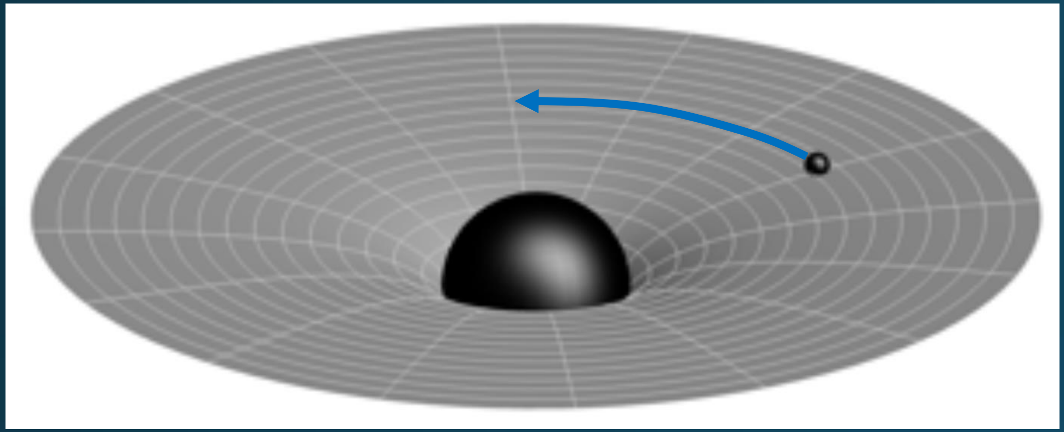
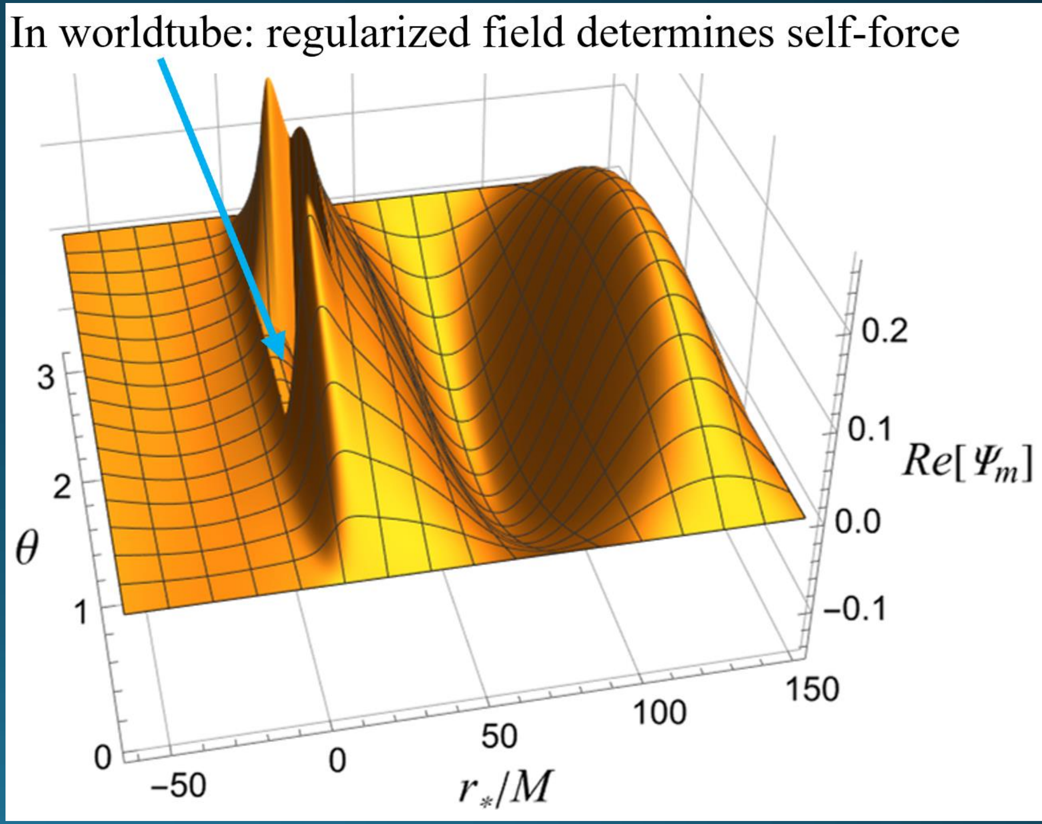
Lorenz gauge Kerr self-force via elliptic PDEs



https://arxiv.org/abs/2206.07031



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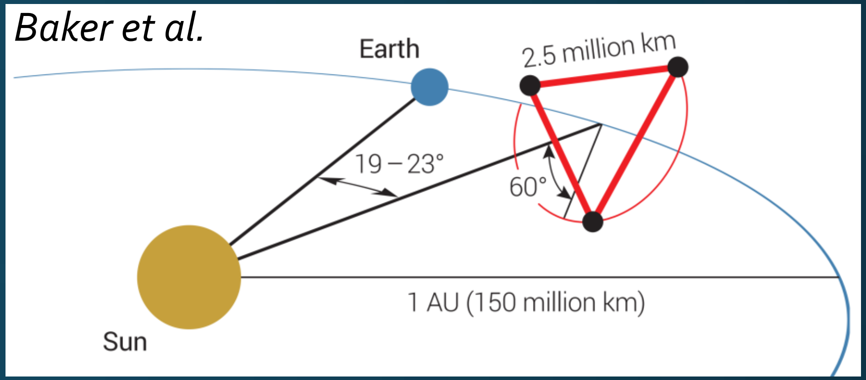
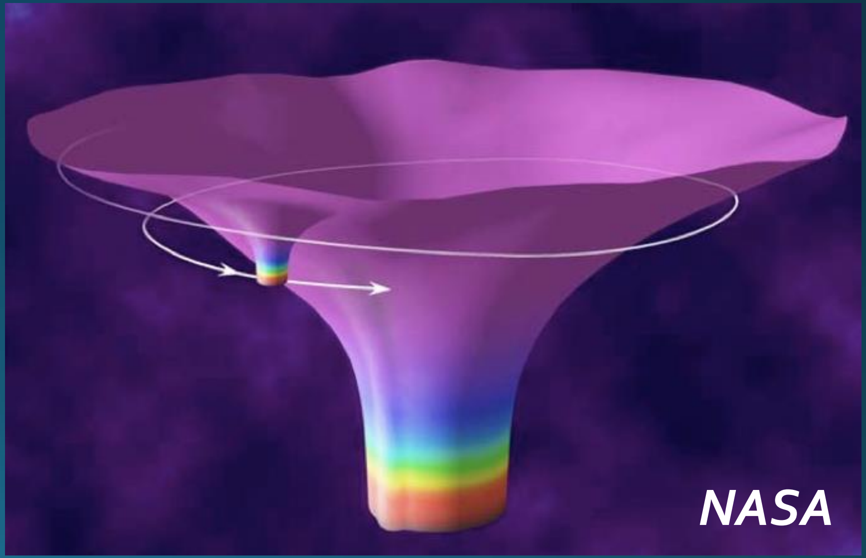
Background

- Extreme Mass-Ratio Inspirals (EMRIs) are **important gravitational wave sources for LISA**
- Precision parameter estimation/science will require templates with **orbital phase error << 1 radian**
- **Broad approach**: Black Hole Perturbation Theory and Self-Force with higher order effects
- Needed higher order effects include **conservative self-force** and **2nd order dissipative self-force**
- **2nd order implemented for Schwarzschild** (*Wardell et al.* 2021, see Warburton's presentation), but **Kerr 2nd order self-force needs development**
- 2nd order self-force understood in **Lorenz gauge**, but prior Kerr work had **instabilities** (*Dolan et al.* 2013)



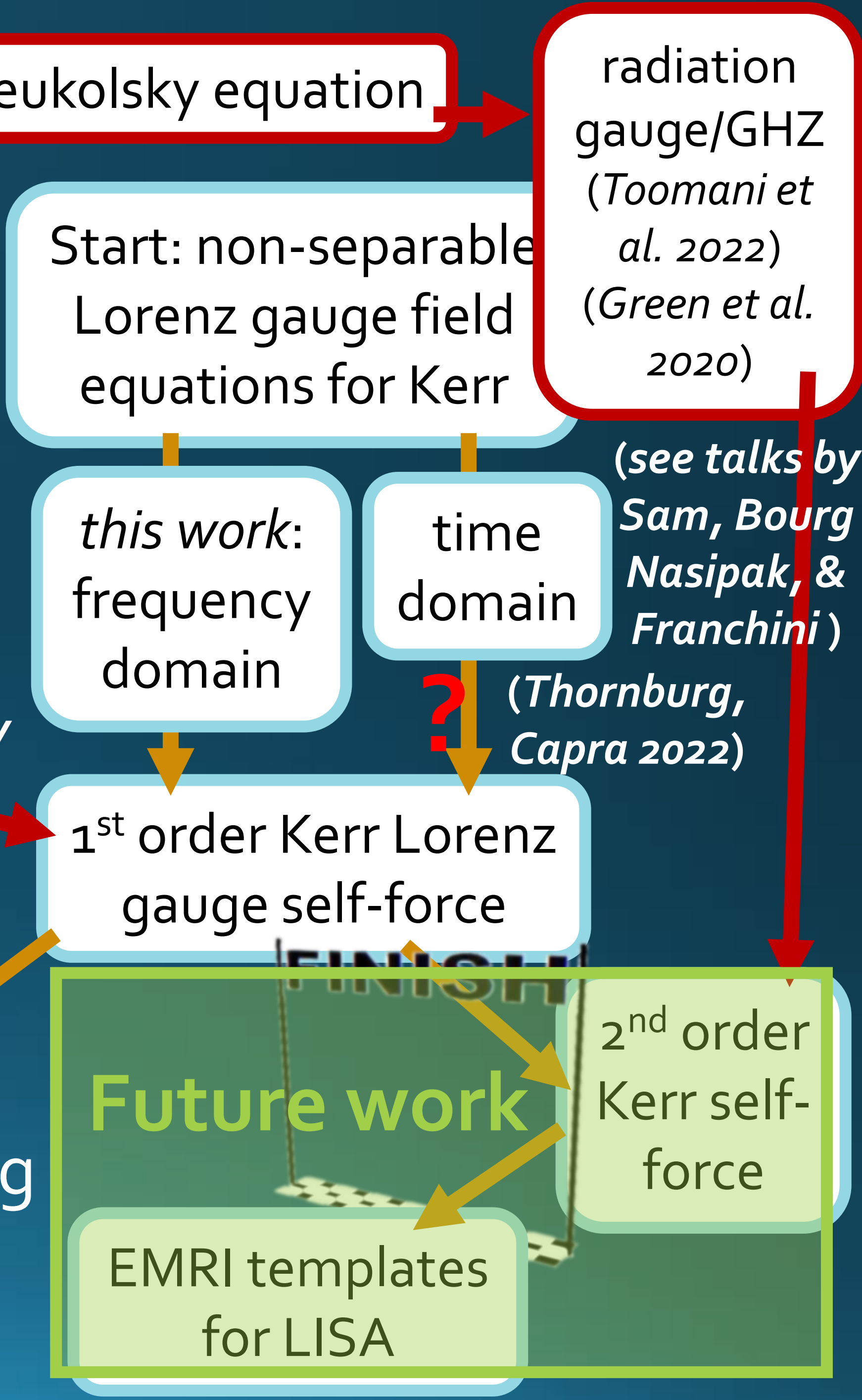


Overview

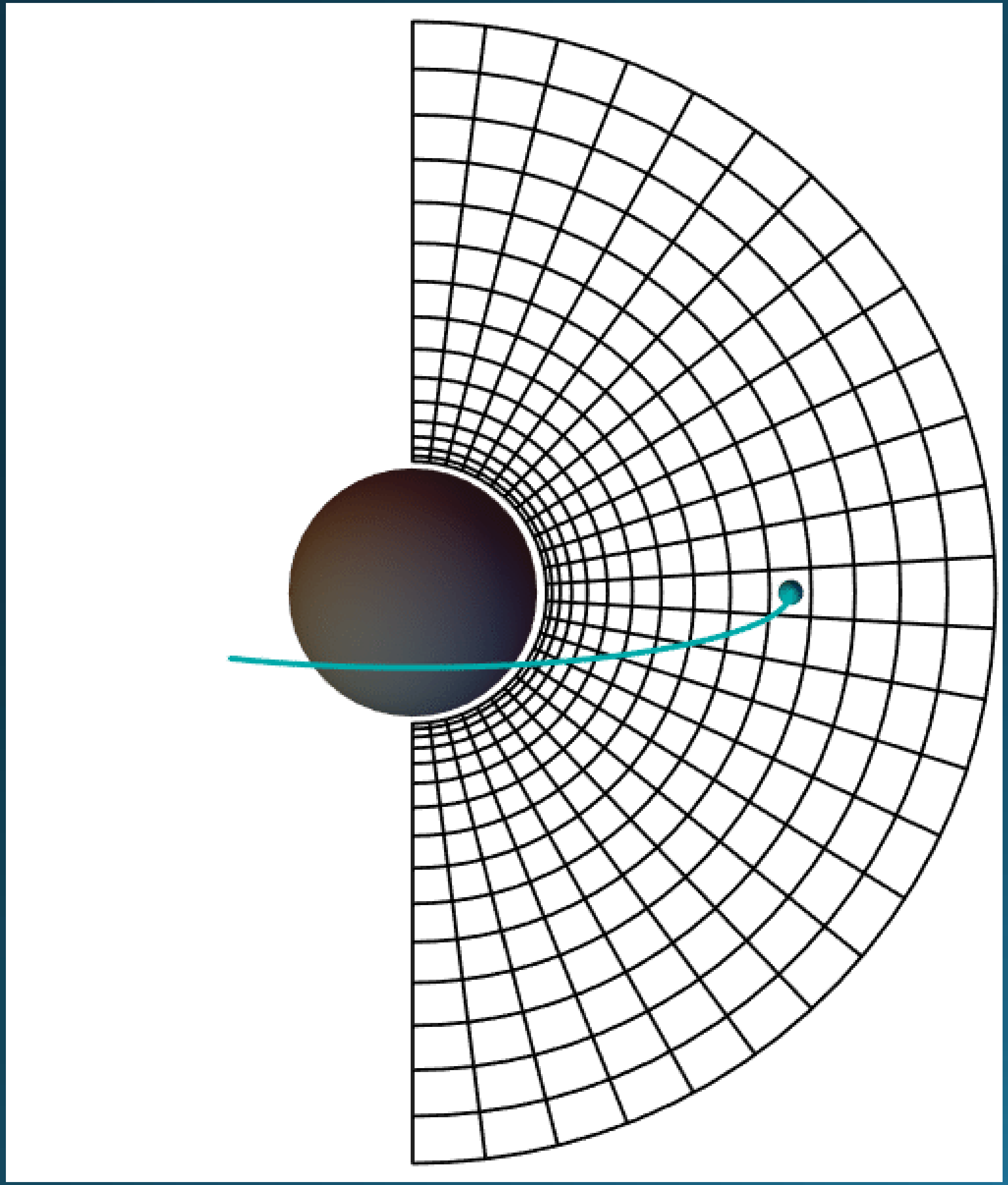
Alternate start: separable Teukolsky equation

- **Goal**: calculate 1st order gravitational self-force for Kerr in Lorenz gauge (then try 2nd order later)
- **Situation**: field equations not directly separable (but, *Dolan et al. 2021* → Lorenz gauge via Teukolsky)
- **Strategy**: Separate φ but do not separate θ → **solve PDEs** ↵
- **Past work**: Time domain ⟶ Hyperbolic PDEs, encountered **time instabilities** (*Dolan et al. 2013*)
- **Our approach**: Frequency domain ⟶ **Elliptic PDEs** with r and θ derivatives, **no time instabilities**
- **This work**: Develop elliptic PDE techniques by finding self-force for circular motion in Kerr spacetime with ***scalar fields first*** and ***then Lorenz gauge gravity***

(see talk by Dolan)



Scalar field in Kerr via elliptic PDEs: $\nabla^\alpha \nabla_\alpha \Phi = -4\pi\rho$



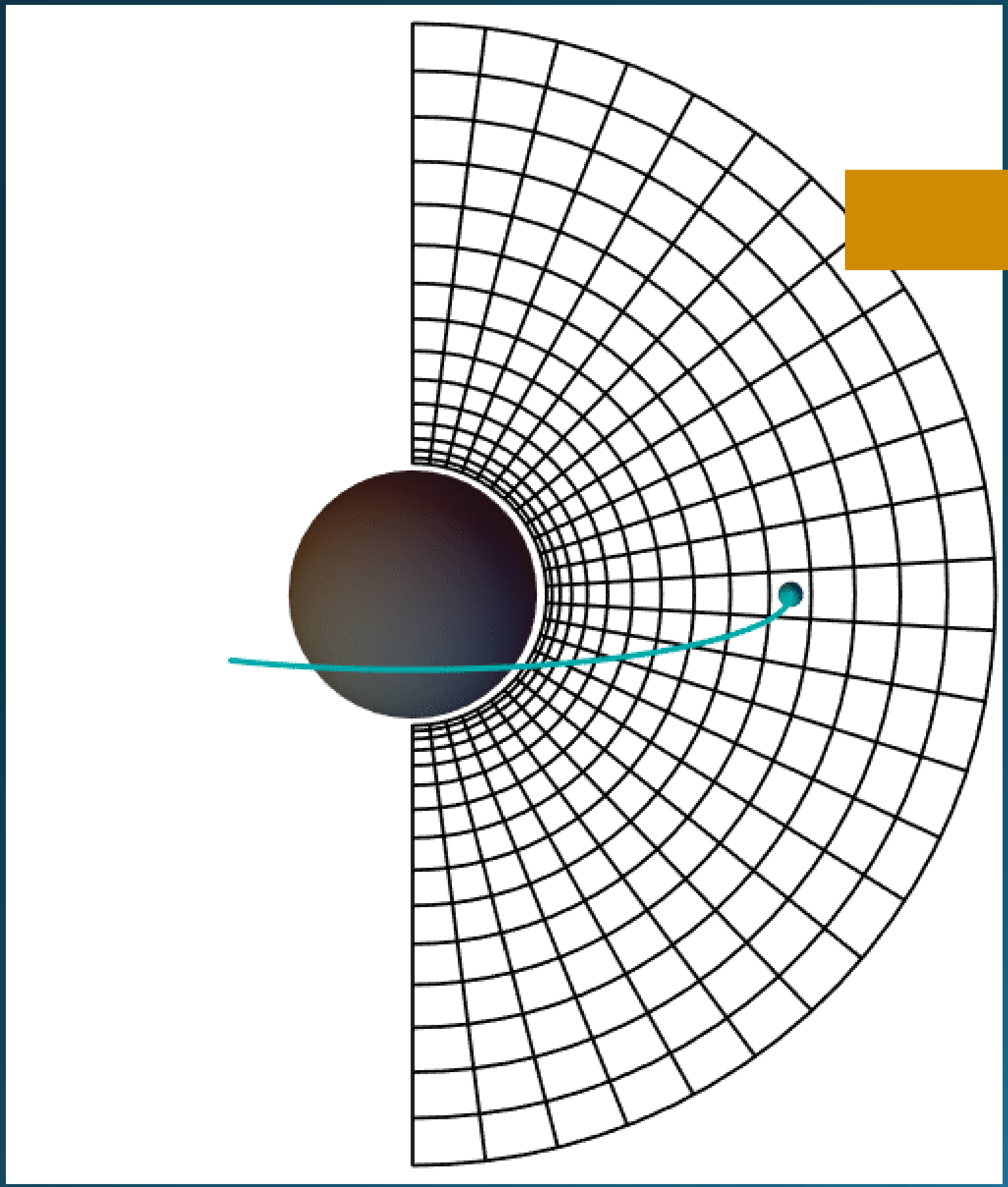
Start from prior time-domain analyses (Dolan et al. 2011, Thornburg et al. 2017), but enter frequency-domain

$$\Phi(t,r,\theta,\varphi) = \frac{1}{r}\sum_m \Psi_m(r,\theta)\; e^{imf(r)} e^{im(\varphi-\Omega t)}$$

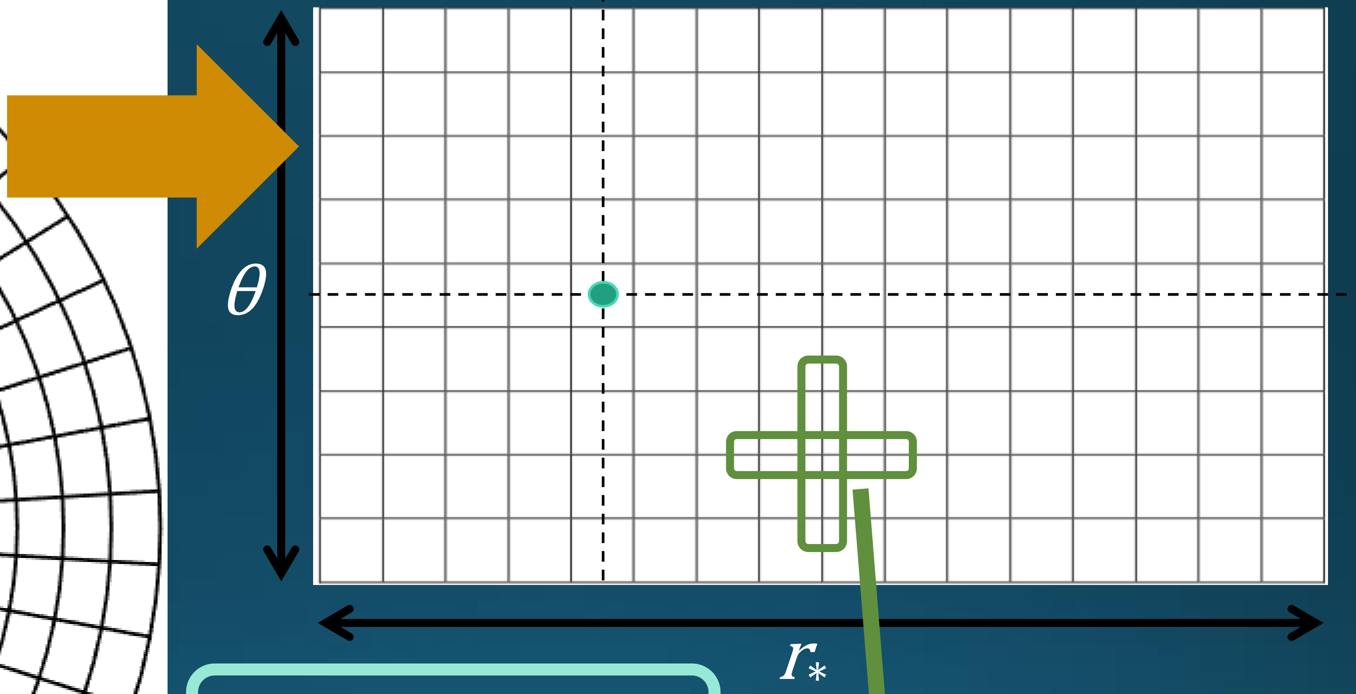
$$\Box_m \equiv -m^2\Omega^2 + \frac{4am^2\Omega Mr}{\Sigma^2} - \frac{\left(r^2+a^2\right)^2}{\Sigma^2}\frac{\partial^2}{\partial r_*^2} - \frac{2iamr\left(r^2+a^2\right)-2a^2\Delta}{r\Sigma^2}\frac{\partial}{\partial r_*} - \frac{\Delta}{\Sigma^2}\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} - \frac{m^2}{\sin^2\theta} - \frac{2M}{r}\left(1-\frac{a^2}{Mr}\right) - \frac{2iam}{r}\right)$$

$$\Box_m \Psi_m = S_m$$

Numerical solution: Finite difference method



Rectangular grid with fixed Δr_* and $\Delta \theta$



$$\Box_m \Psi_m = S_m$$

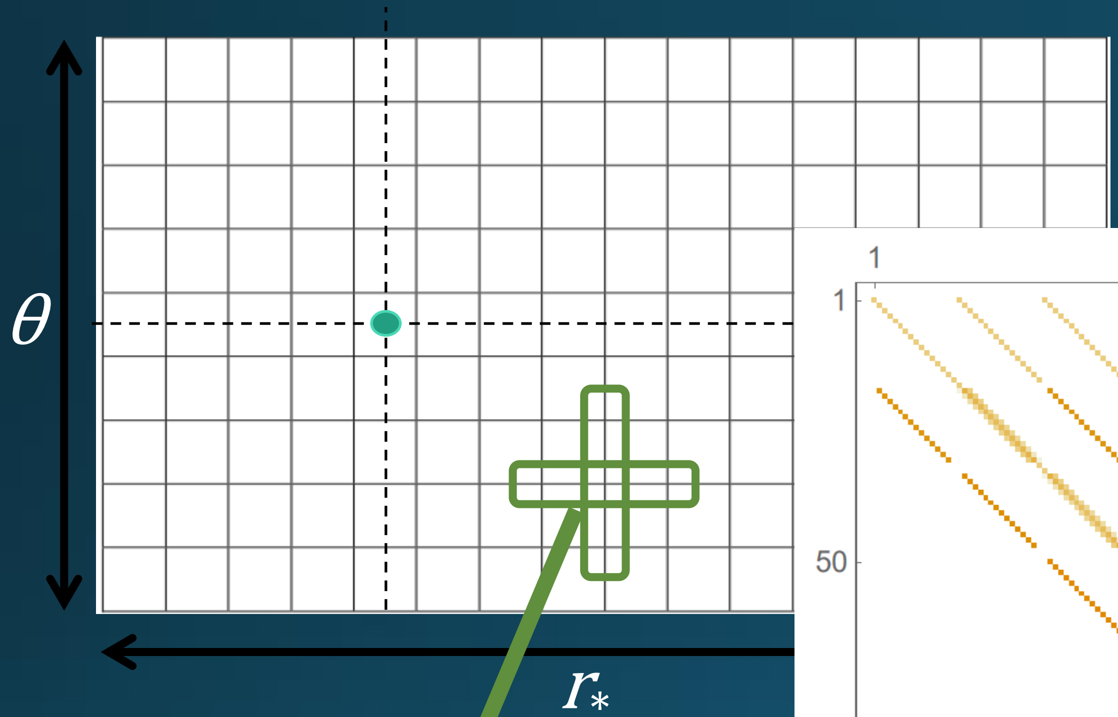
$$A\,\frac{\Psi_m^{27} - 2\Psi_m^{28} + \Psi_m^{29}}{\Delta r_*^2} + B\,\frac{\Psi_m^{18} - 2\Psi_m^{28} + \Psi_m^{38}}{\Delta \theta^2} + \cdots = S$$

Vector of unknowns

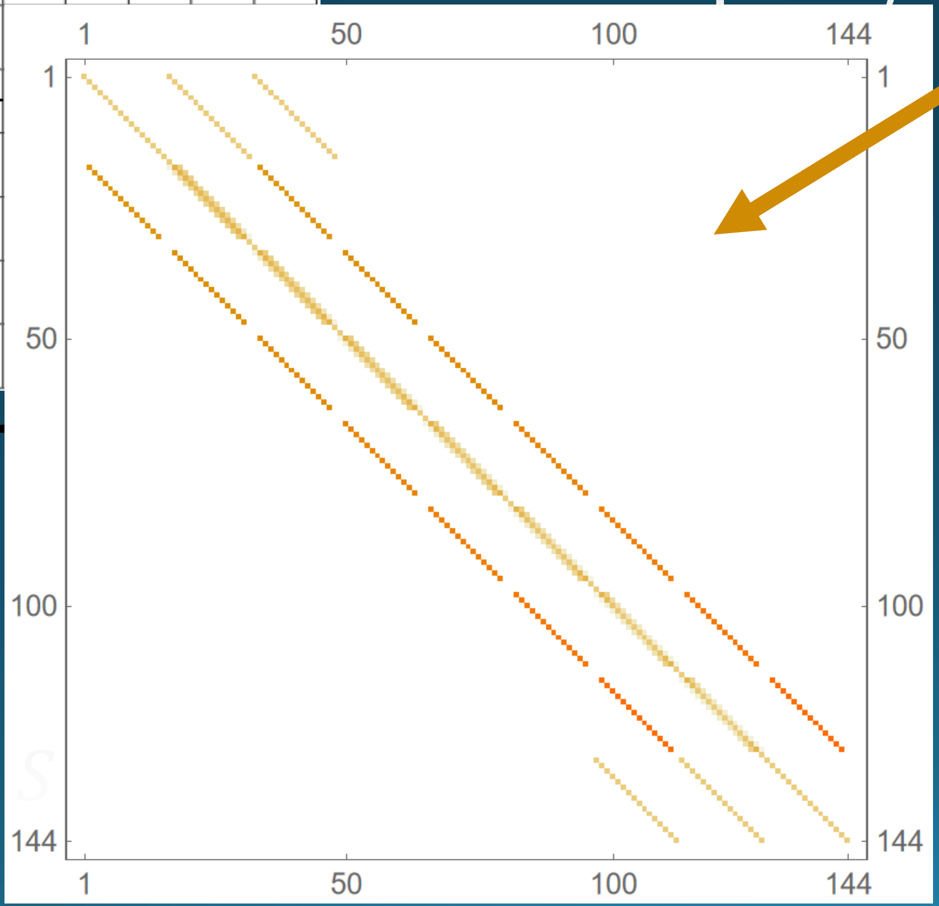
$$\begin{bmatrix} \Psi_m^{00} \\ \Psi_m^{01} \\ \Psi_m^{02} \\ \vdots \\ \Psi_m^{10} \\ \Psi_m^{11} \\ \Psi_m^{12} \\ \vdots \end{bmatrix}$$

Numerical solution: Finite difference method

Rectangular grid with fixed Δr_* and $\Delta \theta$



$$A\,\frac{\Psi_m^{27} - 2\Psi_m^{28} + \Psi_m^{29}}{\Delta r_*^2} + B\,\frac{\Psi_m^{18} - 2\Psi_m^{28} + \Psi_m^{38}}{\Delta \theta^2} + \cdots = S$$



System of algebraic equations

$$\begin{bmatrix} a & b & c & d & \cdots \\ e & f & g & h & \cdots \\ i & j & k & l & \cdots \\ m & n & o & p & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \Psi_m^{00} \\ \Psi_m^{01} \\ \vdots \\ \Psi_m^{10} \\ \Psi_m^{11} \\ \vdots \end{bmatrix} = \begin{bmatrix} S_m^{00} \\ S_m^{01} \\ \vdots \\ S_m^{10} \\ S_m^{11} \\ \vdots \end{bmatrix}$$

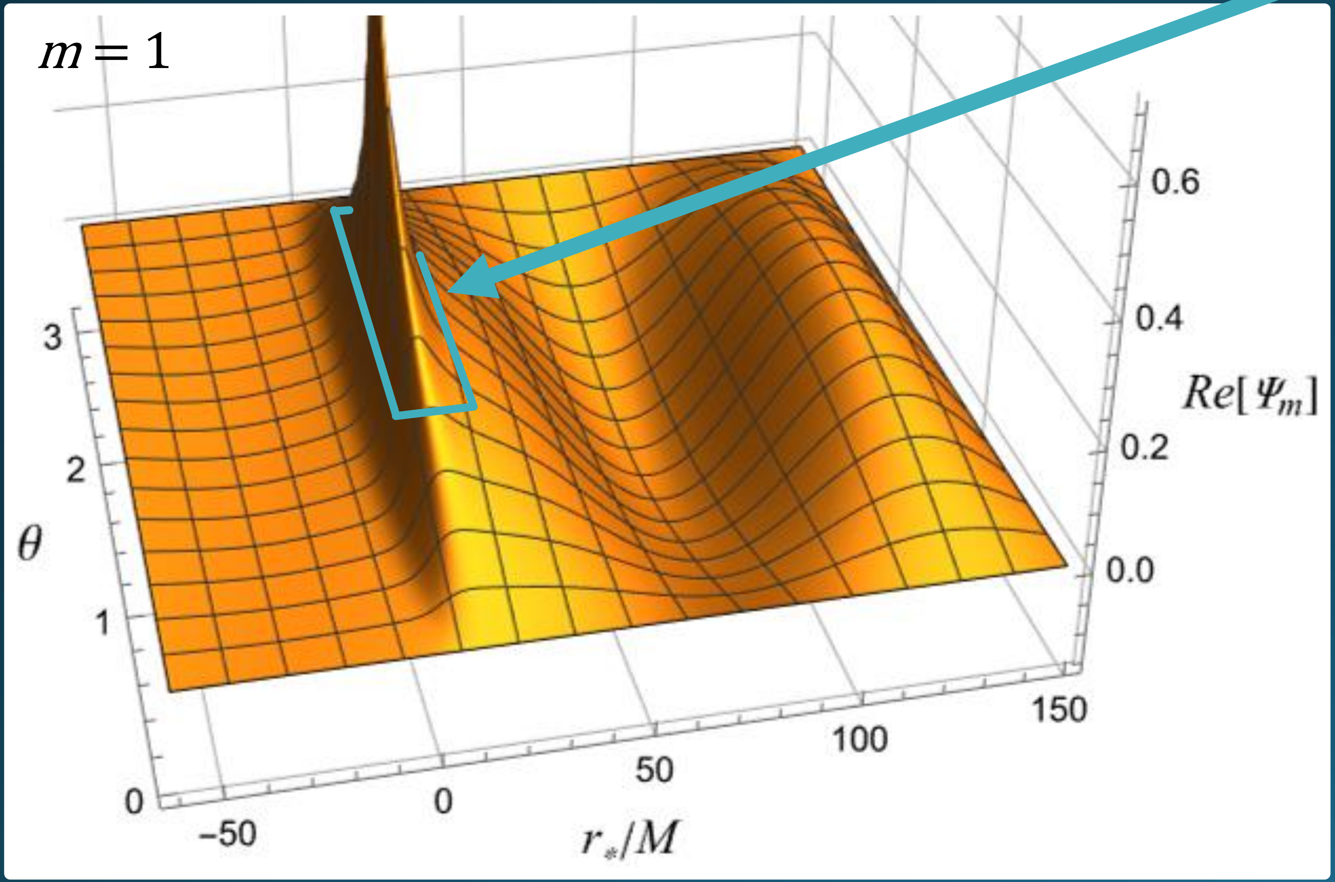
Vector of unknowns

How to handle source?

Linear system with sparse matrix

Self-force regularization: Effective source method

Field from point source blows up:



(see talk by Leather)

- **Strategy**: Introduce worldtube centered on particle
- **Inside worldtube** – Regular/Singular decomposition:

$$\Psi_m = \Psi_m^R + \Psi_m^S$$

- Ψ_m^S is known from an expansion and does not affect the self-force
- Ψ_m^R is finite and causes self-force

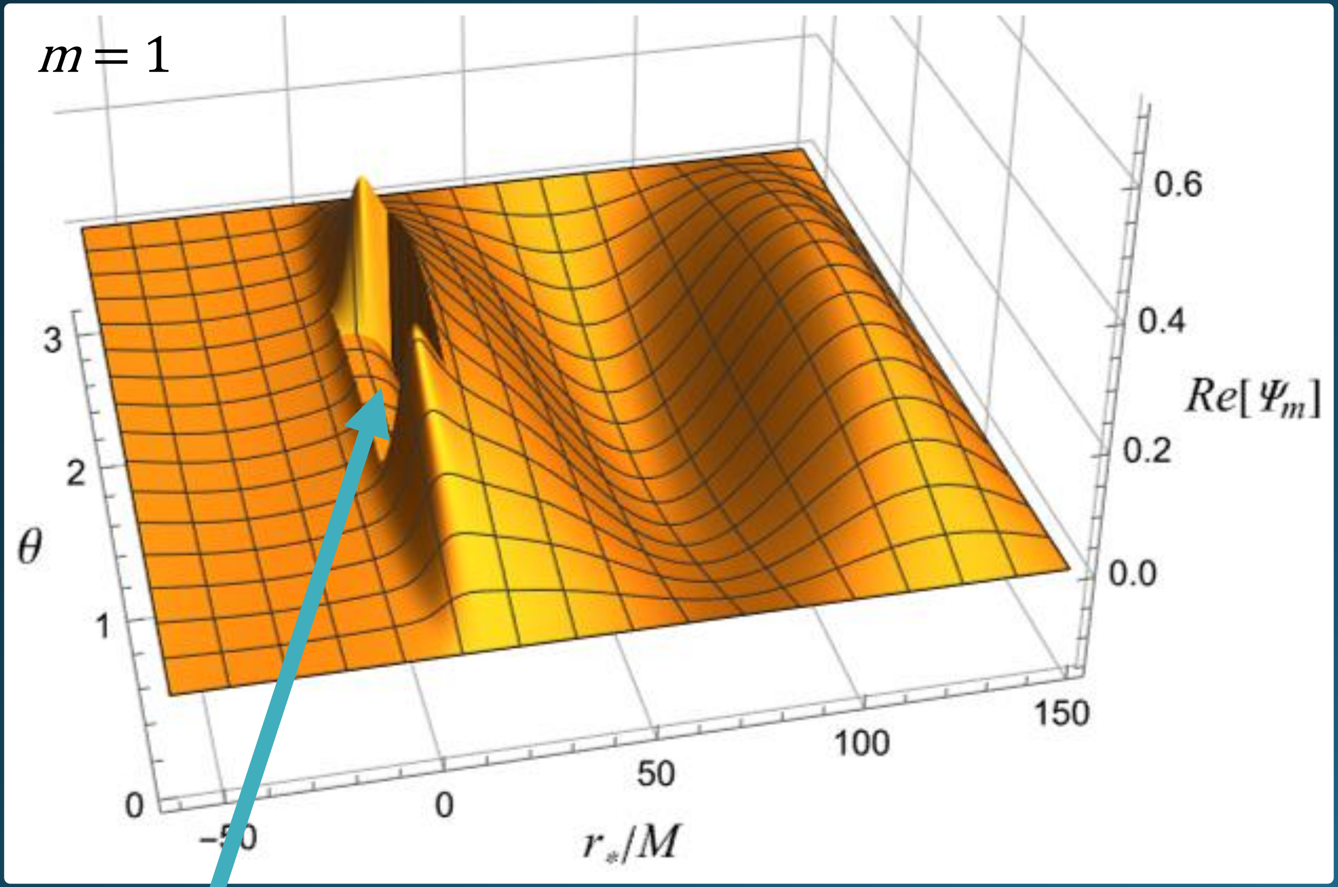
$$\Box_m \left(\Psi_m^R + \Psi_m^S\right) = S_m$$

$$\Box_m \Psi_m^R = S_m - \Box_m \Psi_m^S = S_m^{eff}$$

***Issue**: don't have boundary condition*

Self-force regularization: Effective source method

$S_m^{eff} \equiv S_m - \Box_m \Psi_m^S$ ← *code by Wardell*



Gradient of Ψ_m^R gives self-force

Mathematical strategy:

do have boundary condition

$\Box_m \Psi_m = 0$ (outside worldtube)

$\Psi_m^R = \Psi_m - \Psi_m^S$ (across worldtube)

$\Box_m \Psi_m^R = S_m^{eff}$ (inside worldtube)

(*Detweiler et al. 2002*, *Barack et al. 2007*, *Vega et al. 2008*, *Wardell et al. 2012*)

This affects the vector part of the linear system, but not the matrix part

Scalar self-force calculations converge and agree

https://arxiv.org/abs/2206.07031

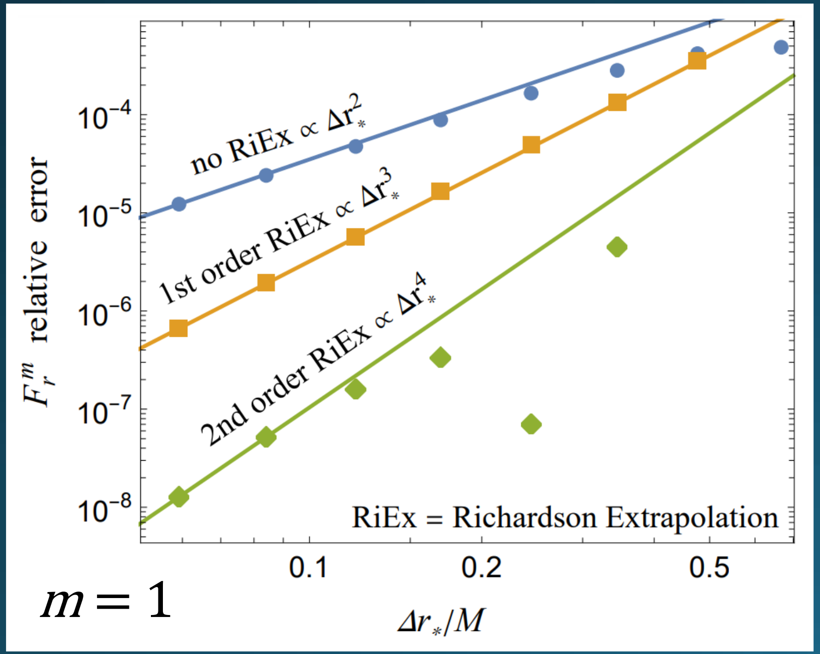
$$F_\alpha = \sum_m \nabla_\alpha \Phi^R_m = \sum_m F^m_\alpha$$

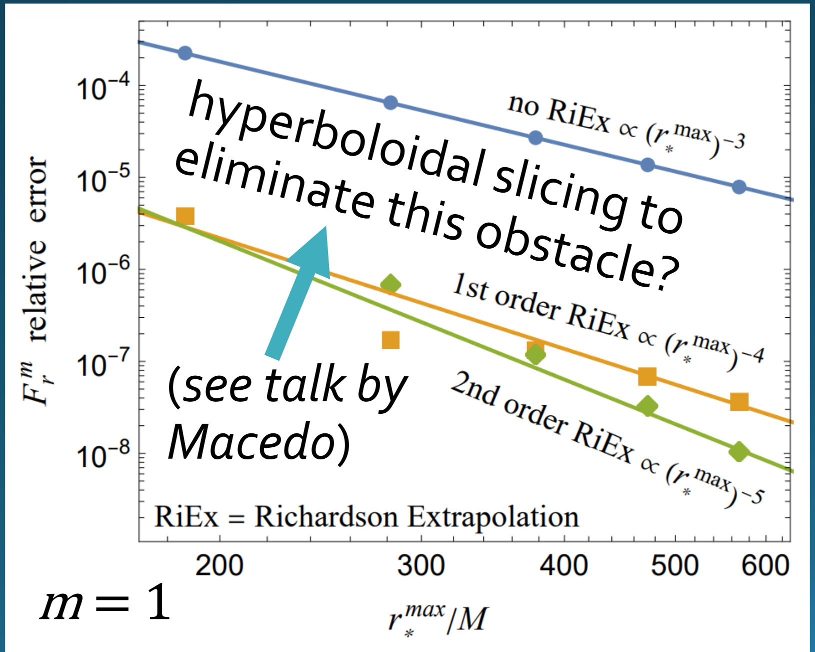
$$r_0 = 6M, \qquad a = 0.5M$$

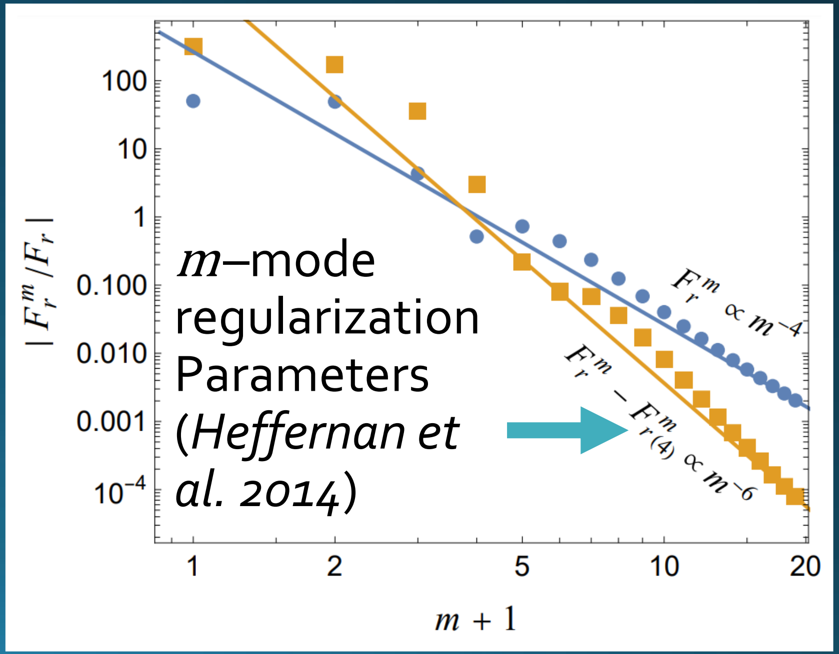
Convergence with decreasing Δr_* and $\Delta\theta$

Convergence with increasing $r_*^{\max}$

Convergence of sum over m-modes







In progress: Lorenz gauge metric perturbations

- Find m-modes of metric perturbation: $g^{\alpha\beta}\nabla_\alpha\nabla_\beta \bar{h}_{\mu\nu} + 2R^{\alpha\ \beta}_{\ \mu\ \nu}\bar{h}_{\alpha\beta} = -16\pi T_{\mu\nu}$
- At each position, solve for the 10 metric components (following Dolan and Barack, use adv Eddington-Finkelstein comps w/ r factors):

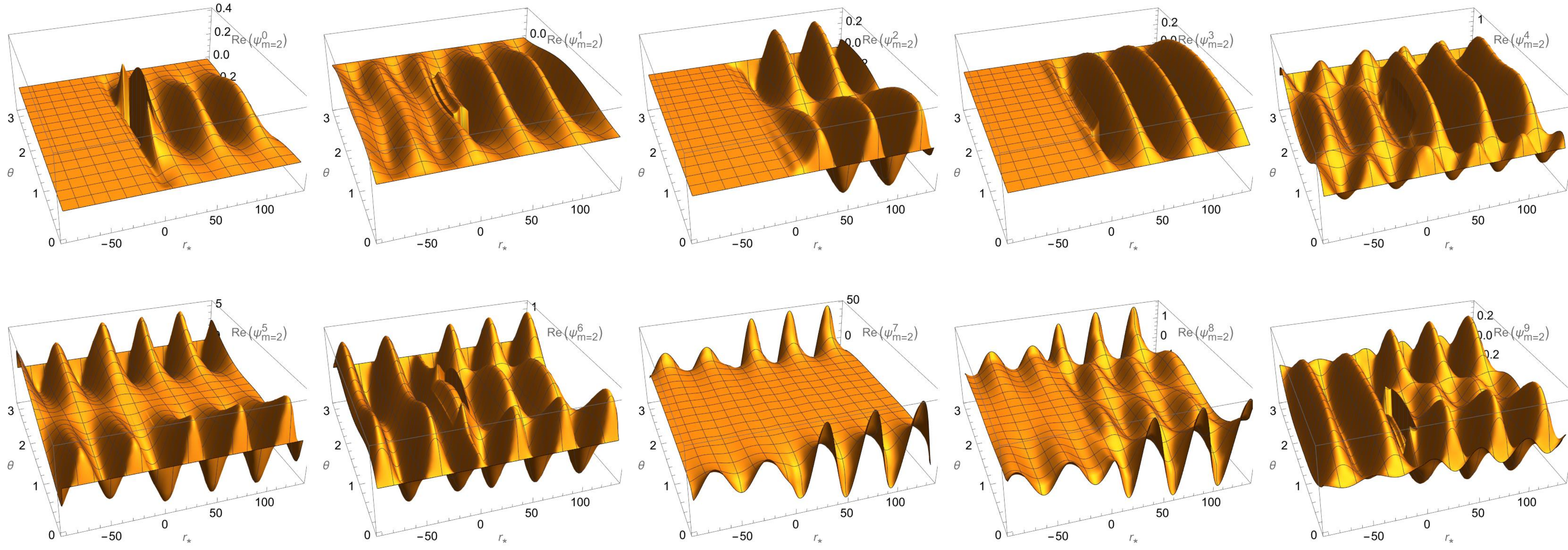
$$\vec{\psi}_m = \begin{pmatrix} \psi_m^0 \\ \psi_m^1 \\ \psi_m^2 \\ \psi_m^3 \\ \psi_m^4 \\ \psi_m^5 \\ \psi_m^6 \\ \psi_m^7 \\ \psi_m^8 \\ \psi_m^9 \end{pmatrix} \simeq \begin{pmatrix} \bar{h}^m_{vv} \\ \bar{h}^m_{vr} \\ \bar{h}^m_{v\theta} \\ \bar{h}^m_{v\varphi} \\ \bar{h}^m_{rr} \\ \bar{h}^m_{r\theta} \\ \bar{h}^m_{v\varphi} \\ \bar{h}^m_{\theta\theta} \\ \bar{h}^m_{\theta\varphi} \\ \bar{h}^m_{\varphi\varphi} \end{pmatrix}$$

- Lorenz gauge field equations can be written in the following form:

$$\left(\frac{\partial^2}{\partial r_*^2} + \frac{\Delta}{(r^2+a^2)^2}\frac{\partial^2}{\partial\theta^2} + \mathbf{A}_m\frac{\partial}{\partial r_*} + \mathbf{B}_m\frac{\partial}{\partial\theta} + \mathbf{C}_m\right)\vec{\psi}_m = \vec{S}_m$$

- For simplicity, carry over methods from scalar project (2nd-order finite difference, effective source code from Barry Wardell, etc.)

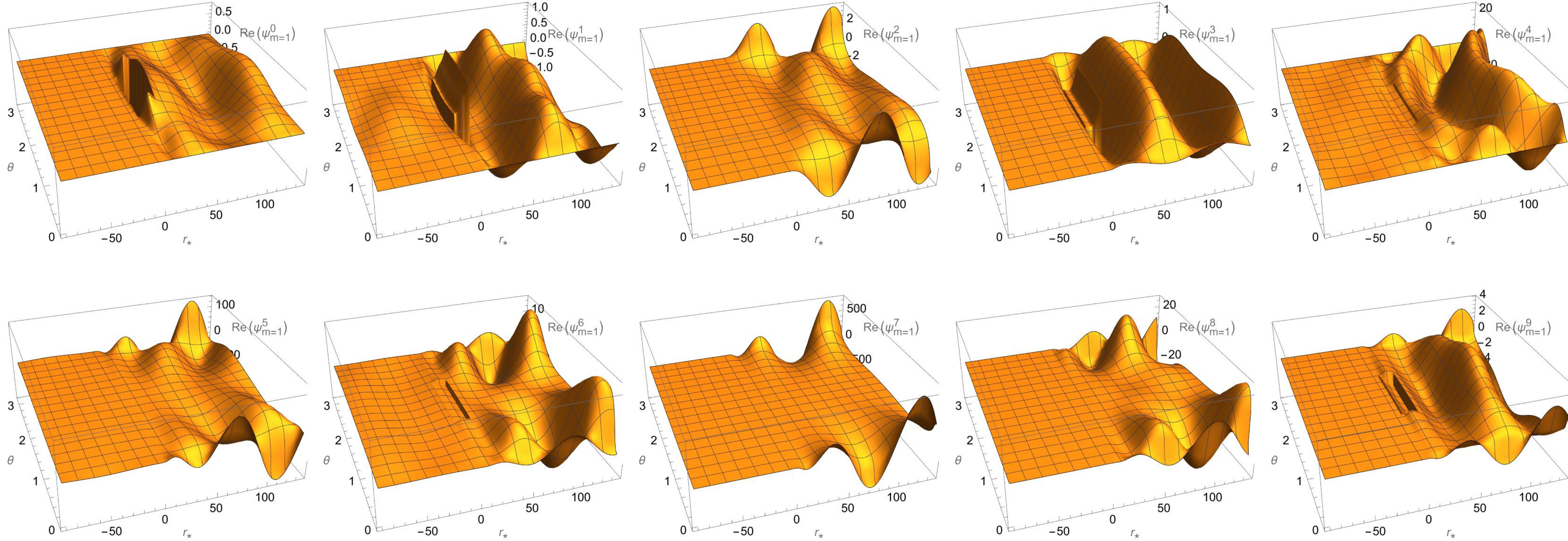
Early results: Lorenz gauge metric perturbation



$m = 2\,,\qquad a = 0\,,\qquad r_0 = 6M$

Have the *m*=1 issues been resolved?

No time instability, but *m=1* grows badly with *r*…



$m = 1$, $a = 0$, $r_0 = 6M$

Needs more investigation…

Concluding notes and summary

- **Goal**: calculate Lorenz gauge Kerr gravitational self-force
- **Progress so far**:
  - Solved r-θ PDEs in frequency domain ($\frac{\partial}{\partial t} = -i\omega \longrightarrow$ elliptic PDEs)
  - Scalar self-force agrees with prior results (see: https://arxiv.org/abs/2206.07031)
  - Preliminary results for gravity in Lorenz gauge, resolving some issues
- **Future directions**:
  - Use faster programming language (petsc4py: *Balor Brennan poster on Sat*)
  - Finish Lorenz gauge Kerr gravitational self-force
  - 2^{nd} order Lorenz gauge self-force in Kerr spacetime?
  - Radiation gauge/Teukolsky handles eccentric/inclined, **can we handle eccentric/inclined with elliptic PDEs?** Source not perfectly smooth → slower Fourier convergence, but maybe there is a clever trick

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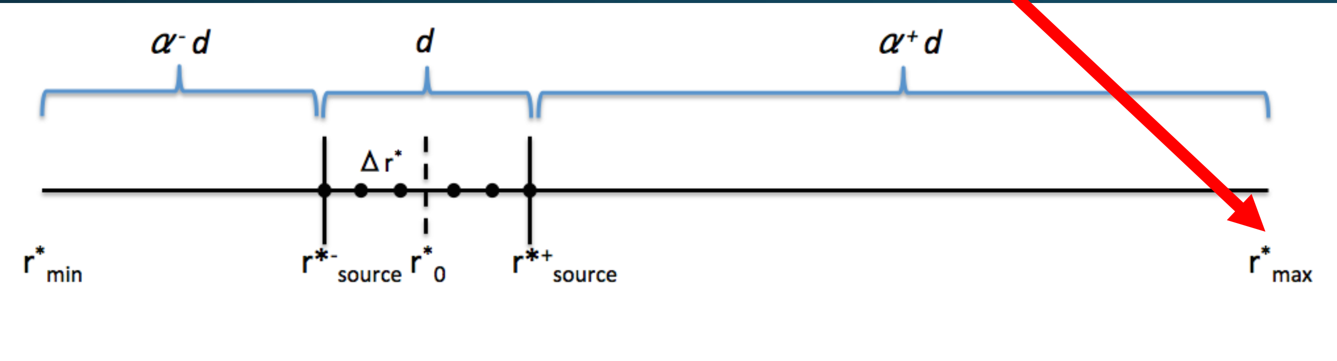
Large r* analysis: $\Psi^m = e^{i\omega r^*}\left(A(\theta) + \frac{B(\theta)}{r} + \frac{C(\theta)}{r^2} + \vartheta(\frac{1}{r^3})\right)$



"naïve" condition -

BC1: $\frac{\partial \Psi^m}{\partial r^*} - i\omega\Psi^m = \vartheta(\frac{1}{r_{max}^2})$

Less naïve - BC2: $\frac{\partial^2 \Psi^m}{\partial {r^*}^2} - 2i\omega\frac{\partial \Psi^m}{\partial r^*} - i\omega\Psi^m = \vartheta(\frac{1}{r_{max}^3})$

Sophisticated? - BC3: $\frac{\partial^3 \Psi^m}{\partial {r^*}^3} - 3i\omega\frac{\partial^2 \Psi^m}{\partial {r^*}^2} - 3\omega^2\frac{\partial \Psi^m}{\partial r^*} + i\omega^3\Psi^m = \vartheta(\frac{1}{r_{max}^4})$

Bayliss-Turkel type boundary conditions