Asymptotic gravitational-wave fluxes from a spinning test body on generic orbits around a Kerr black hole

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Introduction

- Motivation: contribution of the secondary spin to the 1PA term
- Flux-driven inspirals: only asymptotic GW fluxes needed (?)
- Linear order in the secondary spin
- We used Teukolsky equation to calculate GW fluxes from off-equatorial orbits of spinning particle in the Kerr spacetime in frequency domain
- We compared the result with time-domain fluxes

Spinning particle in the Kerr spacetime

- Stress-energy tensor $T^{\mu\nu} = \int \mathrm{d}\tau \left(P^{(\mu} u^{\nu)} \frac{\delta^4}{\sqrt{-g}} - \nabla_\alpha \left(S^{\alpha(\mu} u^{\nu)} \frac{\delta^4}{\sqrt{-g}} \right) \right)$
- Linearized Mathisson-Papapetrou-Dixon equations

$$\mu \frac{\mathrm{D}^2 z^\mu}{\mathrm{d}\tau^2} = -\frac{1}{2} R^\mu{}_{\nu\rho\sigma} \frac{\mathrm{d}z^\nu}{\mathrm{d}\tau} S^{\rho\sigma} \qquad \frac{\mathrm{D}S^{\mu\nu}}{\mathrm{d}\tau} = 0,$$

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- Constants of motion:

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  - $C_Y = \sigma_\parallel \sqrt{K} = Y_{\mu\nu} P^\mu S^\nu/(\mu M)$
  - $K_R = K_{\mu\nu} P^\mu P^\nu - 2 P^\mu S^{\rho\sigma} (Y_{\mu\rho;\kappa} Y^\kappa{}_\sigma + Y_{\rho\sigma;\kappa} Y^\kappa{}_\mu)$

Parallel transport

- Spin vector $S^\mu = -1/2\epsilon^{\mu\nu\kappa\lambda}P_\nu S_{\kappa\lambda}/\mu$ parallel transported along a geodesic
- Solution in Marck tetrad (Marck [1983], van de Meent [2020])

$$S^\mu = \mu M\left(\sigma_\perp(\cos\psi_p e_1^\mu + \sin\psi_p e_2^\mu) + \sigma_\parallel e_3^\mu\right)$$

$$\psi_p = \Upsilon_s\lambda + \Delta\psi_r(\lambda) + \Delta\psi_z(\lambda)$$

$$e_0^\mu = u^\mu = -u_n l^\mu - u_l n^\mu + u_{\bar{m}} m^\mu + u_m \bar{m}^\mu$$

$$e_1^\mu = -r\Xi(r,z)K^{-\frac{1}{2}}(u_n l^\mu - u_l n^\mu) - iaz\Xi^{-1}(r,z)K^{-\frac{1}{2}}(u_{\bar{m}} m^\mu - u_m \bar{m}^\mu)$$

$$e_2^\mu = \Xi(r,z)(-u_n l^\mu - u_l n^\mu) + \Xi^{-1}(r,z)(u_{\bar{m}} m^\mu + u_m \bar{m}^\mu)$$

$$e_3^\mu = azK^{-\frac{1}{2}}(u_n l^\mu - u_l n^\mu) - irK^{-\frac{1}{2}}(u_{\bar{m}} m^\mu - u_m \bar{m}^\mu)$$

- Components of the spin tensor:

$$S_{ln} = -\sigma_\parallel \frac{r(K - a^2z^2)}{\sqrt{K}\Sigma}, \quad S_{nm} = \sigma_\parallel \frac{r - iaz}{\sqrt{K}} u_m u_n, \ldots$$

Solution in frequency domain

- Parametrization of the orbit (Drummond and Hughes [2022a,b]):

$$r = \frac{p}{1 + e\cos(\Upsilon_r\lambda + \delta\hat{\chi}_r(\lambda) + \delta\chi_r^S(\lambda))} + \mathfrak{r}^S(\lambda)$$

$$z = \cos\theta = \sin I \cos\Big(\Upsilon_z\lambda + \delta\hat{\chi}_z(\lambda) + \delta\chi_z^S(\lambda)\Big) + \mathfrak{z}^S(\lambda)$$

$$u_t = -\hat{E} + u_t^S(\lambda)$$

$$u_\phi = \hat{L}_z + u_\phi^S(\lambda)$$

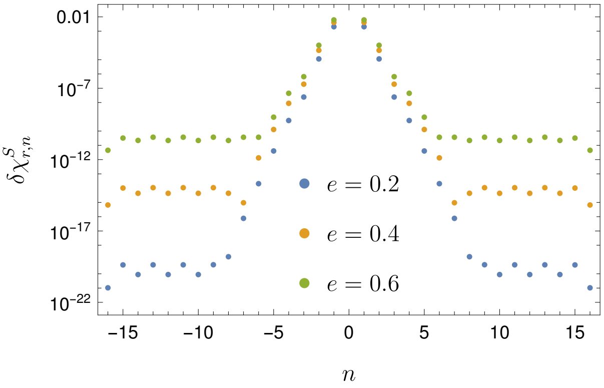
- Expansion in Fourier series $f(\lambda) = \sum_{nk} f_{nkj} e^{-in\Upsilon_r\lambda - ik\Upsilon_z\lambda - ij\Upsilon_s\lambda}$
- Phases $w_\mu = \Upsilon_\mu\lambda$ can be used instead of λ
- $\sigma_\perp$ part fully oscillating $\Rightarrow$ f_{nkj} for $j = \pm 1$ proportional to $\sigma_\perp$, for $j = 0$ proportional to $\sigma_\parallel$

Fourier coefficients

- The MPD equations and normalization of u^μ linearized in σ
- System of linear equations for the Fourier coefficients obtained from Fourier series

$$\mathbf{M} \cdot \mathbf{v} - \mathbf{c} = 0$$

- Overconstrained system solved with least squares method



$a = 0.9M,\ p = 15,\ I = 15^\circ$

Teukolsky equation

- Weyl scalar $\psi_4 = -C_{\alpha\beta\gamma\delta} n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta$
- Teukolsky equation ${}_{-2}\mathcal{O}_{-2}\psi = 4\pi\Sigma T$

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    - Horizon-penetrating hyperboloidal coordinates
    - (2+1)-D PDE
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- Teukolsky equation in frequency domain

$$\psi_4 = (r - ia\cos\theta)^{-4} \sum_{l,m} \int \mathrm{d}\omega \, \psi_{lm\omega}(r) \, {}_{-2}S^{a\omega}_{lm}(\theta) e^{-i\omega t + im\varphi}$$

  - Radial equation $\mathcal{D}_{lm\omega}\psi_{lm\omega}(r) = \mathcal{T}_{lm\omega}$
  - Angular equation for spin-weighted spheroidal harmonics

Radial Teukolsky equation

- Asymptotic behavior of the radial function

$$\psi_{lm\omega}(r) = \begin{cases} C^{+}_{lm\omega} r^3 e^{i\omega r} & \text{as} \quad r \to \infty \\ C^{-}_{lm\omega} \Delta e^{-ik_H r^*} & \text{as} \quad r \to r_+ \end{cases}$$

- Amplitudes found using Green function method

$$C^{\pm}_{lmn} = \frac{1}{W} \int_{r_+}^{\infty} \frac{R^{\mp}_{lmn}(r) \mathcal{T}_{lmn}(r)}{\Delta(r)} \mathrm{d}r$$

- Source term constructed from the derivatives of the stress-energy tensor T_{nn}, $T_{n\bar{m}}$, $T_{\bar{m}\bar{m}}$

$$\sqrt{-g} T_{ab} = \int \mathrm{d}\tau \left(\left(u_a u_b + S^{cd} u_{(b} \gamma_{a)dc} + S^{c}{}_{(a} \gamma_{b)dc} v^d \right) \delta^4 - \partial_\rho \left(S^{\rho}{}_{(a} u_{b)} \delta^4 \right) \right)$$

- Solution can be written as $C^{\pm}_{lm\omega} = \int \mathrm{d}\tau \Sigma^{-1} I^{\pm}_{lm\omega}(\tau) e^{i\omega t(\tau) - im\phi(\tau)}$

Partial amplitudes

- Discrete spectrum of frequencies $\omega_{mnkj} = m\Omega_\phi + n\Omega_r + k\Omega_z + j\Omega_s$

$$C^{\pm}_{lmnkj} = \frac{1}{(2\pi)^2\Gamma} \int \mathrm{d}w_r \mathrm{d}w_z \mathrm{d}w_s I^{\pm}_{lmnkj}(w_r, w_z, w_s) e^{i\omega\Delta t(w_r,w_z,w_s) - im\Delta\phi(w_r,w_z,w_s) + inw_r + ikw_z + ijw_s}$$

- Numerically integrated with the midpoint rule, homogeneous solution from the BHPT
- Waveform

$$h = -\frac{2}{r} \sum_{lmnkj} \frac{C^{+}_{lmnkj}}{\omega^2_{mnkj}} S^{a\omega_{mnkj}}_{lm}(\theta) e^{-i\omega_{mnkj}t + im\phi}$$

- Energy and angular momentum fluxes

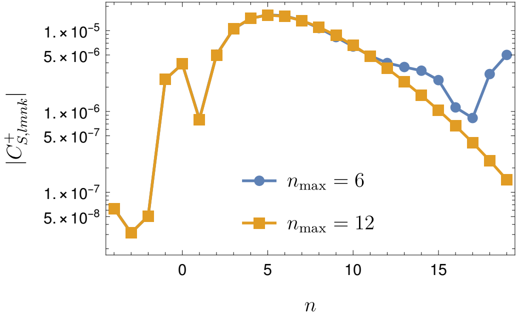
$$\mathcal{F}^E = \sum_{lmnkj} \frac{\left|C^{+}_{lmnkj}\right| + \alpha_{lmnkj}\left|C^{-}_{lmnkj}\right|}{4\pi\omega^2_{mnkj}}, \qquad \mathcal{F}^{J_z} = \sum_{lmnkj} m \frac{\left|C^{+}_{lmnkj}\right| + \alpha_{lmnkj}\left|C^{-}_{lmnkj}\right|}{4\pi\omega^3_{mnkj}}$$

Linearization of the fluxes

- Trajectory calculated up to linear order in σ
- Amplitudes and fluxes valid up to linear order as well
- Numerical linearization with 4th order finite difference formula

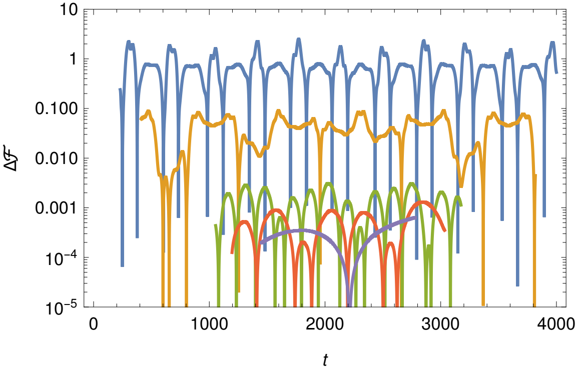
$$f^S = \frac{\frac{1}{12} f(-2\sigma) - \frac{2}{3} f(-\sigma) + \frac{2}{3} f(\sigma) - \frac{1}{12} f(2\sigma)}{\sigma}$$

- Amplitudes for $j = \pm 1$ proportional to $\sigma_\perp$
- Fluxes for $j = \pm 1$ proportional to $\sigma_\perp^2$
- In linear order the fluxes are independent of $\sigma_\perp$



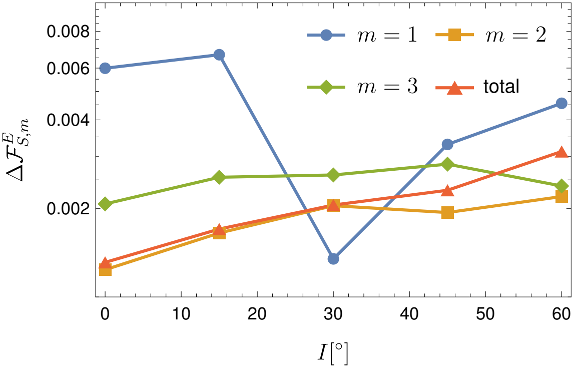
Comparison with time domain results

- E, J_z, r, θ, u^r, S^r, S^θ calculated in frequency domain
- Other components calculated from the constraints
- Nonlinearized MPD equations solved numerically
- Time domain fluxes from this trajectory
- Averaging of the generic fluxes: successive moving averages with periods $2\pi/\Omega_r$, $2\pi/\Omega_z$, $2\pi/(n\Omega_r + k\Omega_z)$



Relative difference between the energy flux and the average value

Comparison with time domain results



Nearly spherical orbits with $a = 0.9M$, $p = 10$

p	e	$I/^\circ$	m	$\mathcal{F}^E_{S,m}$	$\Delta\mathcal{F}^E_{S,m}$
10	0.1	15	2	-2.8259×10^{-6}	1×10^{-3}
12	0.2	30	1	-1.1954×10^{-7}	2×10^{-5}
12	0.2	30	2	-1.0488×10^{-6}	1×10^{-3}
12	0.2	30	3	-1.4210×10^{-7}	3×10^{-3}
12	0.2	60	2	-8.0550×10^{-7}	5×10^{-4}
15	0.5	15	2	-4.2936×10^{-7}	2×10^{-3}

Generic orbits

Summary

- Secondary spin is needed for the post-adiabatic term
- We calculated the fluxes of energy and angular momentum from generic orbits of spinning particles
- We compared the frequency domain and time domain solutions
- Only parallel component of the secondary spin is relevant
- For the inspirals evolution of the Carter-like constant is needed

Thank you

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