Resonant dynamics of extreme mass-ratio inspirals in a perturbed Kerr spacetime

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Outline

- **Phenomenology of near-resonance orbits**
  - Poincare map and rotation number ***(3 kinds of orbits)***
- **Unified description in action-angle variables**
  - Simple harmonics
- **Impact for EMRI evolution**
  - Phase shifts

1. Phenomenology: Poincare map

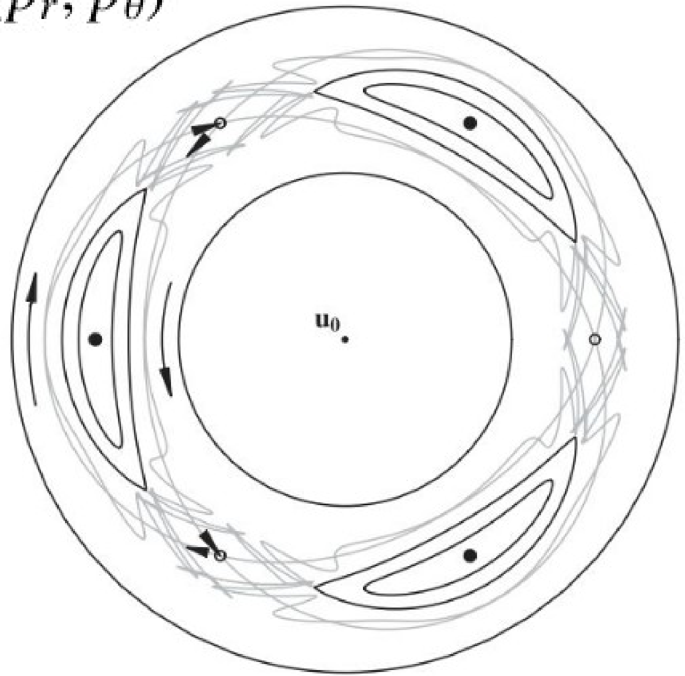
A test particle in Kerr → integrable (r, θ) + (p_r, p_θ)

in perturbed Kerr → non-integrable

Poincare map: $(r, p_r)_{\theta=\pi/2}$

3 kinds of orbits

- *Regular orbits (O1)*
- *Chaotic transitional orbits (O2)*
- *On-resonance orbits (O3)*

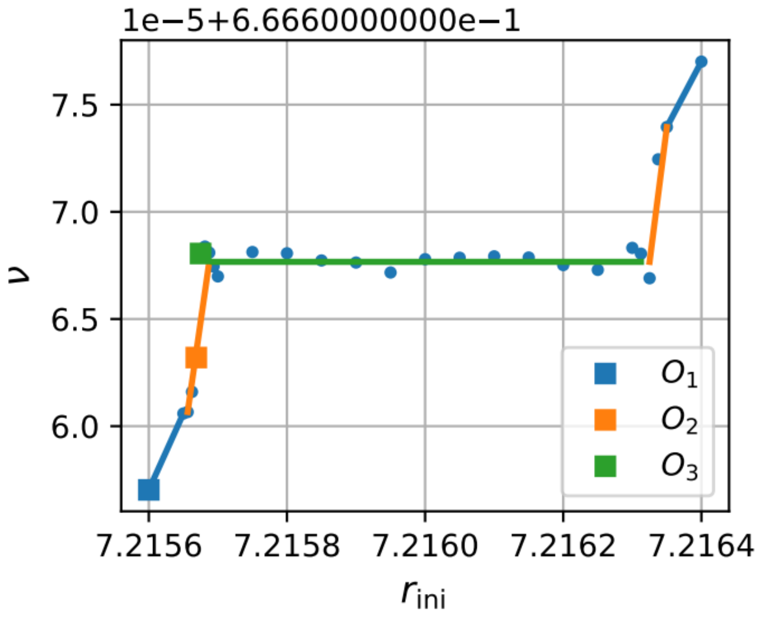


Cartoon Poincare Map
Lukes-Gerakopoulos+ 2010

1. Phenomenology: rotation number

A test particle in a perturbed Kerr: (r, θ) + (p_r, p_θ)

Rotation number: $\nu := \frac{\langle \Omega^r \rangle}{\langle \Omega^\theta \rangle}$



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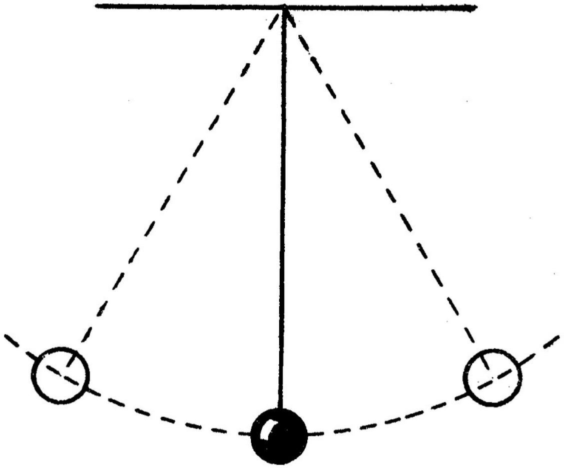
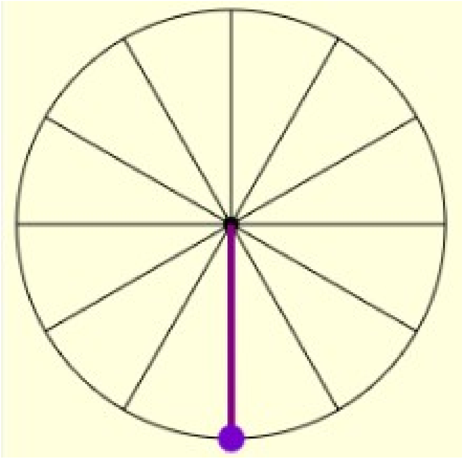
2. Physics and a unified description

Reduced DOF and simple harmonics





Libration → *regular orbits (O1)*

Rotation → *on-resonance orbits (O3)*

Transition → *chaotic transitional orbits (O2)*

Unified description: in action-angle variables

- Kerr spacetime $\mathcal{J}_\alpha = \mathcal{J}_\alpha(\{x^\beta, p_\beta\}), \quad q^\alpha = q^\alpha(\{x^\beta, p_\beta\})\,.$

$$H_0(x^\mu, p_\nu) = \frac{1}{2} g^{\mu\nu}_{\text{Kerr}} p_\mu p_\nu \longrightarrow H_0^{aa} = H_0^{aa}(\mathcal{J}_\alpha)$$

$$\dot{\mathcal{J}}_\alpha = -\frac{\partial H_0}{\partial q^\alpha} = 0, \quad \dot{q^\alpha} = \frac{\partial H_0}{\partial \mathcal{J}_\alpha} = \Omega^\alpha$$

$$\text{Resonance}: \quad n_r \Omega^r + n_\theta \Omega^\theta = 0$$

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- Perturbed Kerr

$$H(x^\mu, p_\nu) = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu = H_0 + \epsilon H_{\mathrm{int}} \longrightarrow H_0(\mathcal{J}_\alpha) + \epsilon H_{\mathrm{int}}(\mathcal{J}_\alpha, q^\beta)$$

$$\frac{d\mathcal{J}_\alpha}{d\tau} = -\epsilon \frac{\partial \mathcal{H}_{\mathrm{int}}}{\partial q^\alpha}, \qquad \frac{dq^\alpha}{d\tau} = \Omega_\alpha + \epsilon \frac{\partial \mathcal{H}_{\mathrm{int}}}{\partial \mathcal{J}_\alpha},$$

Near Identity Transformation:
resonance as a driven force

$$\tilde{q}^\alpha = q^\alpha + \epsilon \sum_{n_\alpha \Omega^\alpha \neq 0} \frac{1}{n_r \Omega^r + n_\theta \Omega^\theta} \frac{\partial H_{n_j}}{\partial \mathcal{J}_\alpha} e^{i(n_r q^r + n_\theta q^\theta)},$$

$$\tilde{J}_\alpha = J_\alpha + \epsilon \sum_{n_\alpha \Omega^\alpha \neq 0} \frac{n_\alpha}{n_r \Omega^r + n_\theta \Omega^\theta} H_{n_j} e^{i(n_r q^r + n_\theta q^\theta)}.$$

We have

$$\frac{d\tilde{q}^\alpha}{d\tau} = \Omega_\alpha + \epsilon \sum_{k \in \mathbb{Z}} \frac{\partial H_{k\mathbf{N}}}{\partial \mathcal{J}_\alpha} e^{ik(N_r q^r + N_\theta q^\theta)},$$

$$\frac{d\tilde{\mathcal{J}}_\alpha}{d\tau} = -i\epsilon N_\alpha \sum_{k \in \mathbb{Z}} k H_{k\mathbf{N}} e^{ik(N_r q^r + N_\theta q^\theta)}.$$

e.g., 2/3 resonance $\mathbf{N} = (N_r, N_\theta) = (-3, 2)$

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Not all action angle variables are independent.

$$\frac{dQ}{d\tau} := \frac{d}{d\tau}(N_r \tilde{q}^r + N_\theta \tilde{q}^\theta) = N_\alpha \Omega^\alpha + O(\epsilon) = \Delta\Omega = O(\epsilon)$$

$$\frac{d\Theta}{d\tau} := \frac{1}{N_r}\frac{d\tilde{J}_r}{d\tau} = \frac{1}{N_\theta}\frac{d\tilde{J}_\theta}{d\tau} = \sum H_{k\mathbf{N}} e^{ikQ} = O(\epsilon)$$

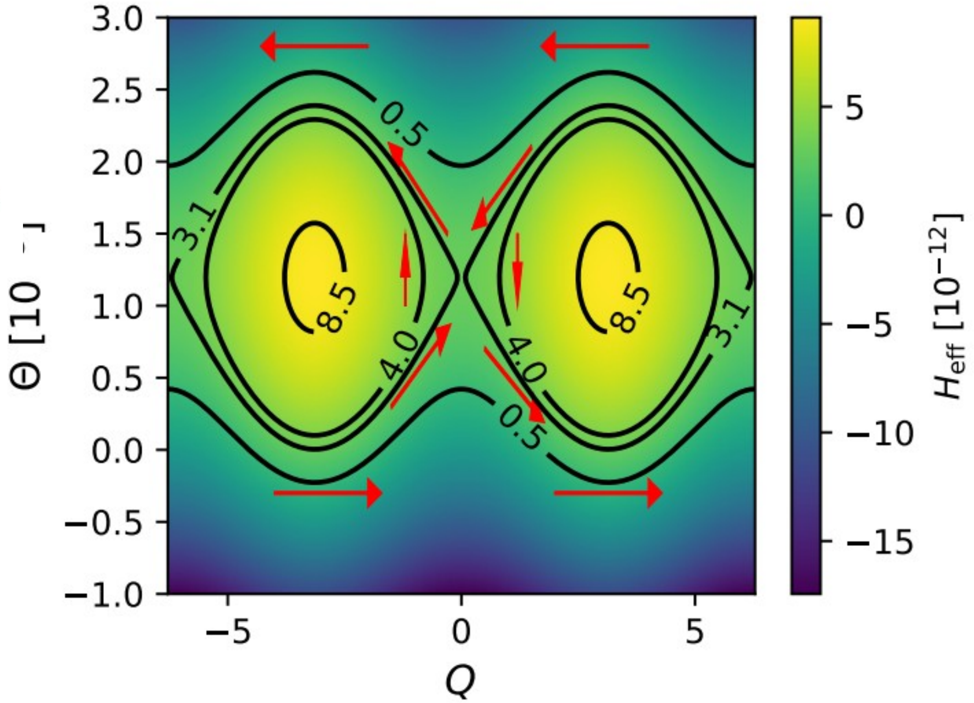
Near-resonance orbits :

1 d.o.f effective Hamiltonian

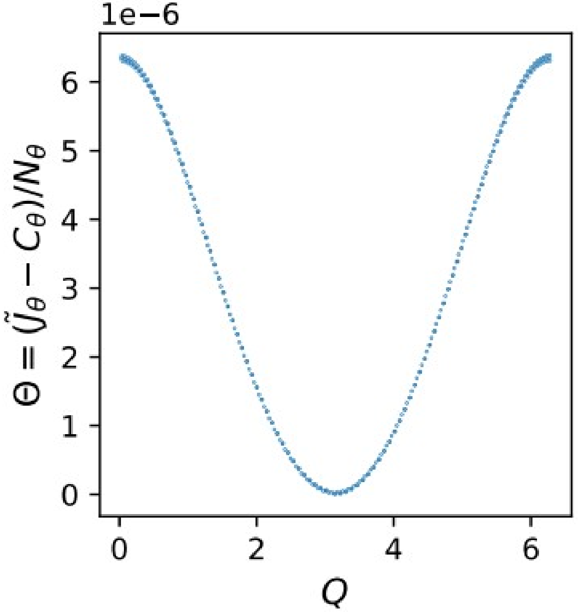
$$H_{\mathrm{eff}} := H_{\mathrm{kin}} + H_{\mathrm{pot}} = \int_0^{\Theta} \Delta\Omega(\Theta)d\Theta + \sum_k H_{k\mathbf{N}} e^{ikQ} \ ,$$

$$\frac{dQ}{d\tau} = \frac{\partial H_{\mathrm{eff}}}{\partial \Theta} \ ,$$

$$\frac{d\Theta}{d\tau} = -\frac{\partial H_{\mathrm{eff}}}{\partial Q} \ .$$



Numerical check: $(x,p) \to (q,J) \to (Q,\Theta)$



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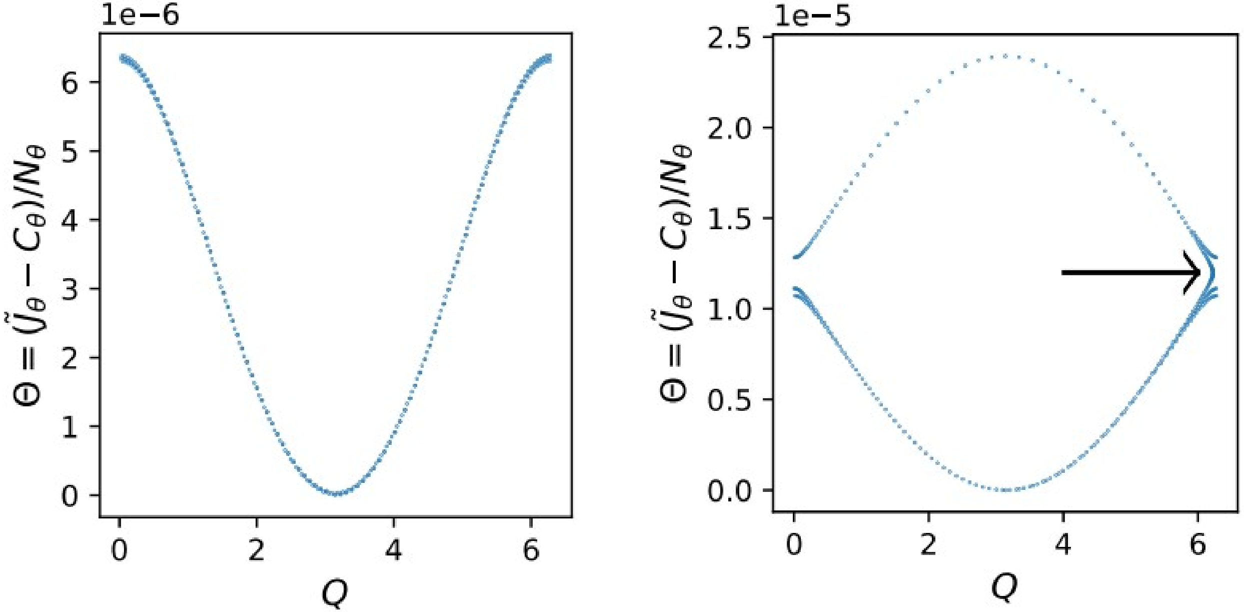

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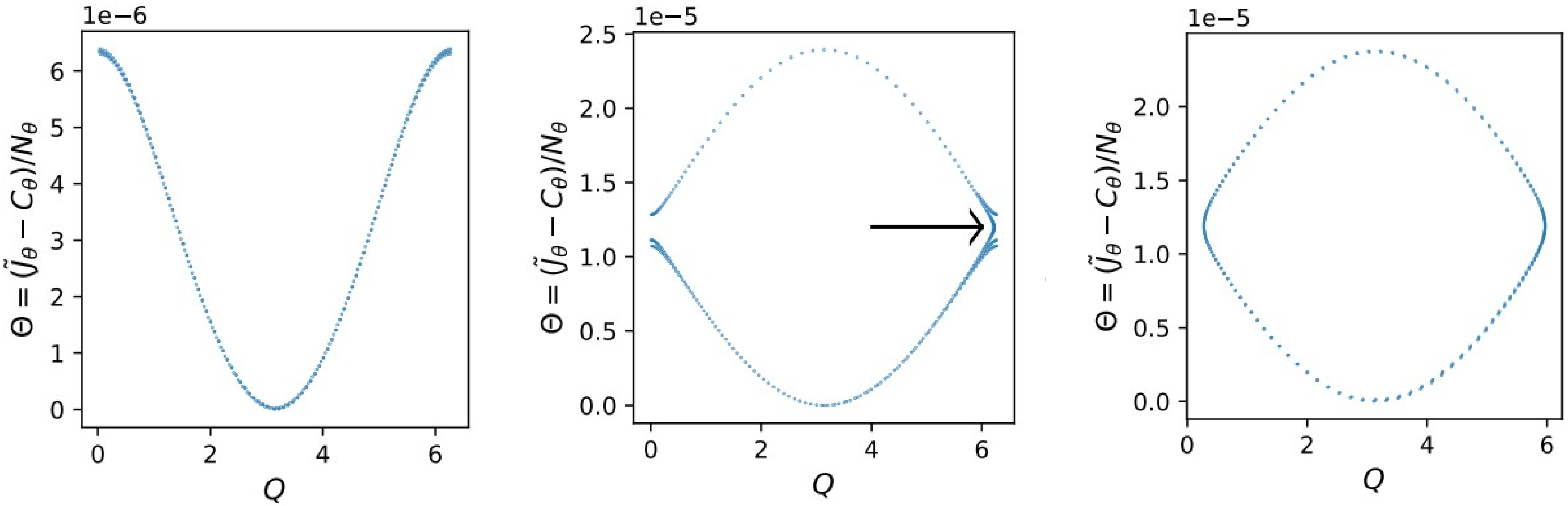


3. Impact for EMRI evolution

$$\frac{dQ}{d\tau} = \sum_{\alpha} a^{\alpha} \tilde{\mathcal{J}}_{\alpha}$$

$$\frac{d\tilde{\mathcal{J}}_{\alpha}}{d\tau} = -N_{\alpha} H_{\mathrm{res}} \sin Q + q\, G_{\alpha}$$

$$\Delta \tilde{\mathcal{J}}_{\alpha} = -N_{\alpha} H_{\mathrm{res}} \int_{-\infty}^{\infty} \sin Q(\tau) d\tau$$

Non-adiabatic limit:

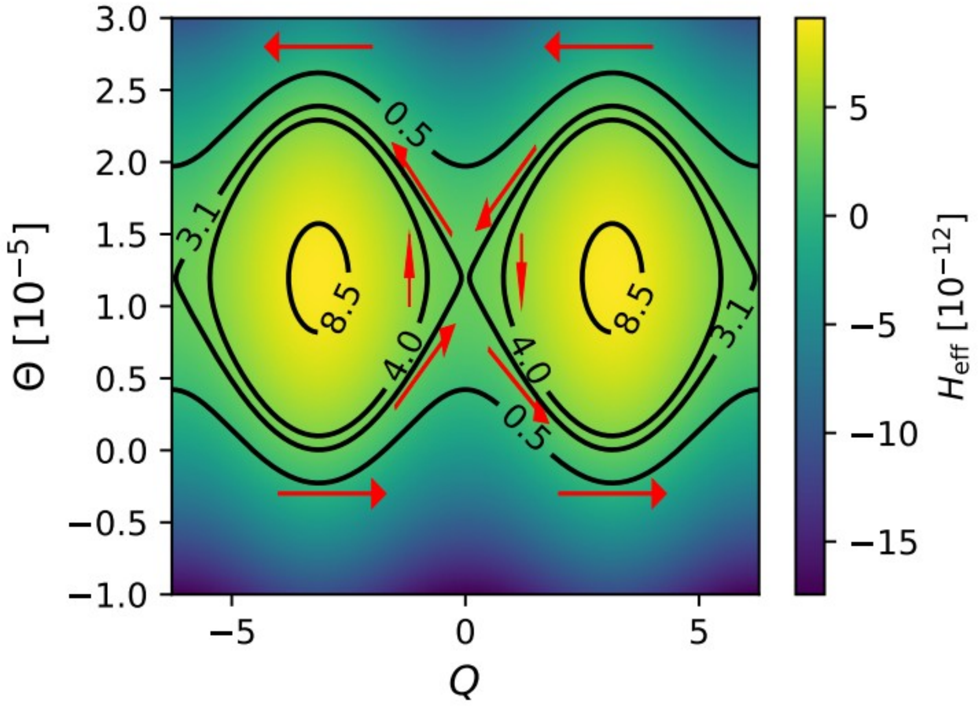
t_cross < t_res (h_res < q = m/M)

Crossing the resonance via passing through all the orbits

$$\delta J \sim h_{\rm res} t_{\rm cross} \sim h_{\rm res}/\sqrt{q}$$

$$\delta\Psi \sim \delta J/q \sim h_{\rm res}/q^{3/2}$$

$$\delta\Psi|_{\rm Kerr} \sim 1/q^{1/2}$$ (Flanagan+2012)



Adiabatic limit: t_cross > t_res (h_res > q)

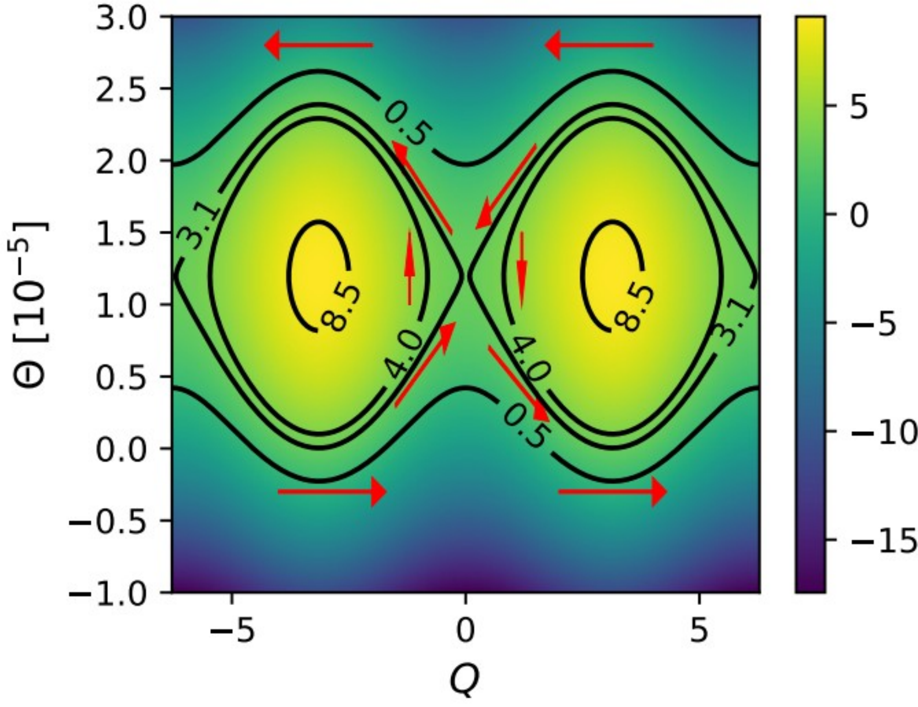
Area conserved as an adiabatic invariant $\int \Theta dQ$

Circumventing resonance via chaotic transitional orbits.

$$\Delta\tilde{J}_\alpha = -N_\alpha H_{\rm res} \int \sin Q(\tau) d\tau + \Delta\tilde{J}_{\alpha,transit}$$

$$\delta J \approx \delta\Theta \sim \sqrt{h_{\rm res}}$$

$$\delta\Psi \sim \delta J/q \sim \sqrt{h_{\rm res}/q}$$



Summary

- **Rich phenomena of resonant dynamics can be understood with effective Hamiltonian of simple harmonics.**
- **Only orbits in the transitional regime are chaotic.**
- **Impact for resonance crossing**
  - *In adiabatic regime, resonance crossing does not happen because resonances are avoided by the transitional orbits.*

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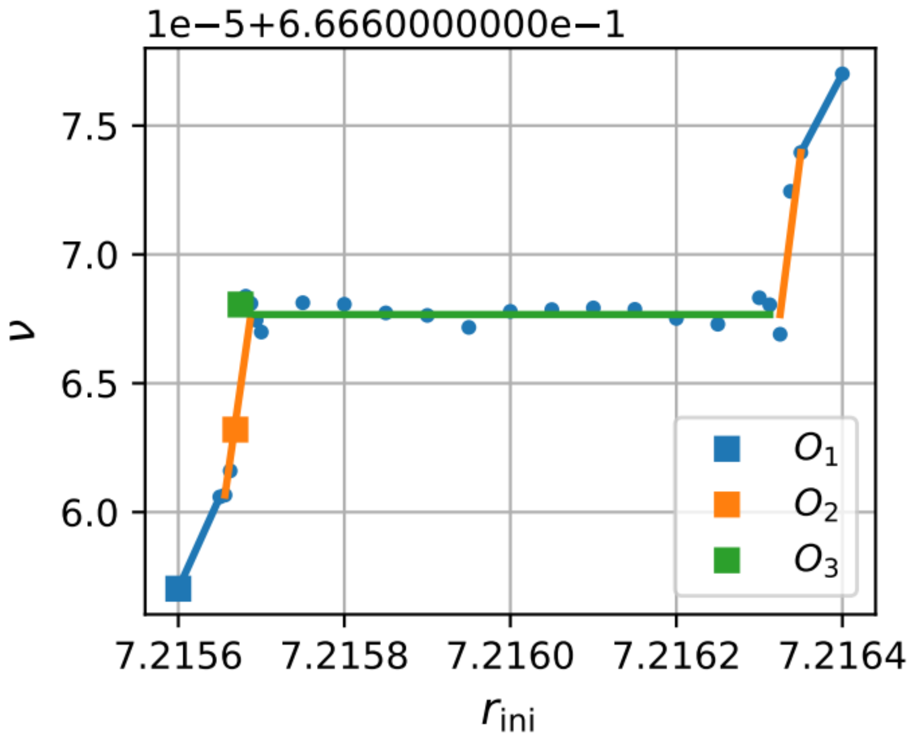
Near-resonance orbits : rotation numbers



$$\nu := \frac{\langle \Omega^r \rangle}{\langle \Omega^\theta \rangle}$$

3 kinds of orbits:

O1, O2, O3

Integrable systems and KAM theorem

- **Integrable system:** # degrees of freedom = # conserved quantities

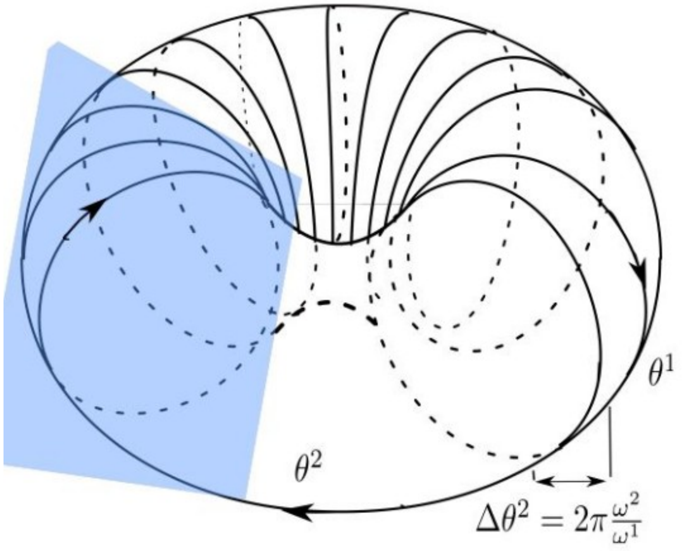
  e.g. a test particle in the Kerr spacetime:

  $$(t, r, \theta, \phi) \longleftrightarrow (H, E, L, C)$$

  e.g. 2-d.o.f. integrable system:

  orbit wraps on a 2-torus

  (Cardenas-Avendano+2018)

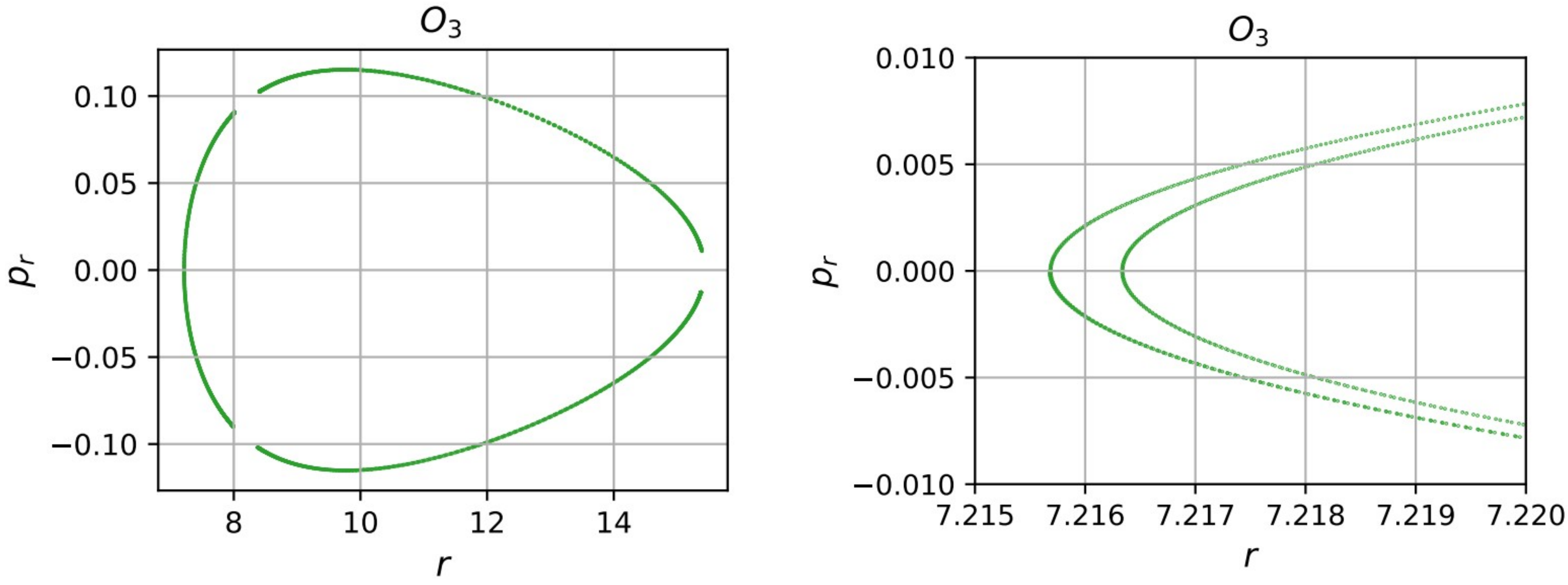


- **KAM theorem:**

  an integrable system + a perturbation → non-integrable

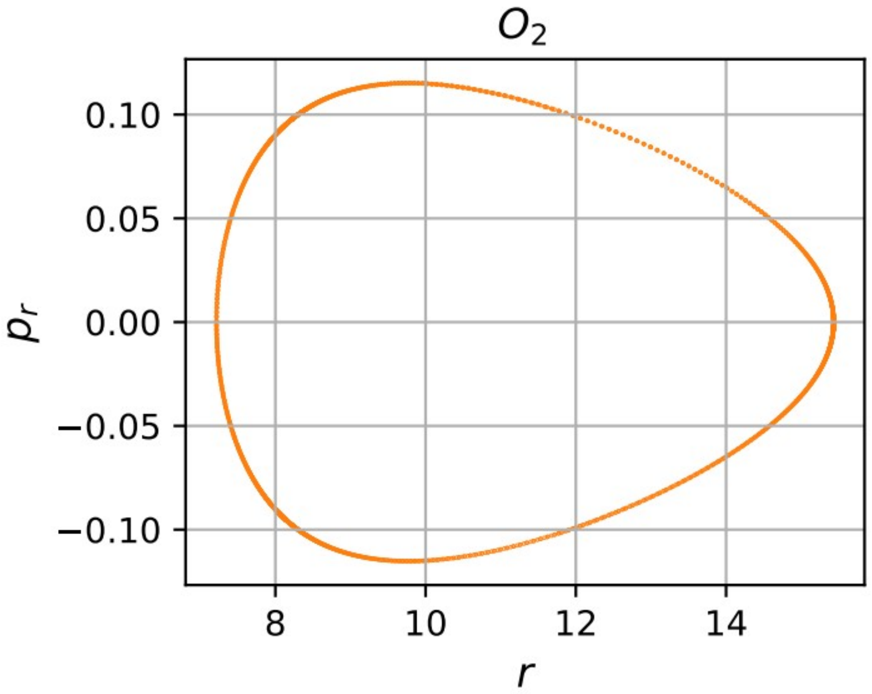
  Chaos can occur at resonances. *e.g. Quadractic Gravity (Donoghue+2021)* $g = g_{\text{Kerr}} + h$

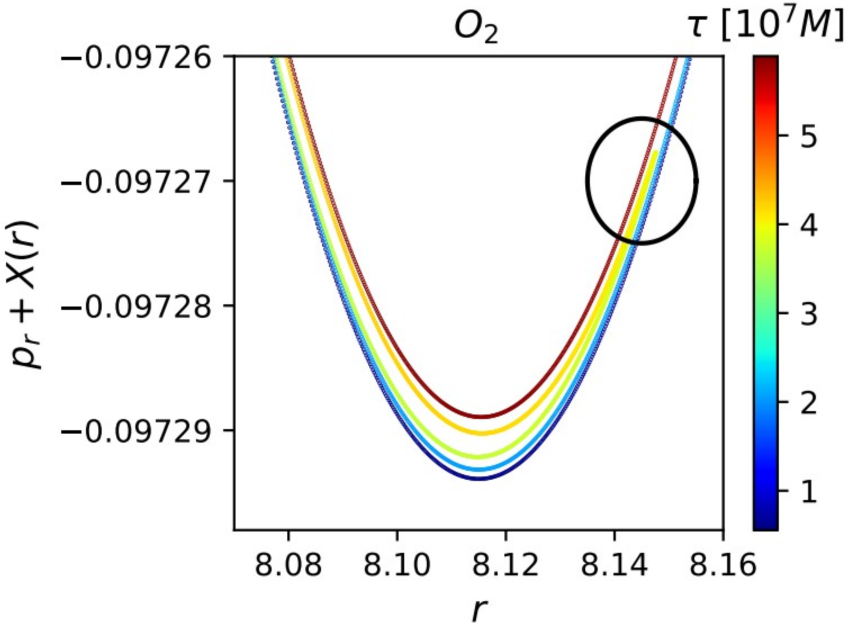
Near-resonance orbits : phenomenology



Poincare maps of O3 → On-resonance Orbit

Near-resonance orbits : phenomenology

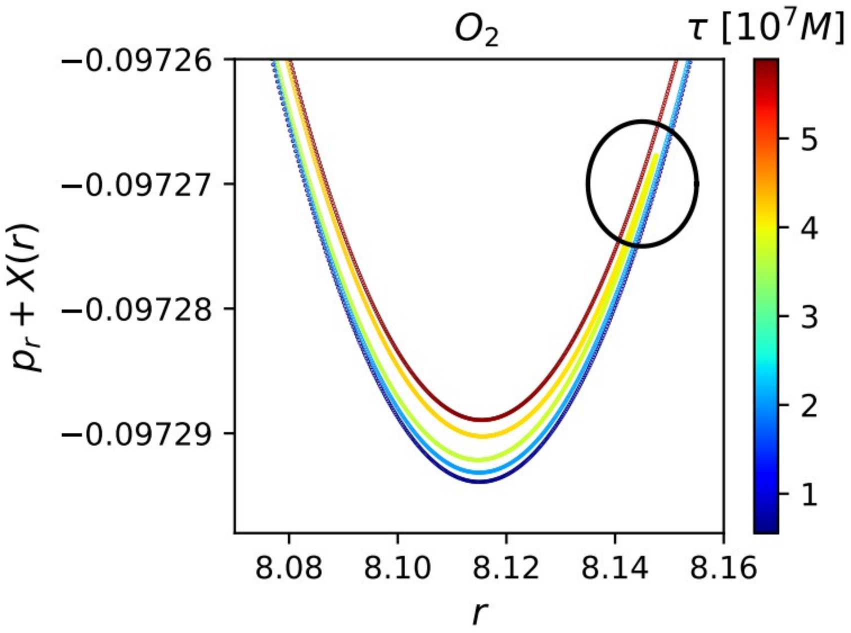




Poincare maps of O2 → Chaotic Transitional Orbit

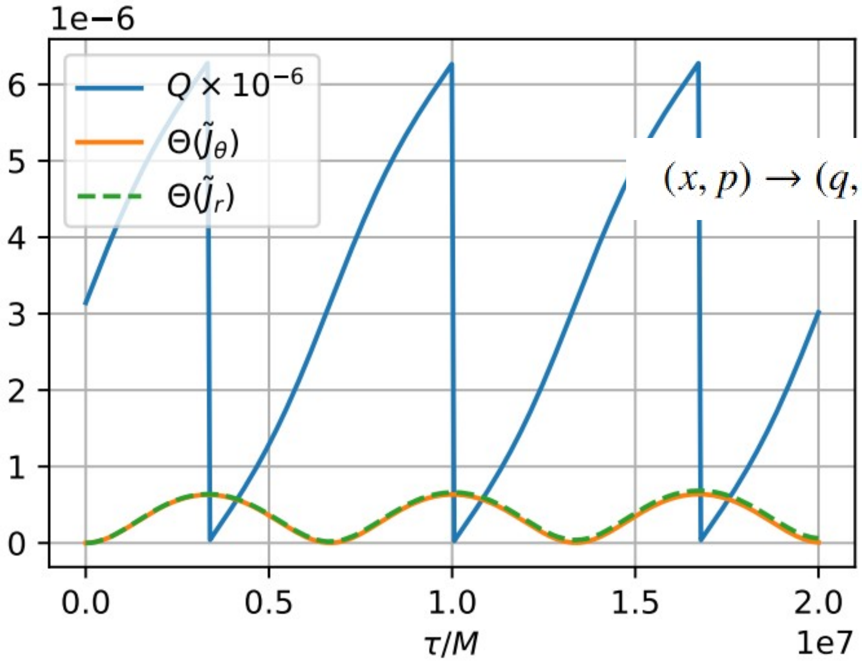
O2 Poincare map:

e.g. spinning BH in Qudractic Gravity *(Donoghue+2021)*

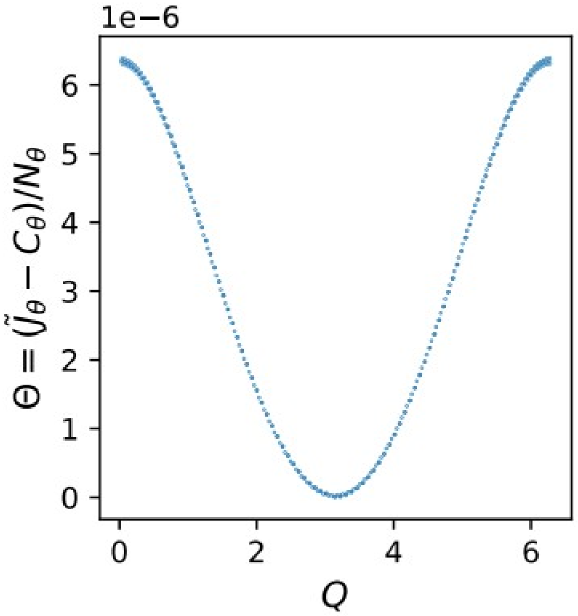


Anti-clockwise → clockwise

Regular orbit O1



$$(x, p) \to (q, J) \to (Q, \Theta)$$



Chaotic transitional orbit O2