A discontinuous Galerkin method for the distributionally-sourced s=0 Teukolsky equation

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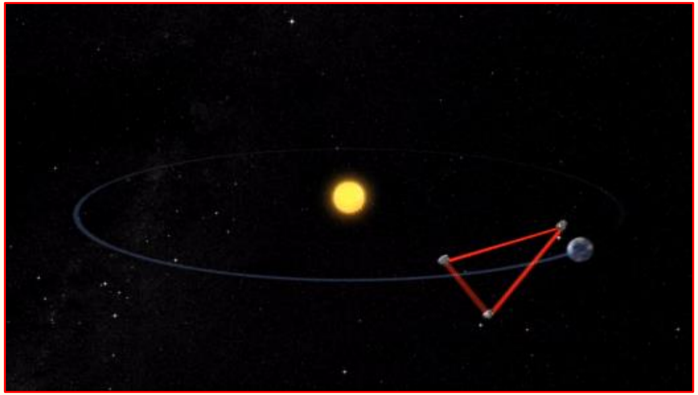
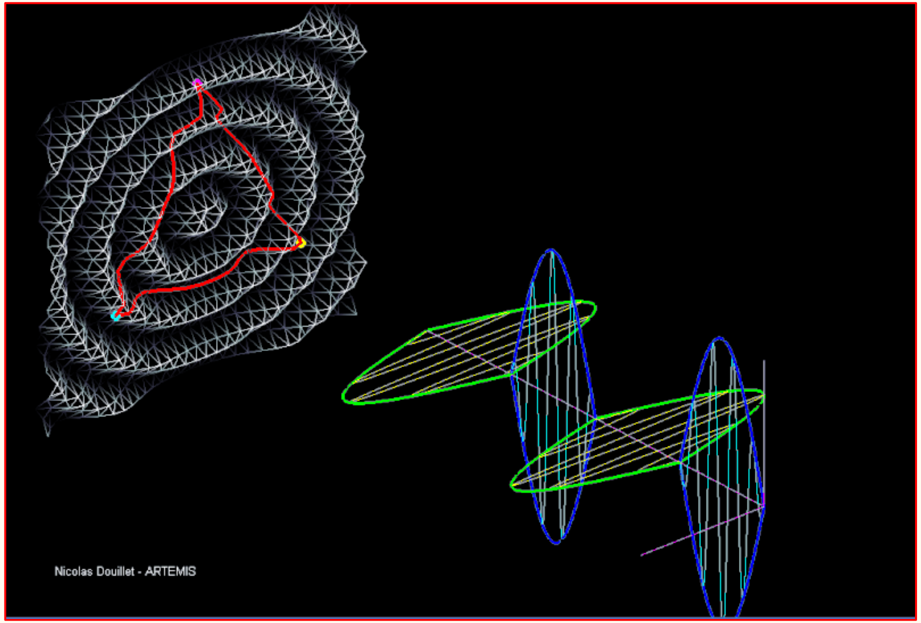
Center for Scientific Computing
& Data Science Research

"The aim of a talk should always be to make the audience feel smarter and not yourself." ~ My dad

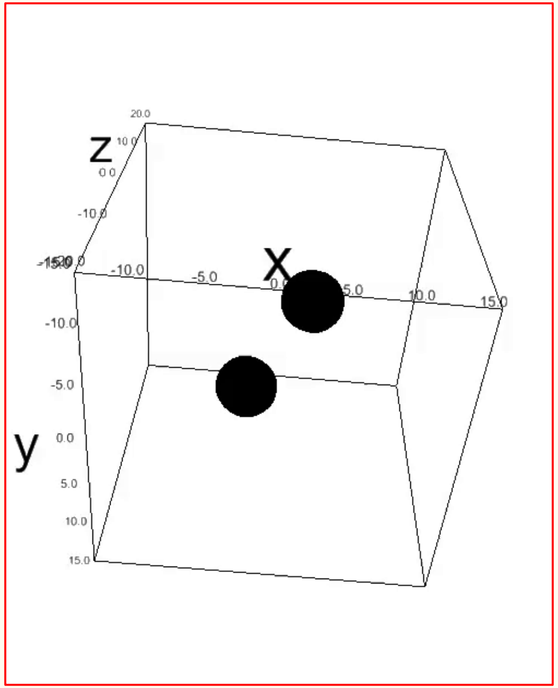
Hey LISA[1]!





[1] Laser Interferometer Space Antenna - Wikipedia

3D Animation from one of our in-house code



s-Cat-ter plot

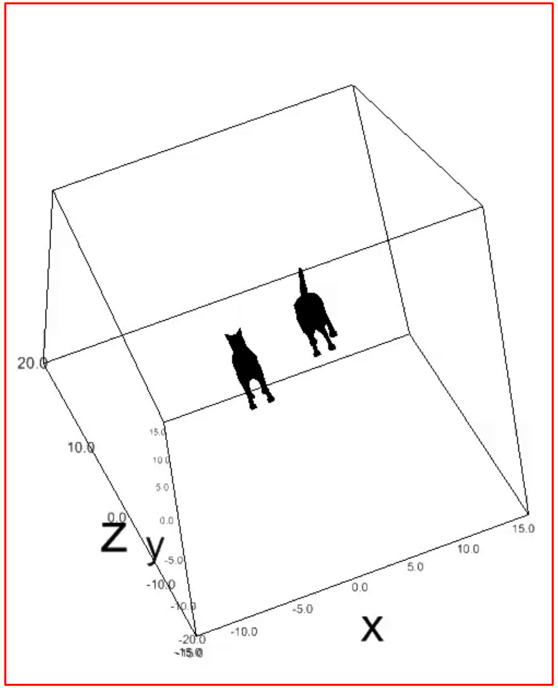
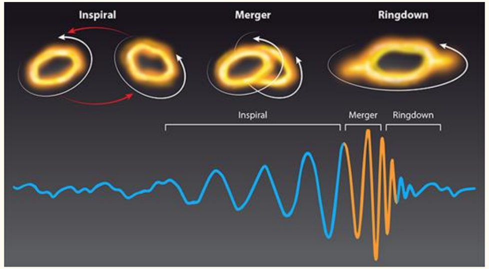


Figure credits : MV+ friends for ICERM Numerical Relativity hackathon at Brown University

Note: The tails of the cats represent the spin on the black holes :)

Matched Filtering 101



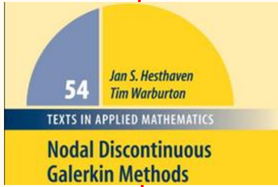
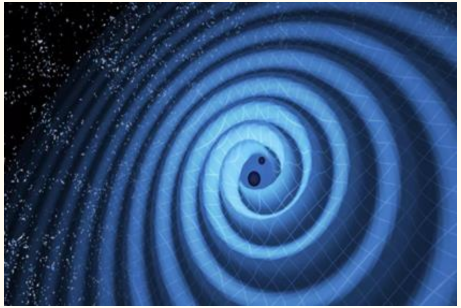
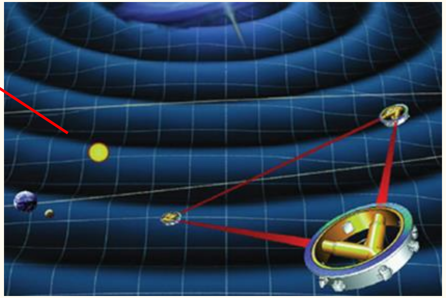
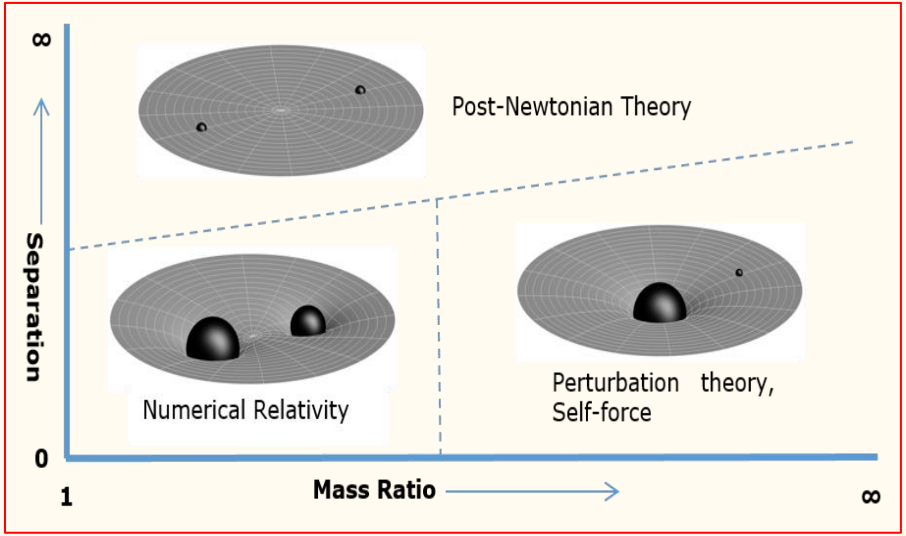


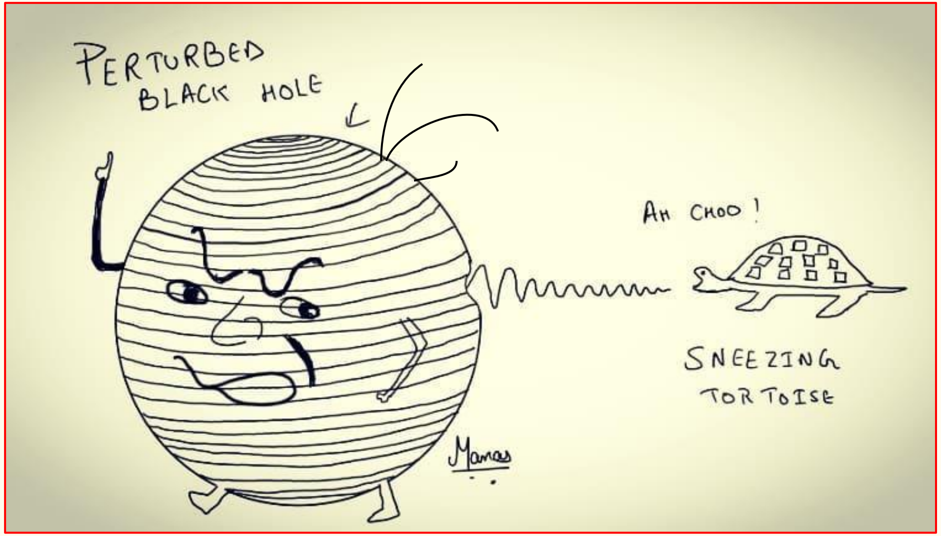
Figure credits : MV for fb.com/GrandUnifiedPhysicsMemes







TimothyRias from Wikipedia



Inspired by C. V. Vishveshwara's original to accommodate "tortoise coordinates"

Teukolsky equation :

$$-\left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right]\partial_{tt}\Psi - \frac{4Mar}{\Delta}\partial_{t\phi}\Psi - 2s\left[r - \frac{M\left(r^2-a^2\right)}{\Delta} + ia\cos\theta\right]\partial_t\Psi - \left(s^2\cot^2\theta - s\right)\Psi$$

$$+\Delta^{-s}\partial_r\left(\Delta^{s+1}\partial_r\Psi\right) + \frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\,\partial_\theta\Psi\right) + \left[\frac{1}{\sin^2\theta} - \frac{a^2}{\Delta}\right]\partial_{\phi\phi}\Psi + 2s\left[\frac{a(r-M)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\right]\partial_\phi\Psi = -4\pi\left(r^2 + a^2\cos^2\theta\right)T$$

where,

$$\Delta = r^2 + a^2 - 2Mr \qquad T = G(t)\cdot\delta\left(r-r_p\right)\cdot\delta\left(\theta-\theta_p\right)\cdot\delta\left(\phi-\phi_p\right)$$

Motivation

- Time domain solvers are generic and flexible, drawbacks include
  - Delta functions are approximated by narrow Gaussians
- All time domain solvers are 2+1D
- Our approach :
  - 1+1D decomposition[1,2]
  - Delta function can be modelled exactly with dG scheme

Ansatz:

$$\Psi(t,r,\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \psi_{\ell,m}(t,r) Y_{\ell,m}(\theta,\phi)$$

Master equation in tortoise coordinates for scalar field:

$$-\ddot{\psi}_\ell + \psi_\ell'' + \frac{\Delta}{(r^2+a^2)^2}\left[\frac{3r^2\Delta}{(r^2+a^2)^2} - \ell(\ell+1) - \frac{r(2r-2M)}{(r^2+a^2)} - \frac{\Delta}{(a^2+r^2)}\right]\psi_\ell = G(t)\delta(r) - \frac{\Delta a^2}{(r^2+a^2)^2}\sum_{L=0}^{\infty} C_{L\ell}\ddot{\psi}_L$$

[1] Nunez, Dario, Juan Carlos Degollado, and Carlos Palenzuela. "One dimensional description of the gravitational perturbation in a Kerr background." Physical Review D 81, no. 6 (2010): 064011.
[2] Stein, L. C. (2012). Probes of strong-field gravity (Doctoral dissertation, Massachusetts Institute of Technology).

The full coupled system by introducing auxiliary variables:

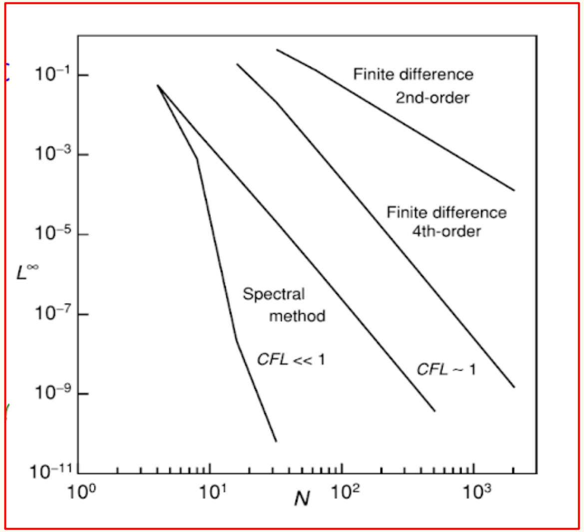
$$\pi_l = -\dot{\Psi}_l \; ; \quad \phi_l = \Psi_l'$$

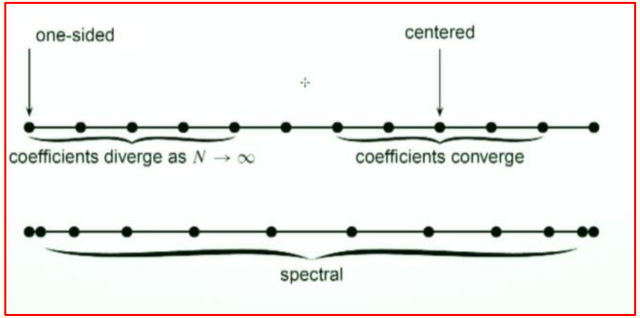
For example, with given l_{max}=6

$$\begin{bmatrix} (1-fC_{00}) & -fC_{02} & 0 & 0 & 0 & 0 & 0 \\ -fC_{20} & (1-fC_{22}) & -fC_{24} & 0 & 0 & 0 & 0 \\ 0 & -fC_{42} & (1-fC_{44}) & -fC_{46} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\pi}_0 \\ \dot{\pi}_2 \\ \dot{\pi}_4 \\ \dot{\phi}_0 \\ \dot{\phi}_2 \\ \dot{\phi}_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \pi_0' \\ \pi_2' \\ \pi_4' \\ \phi_0' \\ \phi_2' \\ \phi_4' \end{bmatrix} + \begin{bmatrix} V_0\psi_0 \\ V_2\psi_2 \\ V_4\psi_4 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0$$

Why discontinuous Galerkin?

- More efficient than finite difference methods
- Easily parallelizable
- Delta function can be exactly incorporated in the scheme
- Exponential decay of error
- Method maintains spectral accuracy for non smooth solution provided the non smoothness is located at interfaces





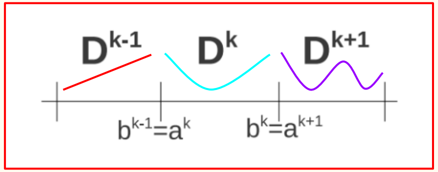
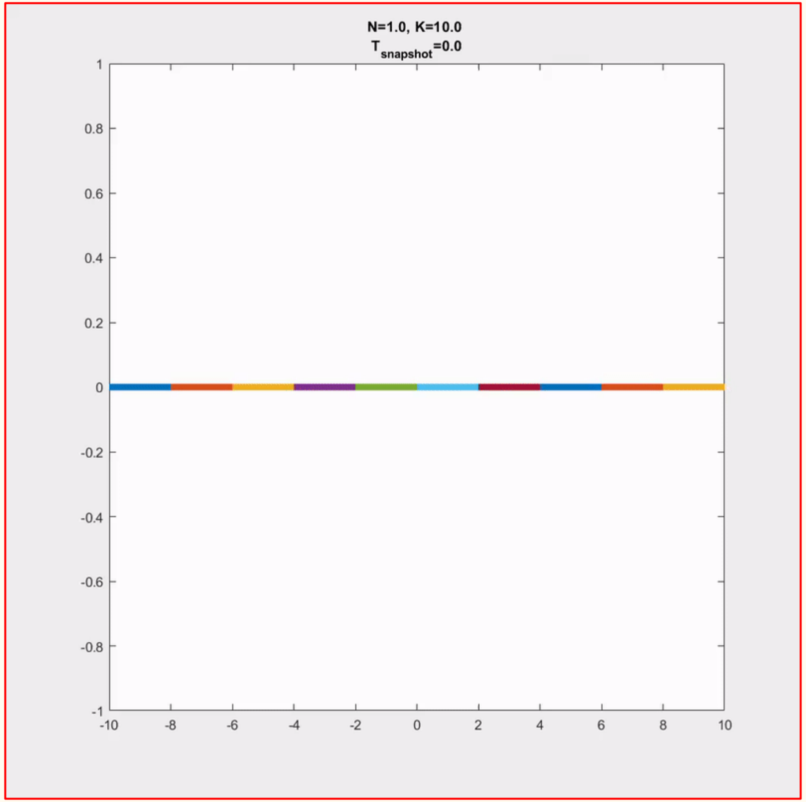
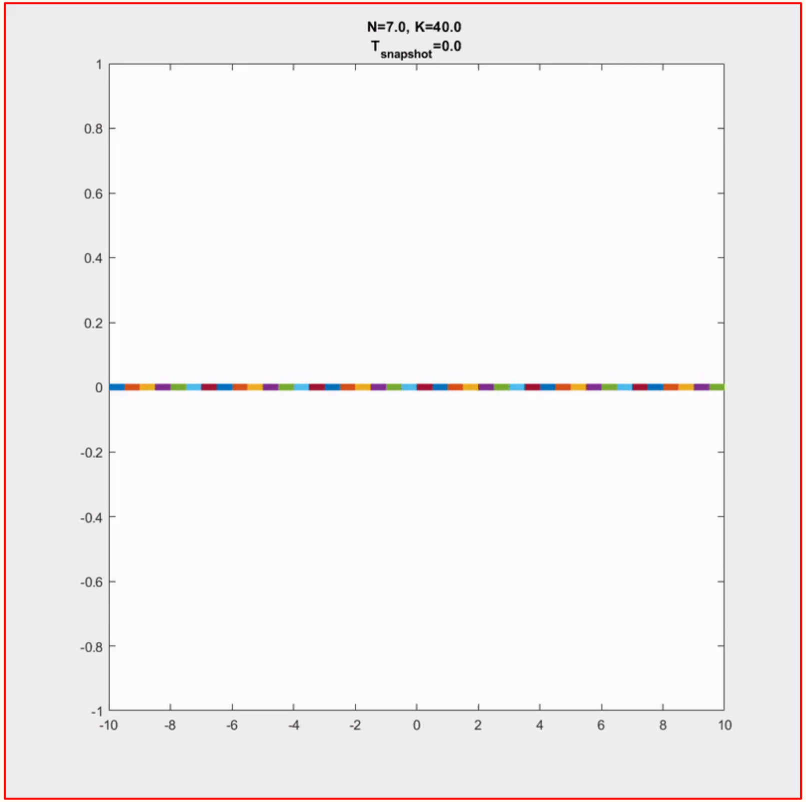


Fig. Credits: Saul Teukolsky

Delta source term in a wave equation

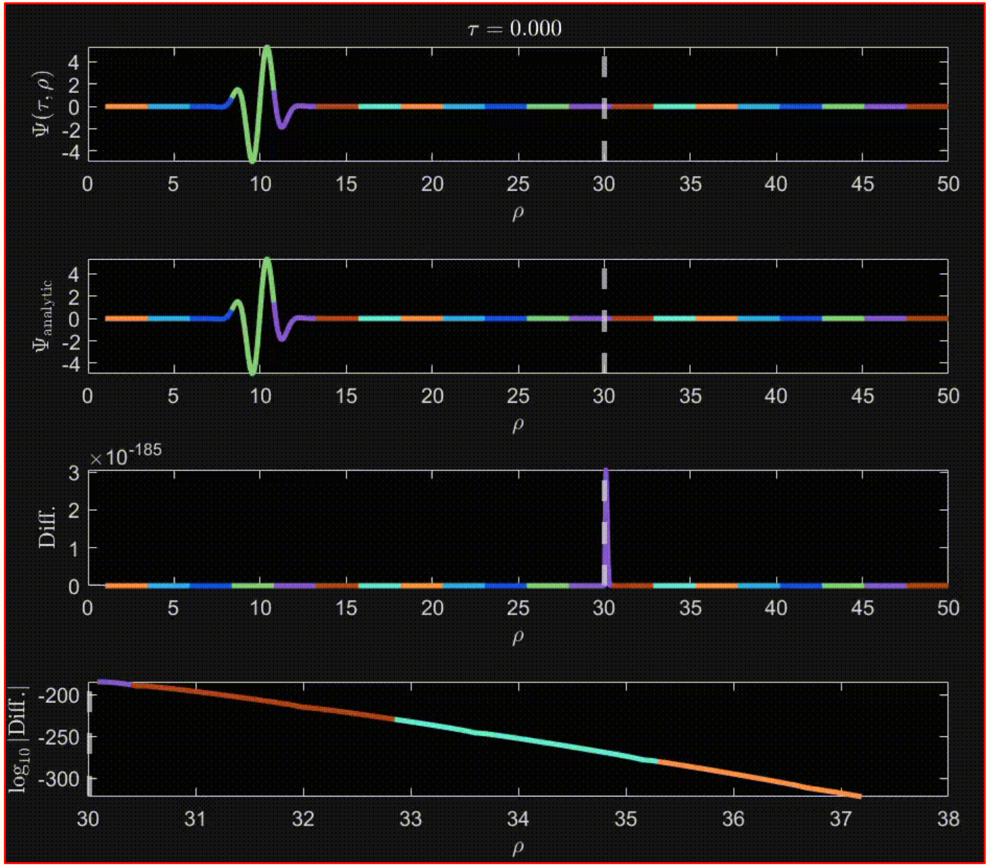
$$-\frac{\partial^2 \psi}{\partial r^2} + \frac{\partial^2 \psi}{\partial t^2} = G(t) \cdot \delta(r - 0)$$





$$-\frac{\partial^2\Psi}{\partial t^2}+\frac{\partial^2\Psi}{\partial r^2}+\frac{\ell(\ell+1)}{r^2}\Psi=0$$

Hyperboloidal Layers[1]



- Null infinity
- Reduces simulation time
- No boundary conditions needed

$(r,t) \longrightarrow (\rho,\tau)$

$$\tau = t - h(r)$$

$$r = \frac{\rho}{\Omega}$$

$$\Omega = 1 - \left(\frac{\rho - R}{s - R}\right)^4 \Theta(\rho - R)$$

$$h(r) = \frac{\rho}{\Omega} - \rho$$

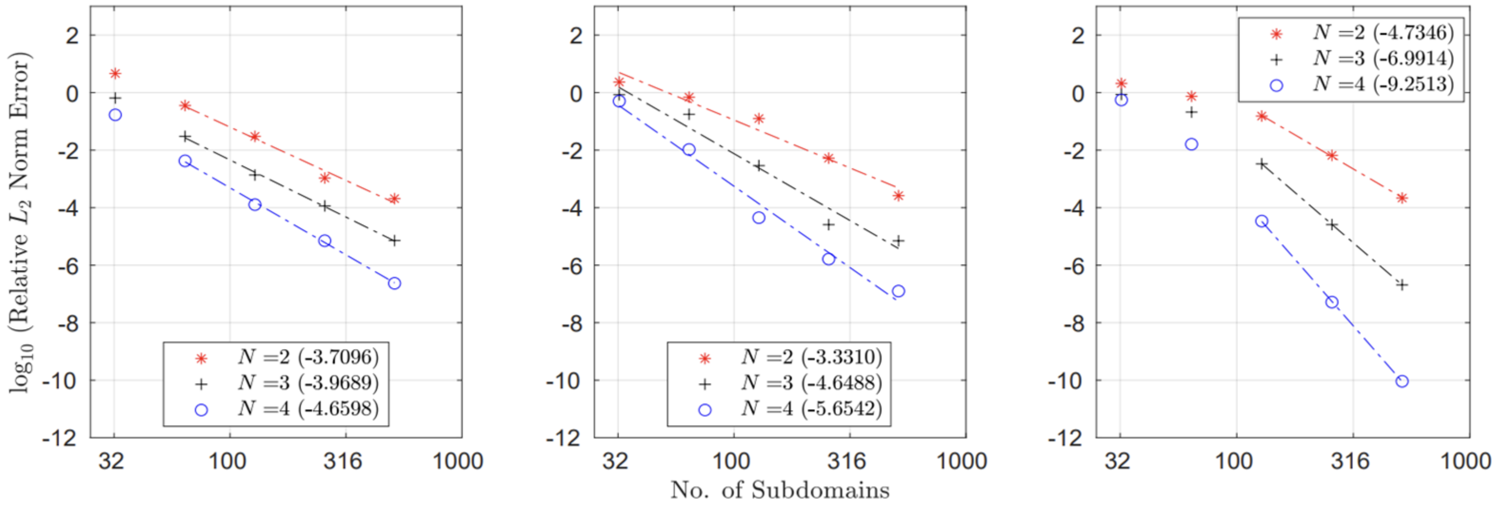
[1]Zenginoğlu, A. (2011). Hyperboloidal layers for hyperbolic equations on unbounded domains. Journal of Computational Physics, 230(6), 2286-2302.

Error and Superconvergence

For sufficiently smooth solutions the error should decay like : $\left\|\Psi - \Psi_h^k\right\|_{\mathrm{D}^k} \leq C(t)\left(\left|\mathrm{D}^k\right|\right)^{N+1}$

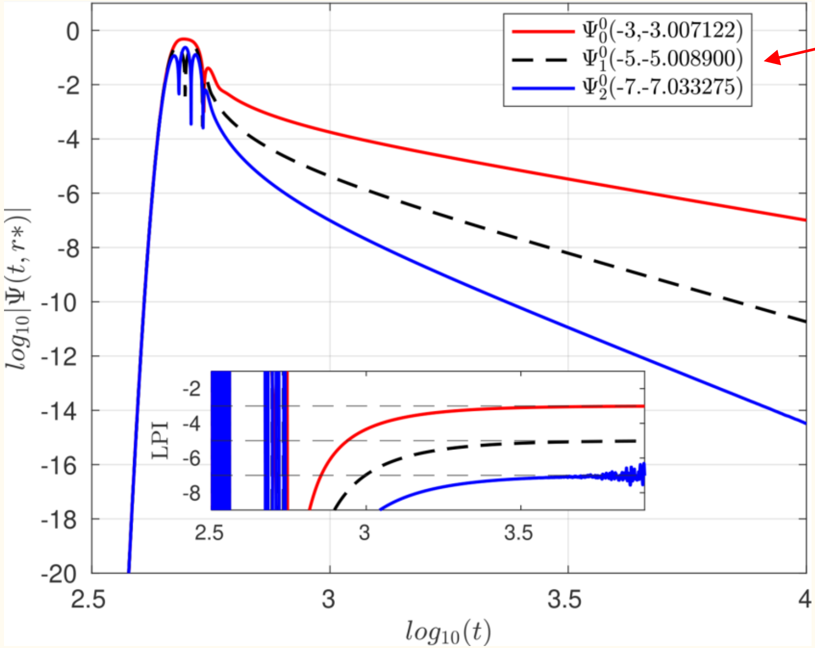
$$-\frac{\partial^2\psi}{\partial r^2} + \frac{\partial^2\psi}{\partial t^2} + \frac{\ell(\ell+1)}{r^2} = 0; \qquad \ell = 2$$

Left of layer | Inside layer | At future null infinity

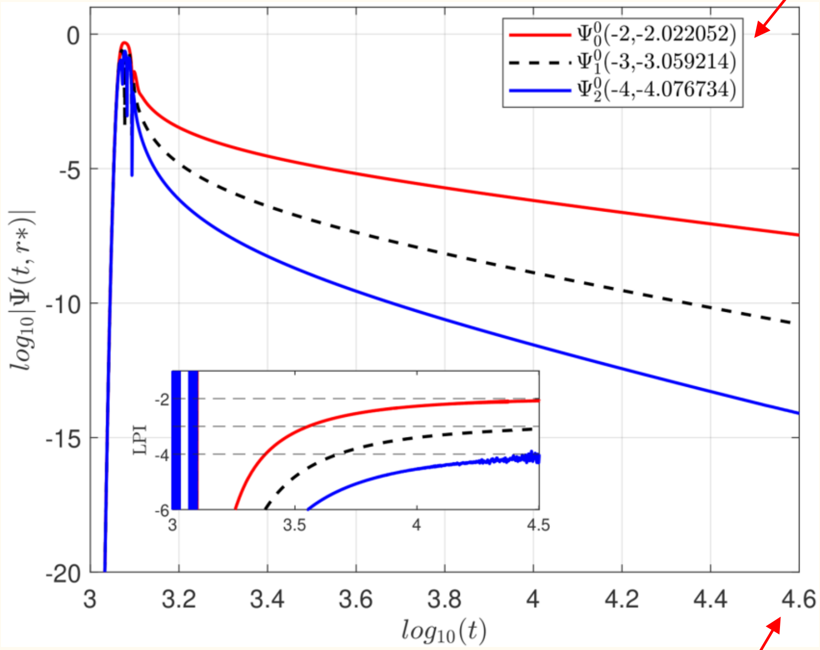


Results and code test I

The tale of tails : Schwarzschild Price Tail



$\Psi \propto t^{-p}$

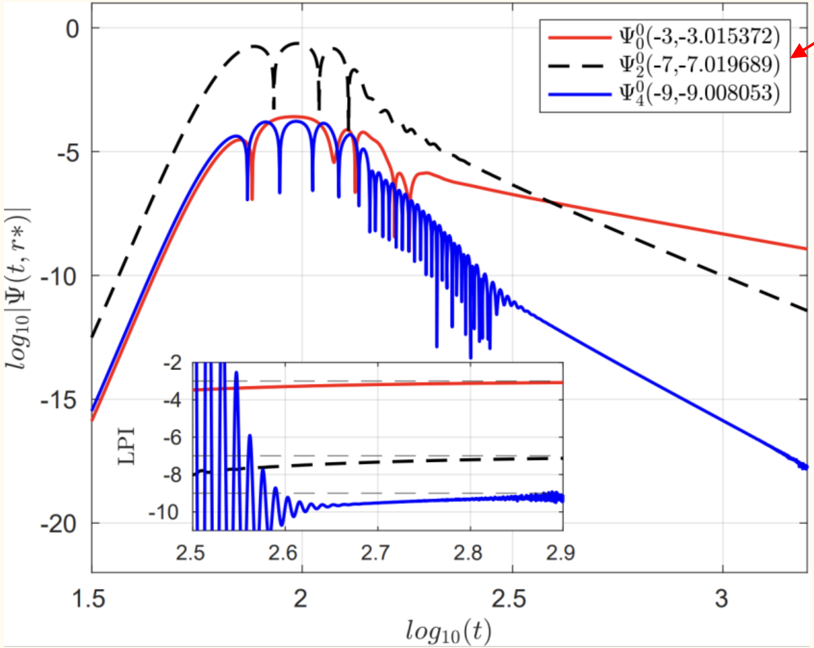


N=10,
subdomains=400,
Signal extracted at a) r*=500 and b) at future null infinity
With zero initial data and a Gaussian(μ=30,σ=10) momentum
Grid [-200,1200] with R=200

[1] Burko, Lior M., and Gaurav Khanna. "Late-time Kerr tails: generic and non-generic initial data sets,'up'modes, and superposition." Classical and Quantum Gravity 28, no. 2 (2011): 025012.

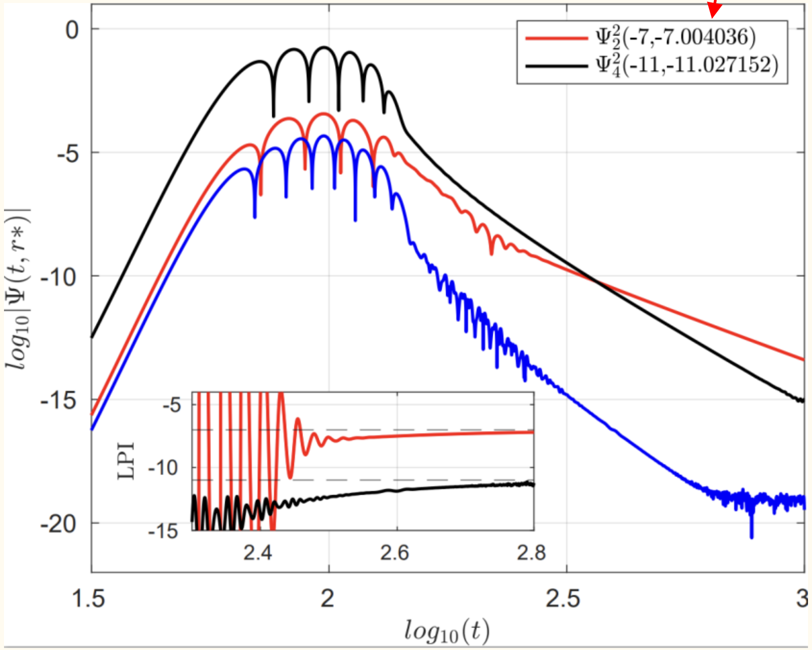
Results and code test II

The tale of tails : Kerr Price Tail



m=0
N=21, subdomains=600,
Signal extracted at r*=100 with primary spin a=0.99995M
With zero initial data and a Gaussian(μ=25,σ=6) momentum
Grid [-2100,1200]

$\Psi \propto t^{-p}$

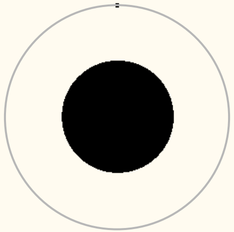


m=2
N=10, subdomains=700,
Signal extracted at r*=100 with primary spin a=0.7M
With zero initial data and a Gaussian(μ=25,σ=6) momentum
Grid [-1100,400] with R=150

[1] Burko, Lior M., and Gaurav Khanna. "Late-time Kerr tails: generic and non-generic initial data sets,'up'modes, and superposition." Classical and Quantum Gravity 28, no. 2 (2011): 025012.

Results and code test II

Scalar Energy Fluxes I



$$
\begin{aligned}
\dot{\psi}_{\ell m} &= -\,\pi_{\ell m} \\
-\left(1+\frac{\Delta a}{(r^2+a^2)}c^{\ell}_{\ell m}-H^2\right)\dot{\pi}_{\ell m} &= -\,(1-H)^2\,\phi'_{\ell m}-2H\,(1-H)\,\pi'_{\ell m} \\
&\quad -H'\,(1-H)\,\pi_{\ell m}-\frac{4imMar}{(r^2+a^2)^2}\pi_{\ell m} \\
&\quad +H'\,(1-H)\,\phi_{\ell m}-V_{\ell m}(r)\psi_{\ell m}+g_{\ell m}(t,r) \\
\dot{\phi}_{\ell m} &= -\,\pi'_{\ell m}.
\end{aligned}
$$

Where for a particle in circular orbit,

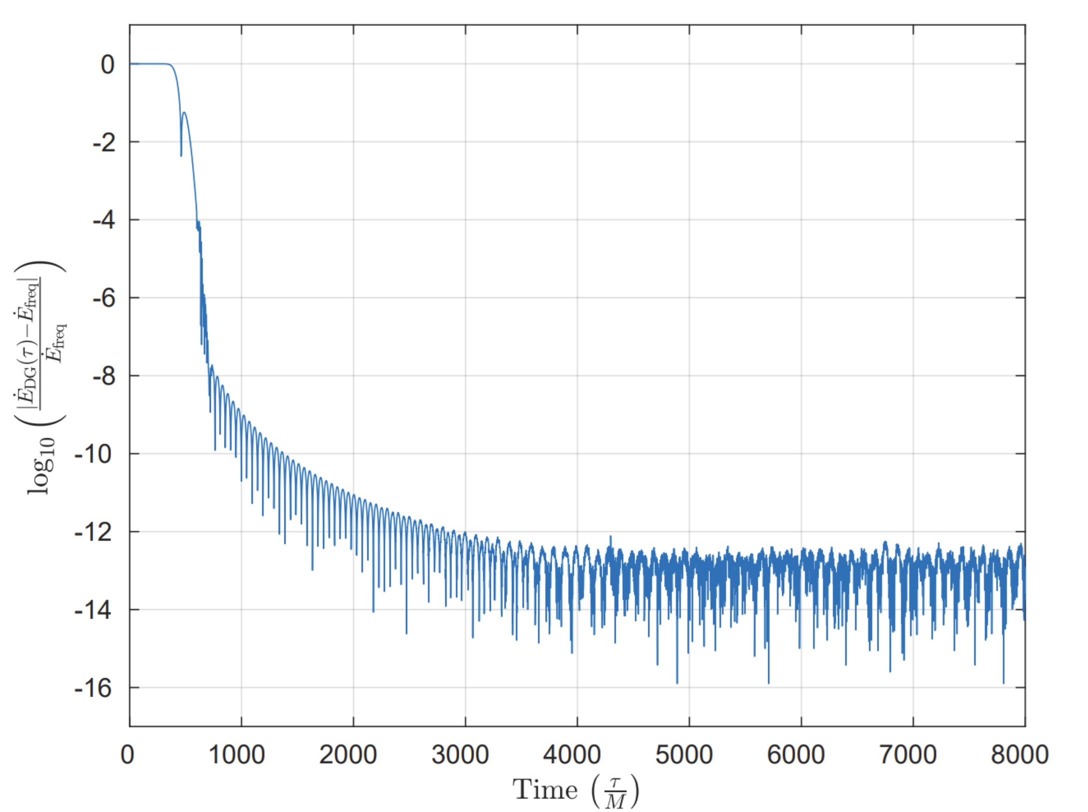
$$
g_{\ell m}(t,r)=\frac{-4\pi q}{u^t\left(r_p^2+a^2\right)^{1/2}}\overline{Y_{\ell m}}\left(\frac{\pi}{2},\phi_p(t)\right)\delta\left(r_*-r_{*,p}\right).
$$

Results and code test II

Scalar Energy Fluxes II

$$\dot{E} = -\frac{1}{4\pi}\sum_{\ell,m}\psi^{\ell m}_{,t}\overline{\psi^{\ell m}_{,r}} = \frac{1}{4\pi}\sum_{\ell,m}\left|\psi^{\ell m}_{,\tau}\right|^2$$

(ℓ,m)	Alg.	Energy Flux	Rel. Error
(2,2)	BHPTK DG	$2.1676683889035\times10^{-6}$ $2.1676683889071\times10^{-6}$	1.7×10^{-12}
(4,2)	BHPTK DG	$3.7609900151243\times10^{-11}$ $3.7609900156162\times10^{-11}$	1.3×10^{-10}
(5,3)	BHPTK DG	$8.8539629089954\times10^{-13}$ $8.8539629113568\times10^{-13}$	2.7×10^{-10}
(9,7)	BHPTK DG	$3.5707073944101\times10^{-14}$ $3.5707073942955\times10^{-14}$	3.2×10^{-11}
(15,15)	BHPTK DG	$2.1814822732028\times10^{-16}$ $2.1814822730317\times10^{-16}$	7.9×10^{-11}



Results and code test II

Scalar Energy Fluxes II

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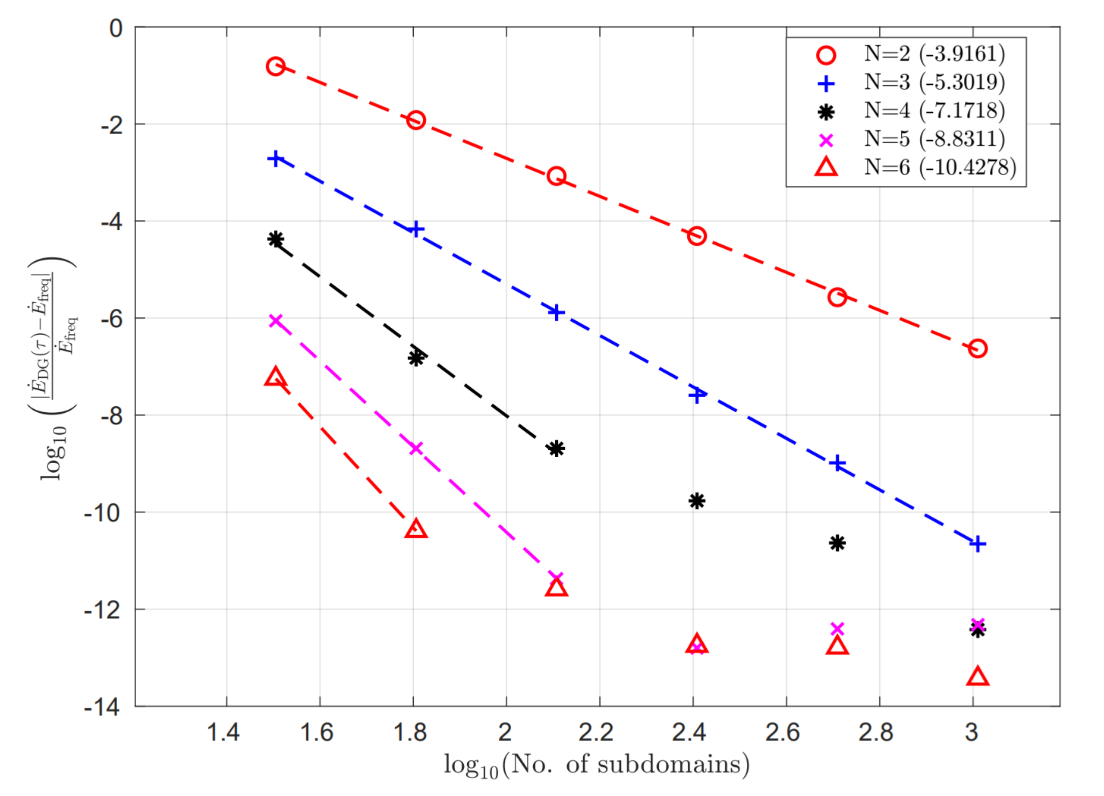
	DG	$3.5707073942955 \times 10^{-14}$	
(15,15)	BHPTK	$2.1814822732028 \times 10^{-16}$	7.9×10^{-11}
	DG	$2.1814822730317 \times 10^{-16}$	



Conclusion (key takeaways and tl;dr)

Work done

- Solved scalar Teukolsky equation in Kerr, with and without the hyperboloidal layers in time-domain
- Modeled the secondary black hole as a delta function in the dG scheme
- Price law verified
- Verified the spectral (super)convergence

Work in progress

- Implementing in C++
- Adding parallelization

Future works

- Computation of gravitational waveforms
- Eccentric orbits
- Efficient, accurate and flexible time domain solver for EMRIs

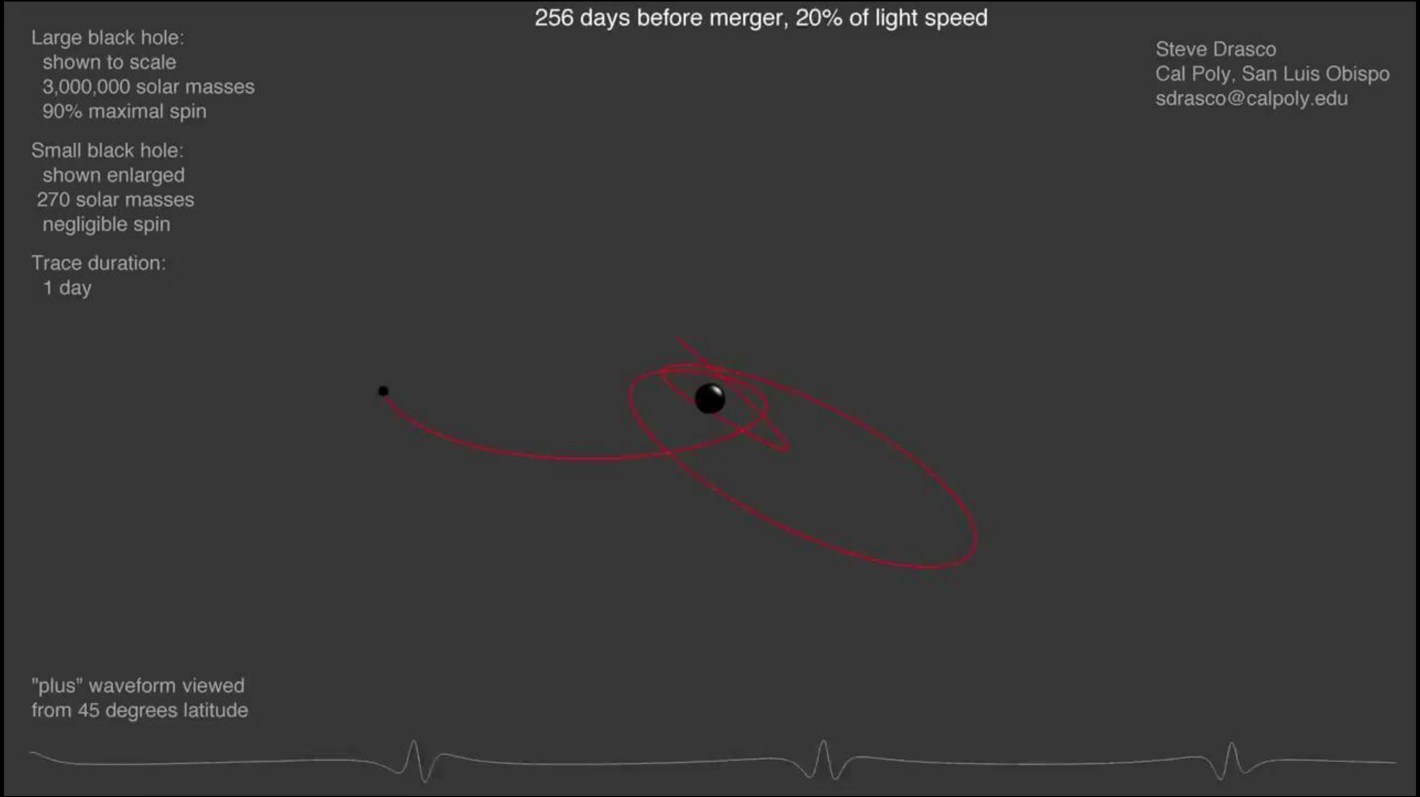
That's all folks!

Questions /Comments?

Relevant paper coming soon to your nearest arxiv server...

Extra Slides

Extreme Mass Ratio Inspirals

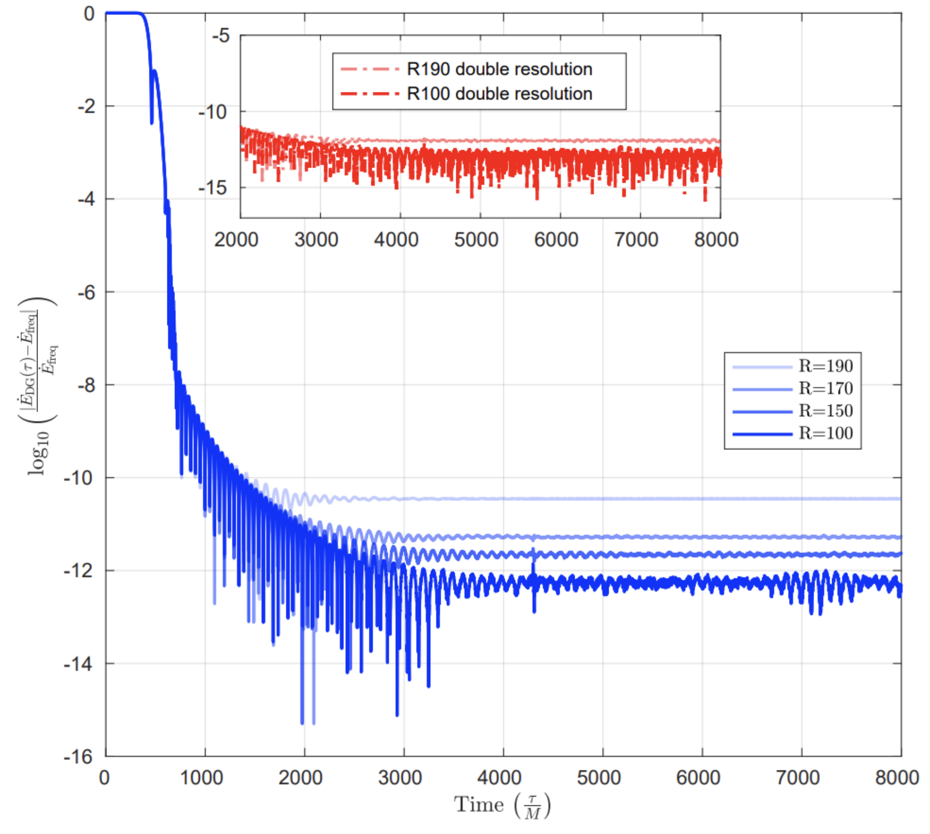


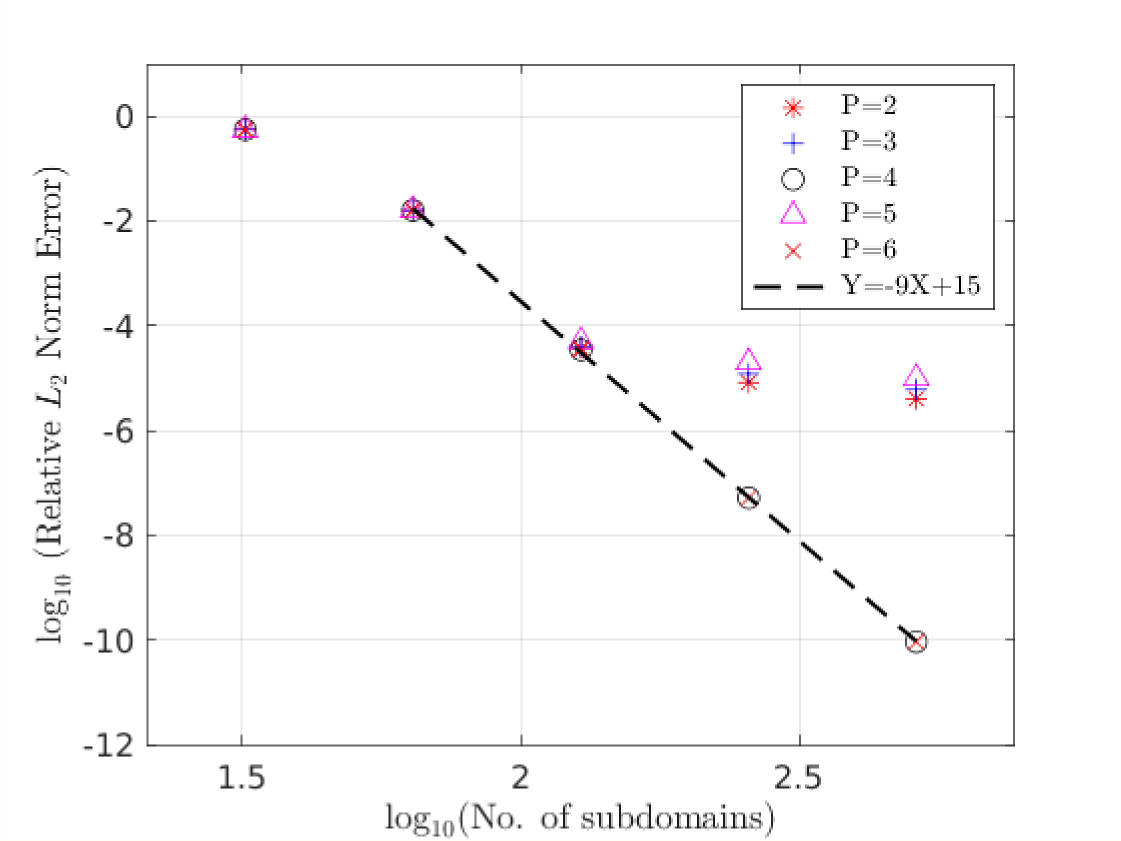
Results and code test II

Scalar Energy Fluxes III

Effects of hyperboloidal layer parameters on the solution :

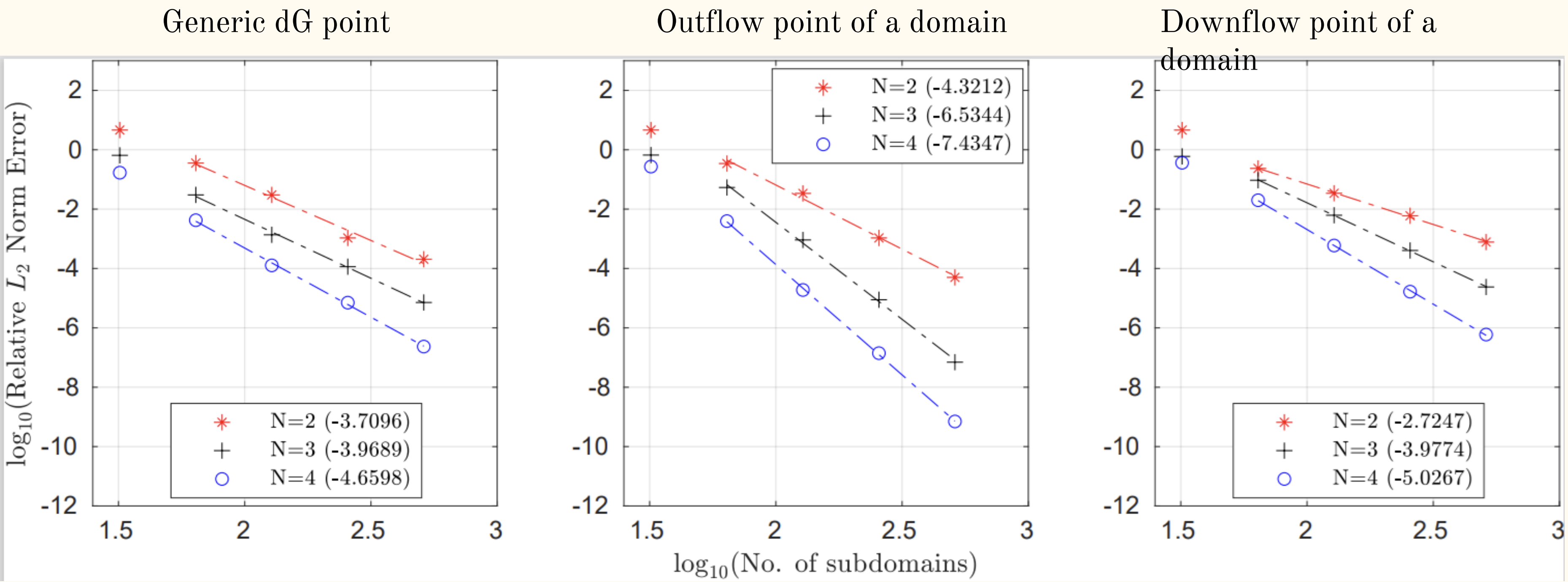
- Values at the null infinity depend on R, S and the resolution
- Can be explained by how well the compactification function is captured





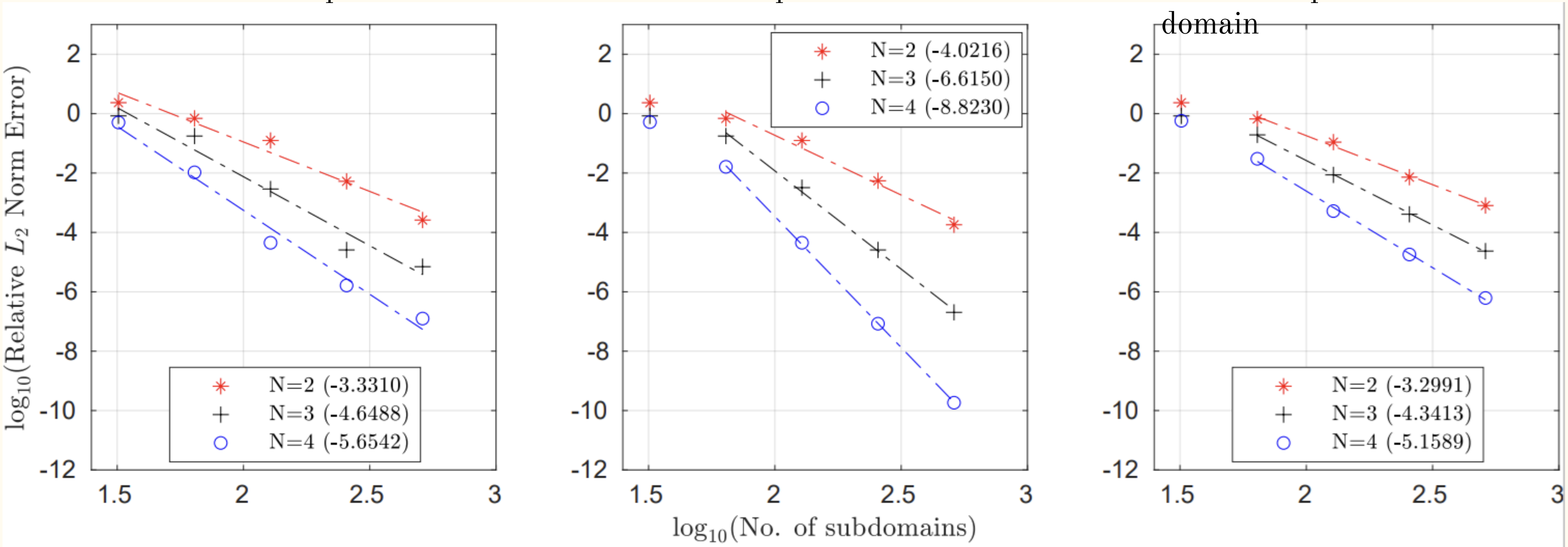
0
-2
-4
-6
-8
-10
-12
1.5
2
2.5
log10 (Relative L2 Norm Error)
log10(No. of subdomains)
P=2
P=3
P=4
P=5
P=6
Y=-9X+15

Error convergence at different points (to left of layers)



Error convergence at different points (inside the layers)

Generic dG point | Outflow point of a domain | Downflow point of a domain



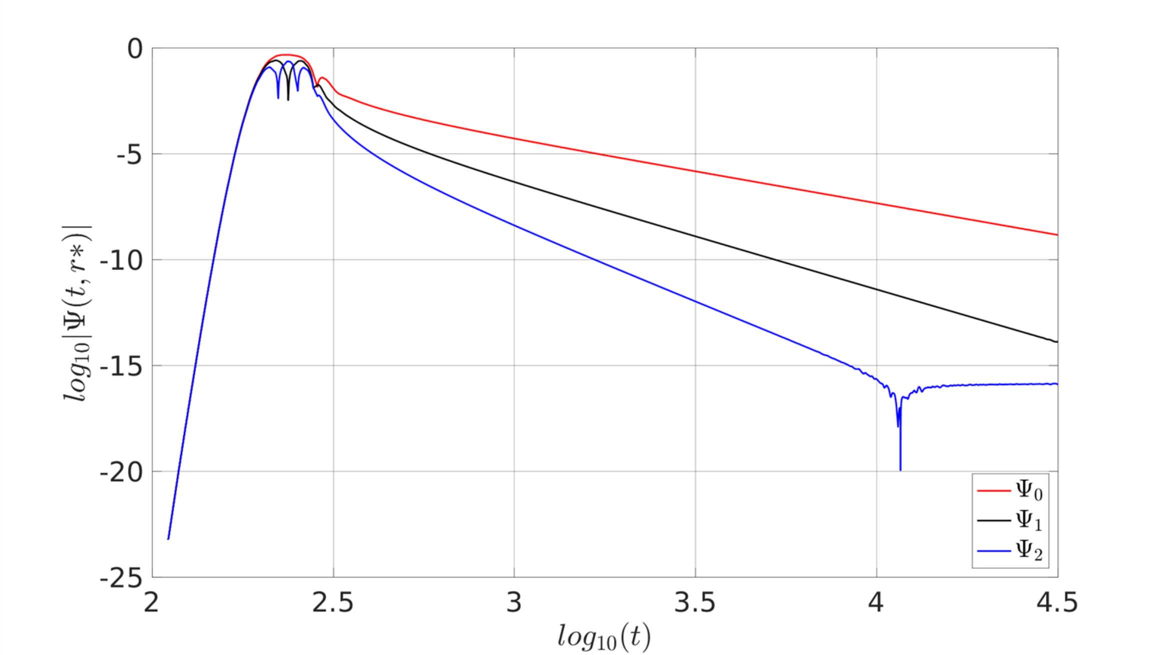
Results and code test : Schwarzschild Price Tail[1] I

Solution extracted inside the layer

N=7,
subdomains=400,
Signal extracted at r*=240M
With moving initial data as Gaussian(30,10)
Grid [-200,1200] with R=200

$$\Psi \propto t^{-p}$$

For moving initial conditions $p = -2l - 3$



[1] Price, Richard H. "Nonspherical perturbations of relativistic gravitational collapse. I. Scalar and gravitational perturbations." Physical Review D 5, no. 10 (1972): 2419.

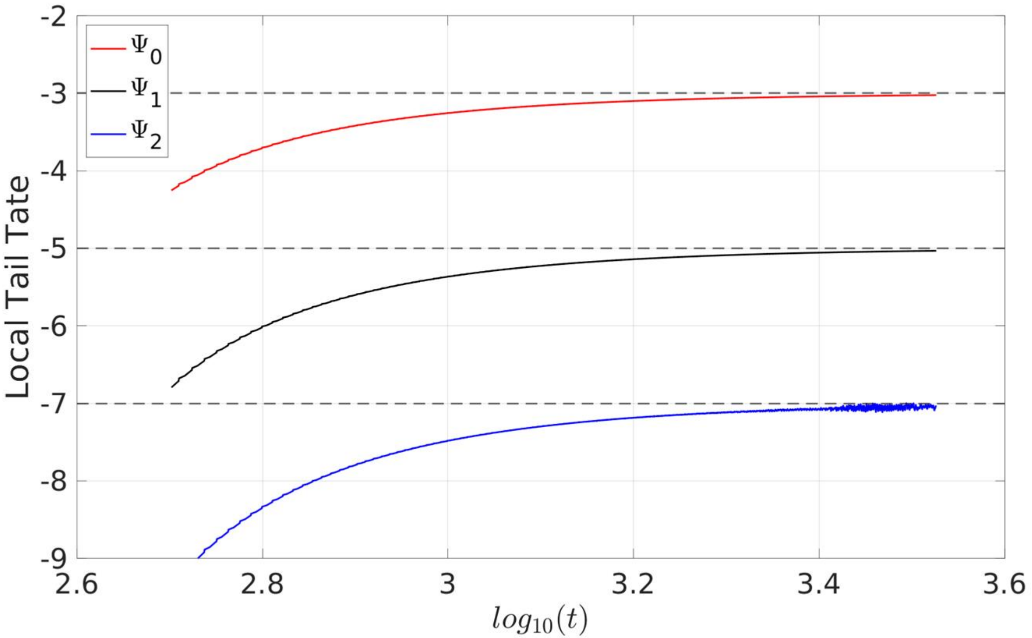
Results and code test : Schwarzschild Price Tail[1] II

Solution extracted inside the layer

N=7,
subdomains=400,
Signal extracted at r*=240M
With moving initial data as Gaussian(30,10)
Grid [-200,1200] with R=200

$$\Psi \propto t^{-p}$$

For moving initial conditions $p = -2l - 3$



[1] Price, Richard H. "Nonspherical perturbations of relativistic gravitational collapse. I. Scalar and gravitational perturbations." Physical Review D 5, no. 10 (1972): 2419.

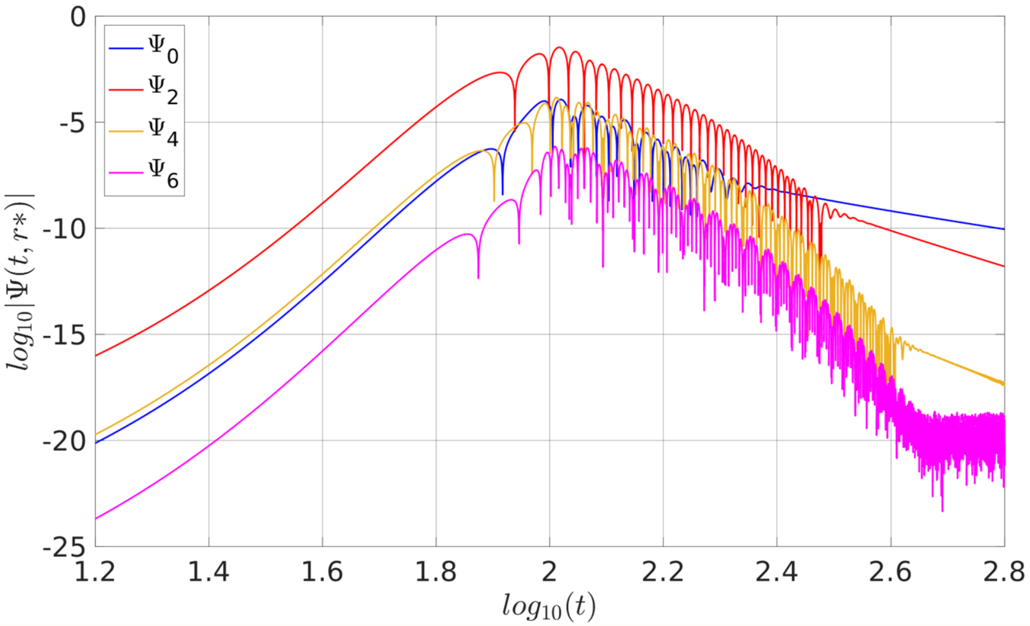
Results and code test : Kerr Price Tail I

N=16,
Subdomains=1000,
Signal extracted at r*=99M and a=0.8

With static Gaussian(2,10) initial data
in l=2 mode

$$\Psi \propto t^{-p}$$

For static initial
conditions $p = -2l - 4$



Results and code test : Kerr Price Tail I

N=16,
Subdomains=1000,
Signal extracted at r*=99M and a=0.8

With static Gaussian(2,10) initial data in l=2 mode

$\Psi \propto t^{-p}$

For static initial conditions $p = -2l - 4$