Distributional sources for the second-order Einstein field equations

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  - Derive a counter term that cancels divergence in the second-order source at the worldline
  - Demonstrate that counter term is equivalent to result obtained through Hadamard finite part regularisation

Detweiler stress-energy tensor: Highly regular gauges [SDU & AP, 2101.11409]

- Well-defined EFEs at second order

$$\delta G^{\mu\nu}[h^2] + \delta^2 G^{\mu\nu}[h^1, h^1] = 8\pi T_2^{\mu\nu}$$

where

$$T_2^{\mu\nu} = -\frac{m}{2}\int u^\mu u^\nu (g^{\alpha\beta} - u^\alpha u^\beta) h^{\mathrm{R}1}_{\alpha\beta}\delta^4(x, z)\, d\tau$$

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- Weak divergence of metric perturbations means automatically true in highly regular gauge
  - Most singular parts of EFEs cancel

$$\delta^2 G^{\mu\nu}[h^{\mathrm{S1}}, h^{\mathrm{S1}}] = -\delta G^{\mu\nu}[h^{\mathrm{SS}}], \qquad \forall r$$

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$$\delta^2 G^{\mu\nu}[h^{\mathrm{S1}}, h^{\mathrm{S1}}] \coloneqq -\delta G^{\mu\nu}[h^{\mathrm{SS}}], \qquad \forall r$$

- Motivated by highly regular gauge where this is automatically true

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where

$$\begin{aligned}\delta^2 G_s^{\mu\nu}[h^1, h^1] \coloneqq \Big(-\delta G^{\mu\nu}[h^{\rm SS}] + 2\,\overbrace{Q^{\mu\nu}[h^{\rm S1}]}^{\delta^2 G^{\mu\nu}[h^{\rm S1}, h^{\rm R1}]} + \delta^2 G^{\mu\nu}[h^{\rm R1}, h^{\rm R1}]\Big)\theta(s-r) \\ + \delta^2 G^{\mu\nu}[h^1, h^1]\theta(r-s)\end{aligned}$$

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- Leads to a consistent result for $T_2^{\mu\nu}$ in highly regular and Lorenz gauges

Using canonical definition in source for Lorenz gauge

- How can we use this to solve for $h^2_{\mu\nu}$?

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- After imposing Lorenz gauge conditions

$$\mathcal{E}_{\mu\nu}[\bar{h}^2] = -16\pi T^2_{\mu\nu} + 2\delta^2 G_{\mu\nu}[h^1, h^1]$$

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- Want to extract delta content of $\delta^2 G_{\mu\nu}[h^1, h^1]$ to put in practical form

Distributional analysis

- Adjoint of linear operator

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- Using these and Detweiler canonical definition, we get

$$\begin{aligned}\int \phi^{\mu\nu}\delta^2 G_{\mu\nu}[h^1, h^1]\, dV = \lim_{s\to 0}\Bigg[&\int \phi_{\mu\nu}\Big(-\delta G_{\mu\nu}[h^{\rm SS}] + 2Q_{\mu\nu}[h^{\rm S1}] \\ &+ \delta^2 G_{\mu\nu}[h^{\rm R1}, h^{\rm R1}]\Big)\theta(s-r)\, dV \\ &+ \int_{r>s} \phi^{\mu\nu}\delta^2 G_{\mu\nu}[h^1, h^1]\, dV\Bigg]\end{aligned}$$

“Singular times singular” term

- Einstein operator is self-adjoint, $\delta G^{\dagger}_{\mu\nu}[h] = \delta G_{\mu\nu}[h]$ [Wald, PRL, 1978], so

$$\int \phi_{\mu\nu}\delta G^{\mu\nu}[h^{\rm SS}]\theta(s-r)\,dV := \int \delta G_{\mu\nu}[\theta(s-r)\phi]h^{\mu\nu}_{\rm SS}\,dV$$

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- Use distributional identities

$$\begin{aligned}\int \delta G_{\mu\nu}[\theta(s-r)\phi]h^{\mu\nu}_{\rm SS}\,dV &= \lim_{R\to 0}\Big(\int_{r>R}\phi_{\mu\nu}\delta G^{\mu\nu}[h^{\rm SS}]\theta(s-r)\,dV \\ &\qquad - \int_{r=R} K^{\delta G}_{\alpha}[\theta(s-r)\phi, h^{\rm SS}]\,dS^{\alpha}\Big) \\ &= -\frac{4m^2\pi}{3s}\int (7g_{\mu\nu} - 2u_{\mu}u_{\mu})\phi^{\mu\nu}\,dt\end{aligned}$$

Stress-energy counter term

- Write result from previous slide as stress-energy tensor

$$\int \phi^{\mu\nu} T^{\text{counter}}_{\mu\nu}\, dV := -\frac{1}{8\pi}\int \delta G_{\mu\nu}[\theta(s-r)\phi] h^{\mu\nu}_{\text{SS}}\, dV$$

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- As $\phi^{\mu\nu}$ is a test field,

$$T^{\text{counter}}_{\mu\nu} = \frac{m^2}{6s}\int (7g_{\mu\nu} - 2u_\mu u_\mu)\delta^4(x,z)\, d\tau$$

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- We can follow the same procedure for $Q_{\mu\nu}[h^{\text{S1}}]$ to find

$$T^{Q}_{\mu\nu} = \frac{m}{3} \int U^{\alpha\beta}{}_{\mu\nu} h^{\text{R1}}_{\alpha\beta} \delta^4(x,z) \, d\tau$$

where $U^{\alpha\beta}{}_{\mu\nu}$ is a function of the metric and 4-velocity (similar given in [2101.11409] but for indices up)

Final reformulation of the source

- By using canonical definition, we have

$$\mathcal{E}_{\mu\nu}[\bar{h}^2] = -16\pi T^2_{\mu\nu} + 2\delta^2 G_{\mu\nu}[h^1, h^1]$$

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- Equation has solution (for some Green's function ${G_{\mu\nu}}^{\mu'\nu'}(x;x')$)

$$\bar{h}^2_{\mu\nu} = -16\pi\int {G_{\mu\nu}}^{\mu'\nu'}(T^2_{\mu'\nu'} - T^Q_{\mu'\nu'})\,dV' + 2\lim_{s\to 0}\Bigg(\int_{r'=s}^{\infty} {G_{\mu\nu}}^{\mu'\nu'}\delta^2 G_{\alpha'\beta'}[h^1, h^1]\,dV' + \frac{4\pi m^2}{3s}\int_\gamma {G_{\mu\nu}}^{\mu'\nu'}(7g_{\mu'\nu'} - 2u_{\mu'}u_{\nu'})\,d\tau\Bigg)$$

Comparison with regularisation methods: Hadamard finite part

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- Introduce regularising factor $(r/L)^B$ where L has dimensions of length and $B \in \mathbb{C}$
- Finite value for integral

$$\mathop{\mathrm{FP}}_{B=0} \int_{-1}^{1} \left(\frac{r}{L}\right)^{B} \frac{1}{r^2}\, dr = -2$$

Rewriting the limit term

- Compare finite part regularisation of $\delta^2 G_{\mu\nu}[h^1, h^1]$ against limit term

$$\bar{h}^2_{\mu\nu} = -16\pi \int G_{\mu\nu}{}^{\mu'\nu'} (T^2_{\mu'\nu'} - T^Q_{\mu'\nu'})\, dV' + 2 \operatorname*{FP}_{B=0} \int \left(\frac{r'}{L}\right)^B G_{\mu\nu}{}^{\mu'\nu'}(x;x') \delta^2 G_{\mu'\nu'}[h^1, h^1]\, dV'$$

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= & -16\pi \int G_{\mu\nu}{}^{\mu'\nu'}(T^2_{\mu'\nu'} - T^Q_{\mu'\nu'})\, dV' + 2 \lim_{s\to 0}\Bigg(\int_{r'=s}^{\infty} G_{\mu\nu}{}^{\mu'\nu'}\delta^2 G_{\mu'\nu'}[h^1, h^1]\, dV' \\
& + \underset{B=0}{\mathrm{FP}} \int_{r'=0}^{s} \left(\frac{r'}{L}\right)^B G_{\mu\nu}{}^{\mu'\nu'}\delta^2 G_{\mu'\nu'}[h^1, h^1]\, dV'\Bigg)
\end{aligned}
$$

Calculating the finite part

- Calculate remaining FP term

$$\underset{B=0}{\mathrm{FP}} \int_{r'=0}^{s} \left(\frac{r'}{L}\right)^{B} G_{\mu\nu}{}^{\mu'\nu'}(x;x')\delta^2 G_{\mu'\nu'}[h^1,h^1] r'^2 \, dt' \, dr' \, d\Omega'$$
$$= \frac{4\pi m^2}{3s} \int G_{\mu\nu}{}^{\mu'\nu'}(x;t',0)(7g_{\mu'\nu'} - 2u_{\mu'}u_{\nu'} + \mathcal{O}(s^2)) \, dt'$$

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- Exactly the same counter term as before

Conclusions and future work

- Reformulated second-order EFEs to use Detweiler stress-energy tensor to solve for retarded field

$$\mathcal{E}_{\mu\nu}[\bar{h}^2] = -16\pi(T^2_{\mu\nu} - T^Q_{\mu\nu}) + 2\delta^2 G_{\mu\nu}[h^1, h^1]$$

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    - Compare with other regularisation schemes, e.g. dimensional regularisation
    - Can we write this in a mode-decomposed form?