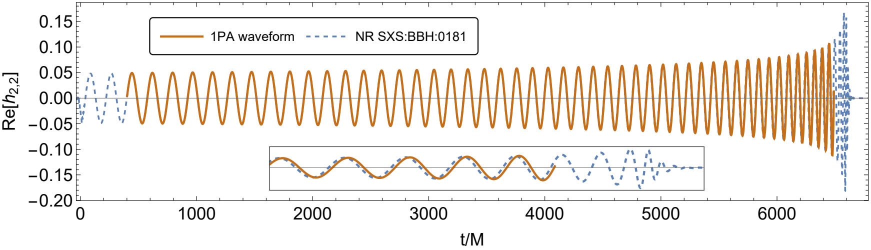


1PA waveforms + PN information

Mass ratio $\varepsilon = 1/4$

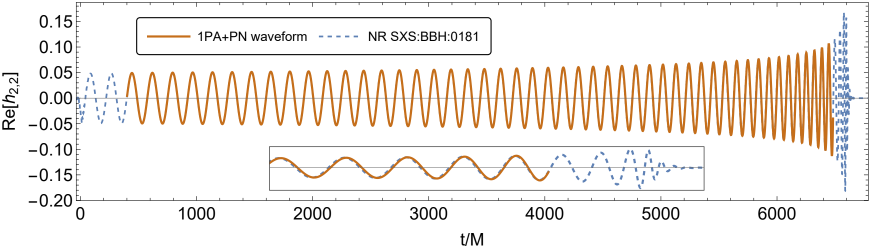


Mass ratio $\varepsilon = 1/6$



1PA waveforms + PN information

Mass ratio $\varepsilon = 1/6$



Conclusion

- Progress toward completing 1PA waveforms
- What's next?
  - merger & ringdown implementation (talk by Lorenzo Kuchler)
  - Kerr (morning talks, discussion session)
  - eccentricity (talk by Ben Leather)

Second-order discussion session

- public release of 2SF tools on BHPToolkit
- upcoming workshops:
  - online series on 1PA waveform generation
  - Princeton, Oct 10-13 (`https://pcts.princeton.edu/events/2023/nonlinear-aspects-general-relativity`)
- paths to Kerr

Paths to Kerr

- Difficulties
  - standard CCK reconstruction produces pathological singularities
    $\Rightarrow$ problematic source at second order
  - numerical burden in any case is calculation of quadratic source term
    – Have we made our lives harder by seeking maximum separability?
- Possibilities
  - Easier in an m-mode decomposition? (Osburn, Panosso Macedo)
  - GHZ puncture scheme (Bourg)
  - Use Lorenz gauge, either reconstructed (Dolan) or directly (Osburn)

Multiscale expansion

[Miller & AP; AP & Wardell; Flanagan, Hinderer, Moxon, AP]

- all time dependence in $h_{\mu\nu}$ follows from puncture's motion and black hole's evolution
- recall $(z^\mu, u^\mu) \to (\varphi_A, J_A)$. Define full set of system parameters $\mathcal{J}_A \sim (J_A, M, a)$

$$\frac{d\varphi_A}{ds} = \Omega_A(\mathcal{J}_B)$$
$$\frac{d\mathcal{J}_A}{ds} = \varepsilon F_A^{(0)}(\mathcal{J}_B) + \varepsilon^2 F_A^{(1)}(\mathcal{J}_B) + O(\varepsilon^3)$$

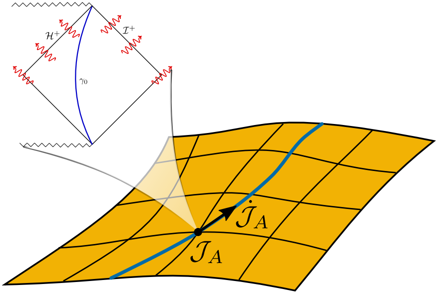
- treat $h_{\mu\nu}$ as function on extended manifold: in spacetime coords (s, x^i),

$$h_{\mu\nu} = \varepsilon h^{(1)}_{\mu\nu}(\varphi_A, \mathcal{J}_A, x^i) + \varepsilon^2 h^{(2)}_{\mu\nu}(\varphi_A, \mathcal{J}_A, x^i) + O(\varepsilon^3)$$

- in Einstein equations, $\dfrac{\partial}{\partial s} = \Omega_A \dfrac{\partial}{\partial \varphi_A} + \dfrac{d\mathcal{J}_A}{ds}\dfrac{\partial}{\partial \mathcal{J}_A}$

Multiscale expansion

[Miller & AP; AP & Wardell; Flanagan, Hinderer, Moxon, AP]



- Fourier series:

$$h^n_{\mu\nu} = \sum_{k^A} h^{(n,\omega_k)}_{\mu\nu}(\mathcal{J}_A, x^i) e^{-ik^A\varphi_A}$$

$$\omega_k := k^A \Omega_A$$

- solve field equations for amplitudes $h^{(n,\omega_k)}_{\mu\nu}$ on grid of $\mathcal{J}_A$ values

- millisecond waveform generation when combined with FastEMRIWaveforms tools [Katz, Chua, Speri, Warburton, Hughes]

Field equations

$$G^{(1)}_{\mu\nu}[h^{\mathcal{R}(1)}] = -G^{(1)}_{\mu\nu}[h^{\mathcal{P}(1)}]$$

$$G^{(1)}_{\mu\nu}[h^{\mathcal{R}(2)}] = -G^{(2)}_{\mu\nu}[h^{(1)}, h^{(1)}] - G^{(1)}_{\mu\nu}[h^{\mathcal{P}(2)}]$$

Field equations

$$G^{(1,0)}_{\omega_k\ell m}[h^{\mathcal{R}(1)}_{\omega_k\ell m}] = -G^{(1,0)}_{\omega_k\ell m}[h^{\mathcal{P}(1)}_{\omega_k\ell m}]$$

$$G^{(1)}_{\mu\nu}[h^{\mathcal{R}(2)}] = -G^{(2)}_{\mu\nu}[h^{(1)}, h^{(1)}] - G^{(1)}_{\mu\nu}[h^{\mathcal{P}(2)}]$$

Field equations

$$G^{(1,0)}_{\omega_k\ell m}[h^{\mathcal{R}(1)}_{\omega_k\ell m}] = -G^{(1,0)}_{\omega_k\ell m}[h^{\mathcal{P}(1)}_{\omega_k\ell m}]$$

$$\begin{aligned} G^{(1,0)}_{\omega_k\ell m}[h^{\mathcal{R}(2)}_{\omega_k}] = -G^{(2,0)}_{\omega_k\ell m}[h^{(1)}_{\omega_{k'}\ell' m'}, h^{(1)}_{\omega_{k''}\ell'' m''}] - G^{(1,0)}_{\omega_k\ell m}[h^{\mathcal{P}(2)}_{\omega_k\ell m}] \\ - G^{(1,1)}_{\omega_k\ell m}[h^{(1)}_{\omega_k\ell m}] \end{aligned}$$

Field equations

$$G^{(1,0)}_{\omega_k\ell m}[h^{\mathcal{R}(1)}_{\omega_k\ell m}] = -G^{(1,0)}_{\omega_k\ell m}[h^{\mathcal{P}(1)}_{\omega_k\ell m}]$$

$$G^{(1,0)}_{\omega_k\ell m}[h^{\mathcal{R}(2)}_{\omega_k}] = -G^{(2,0)}_{\omega_k\ell m}[h^{(1)}_{\omega_{k'}\ell' m'}, h^{(1)}_{\omega_{k''}\ell'' m''}] - G^{(1,0)}_{\omega_k\ell m}[h^{\mathcal{P}(2)}_{\omega_k\ell m}] \\ - G^{(1,1)}_{\omega_k\ell m}[h^{(1)}_{\omega_k\ell m}]$$

Necessary inputs:

- $h^{\mathcal{P}(2)}_{\omega_k\ell m} \to h^{\mathrm{R}(1)}_{\mu\nu}(r_0)$, $\nabla_\rho h^{\mathrm{R}(1)}_{\mu\nu}(r_0)$
- $G^{(2,0)}_{\omega_k\ell m} \to h^{(1)}_{\omega_k\ell m}(r)$, $h^{\mathcal{R}(1)}_{\omega_k\ell m}(r)$, $\partial_r h^{\mathcal{R}(1)}_{\omega_k\ell m}(r)$, $\partial_r^2 h^{\mathcal{R}(1)}_{\omega_k\ell m}(r)$
- $G^{(1,1)}_{\omega_k\ell m}[h^{(1)}_{\omega_k\ell m}] \to F^{(0)}_A \frac{\partial}{\partial \mathcal{J}_A} h^{(1)}_{\omega_k\ell m}$