post-Newtonian expansions of equatorial eccentric Kerr EMRIs

Using the s = +2 Teukolsky functions

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Outline

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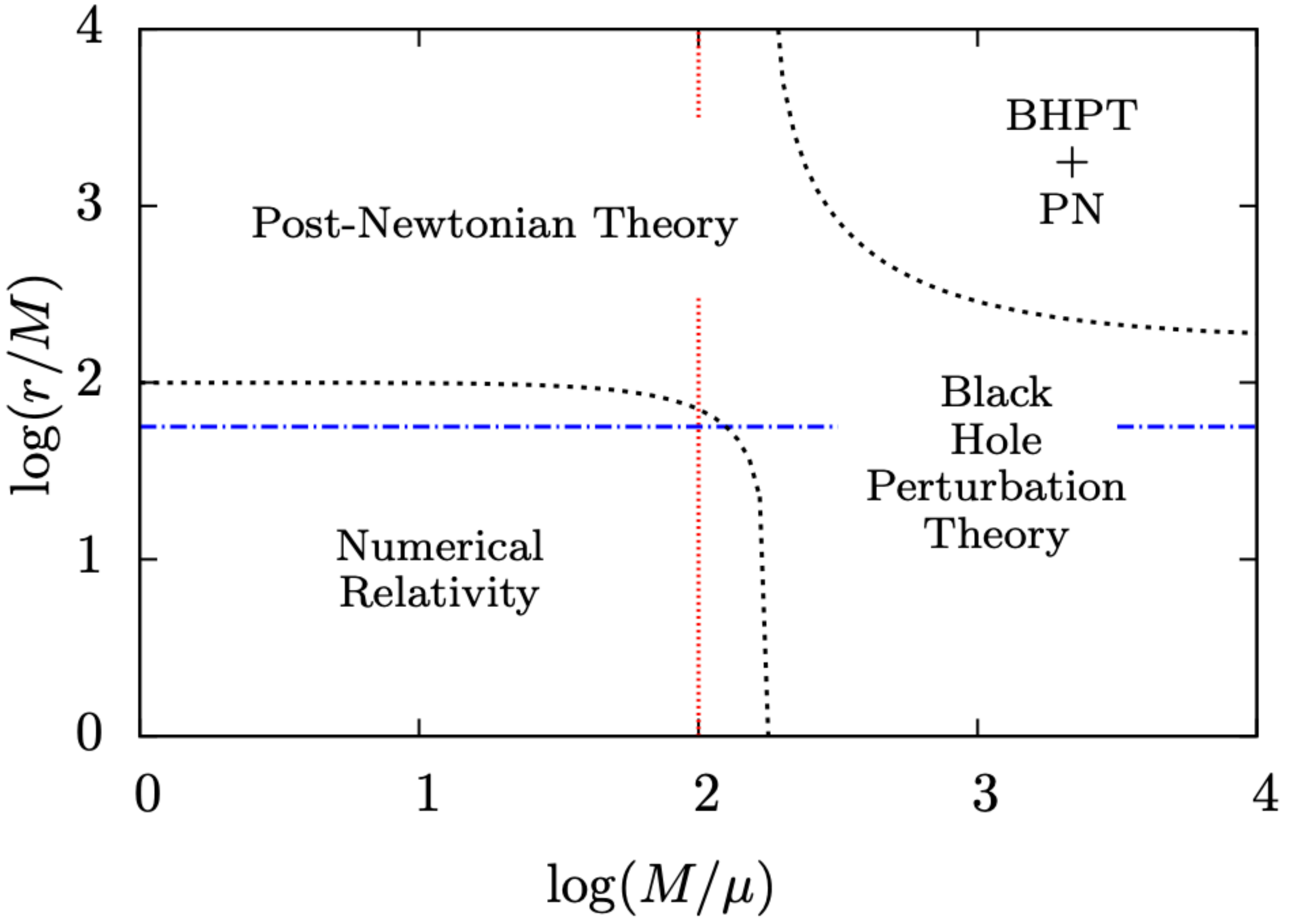
Motivation: BH-PT overlap



Motivation: Transition to Kerr

- Previous works of the UNC Gravity Group involve orbits in Schwarzschild
  - Forseth/Evans/Hopper: 7 PN, e^2.
  - Munna/Evans/Hopper/Forseth: 8.5 PN, e^2.
  - Munna/Evans (2019): select $\log$ coefficients, AO in e.
  - Munna (2020): 19 PN, e^{20} $(\leq 10 \text{ PN})$, e^{10} $(> 10\text{PN})$.
  - Munna/Evans (2020): select coefficients, AO and CF in e.
  - Munna/Evans (2022): redshift invariant
  - Munna/Evans (2022): spin-precession invariant
  - Munna/Evans/Forseth (2023): tidal-heating

Motivation: A Parallel Approach

- Motivated by developing a track parallel to analytic calculations involving $s = -2$ Teukolsky functions.
    - Initially used to compare quantities calculated using the $s = -2$ Teukolsky functions
    - Flexibility in calculations, as some calculations may prefer a specific Teukolsky function.

Motivation: Practicality

- When using $s = -2$ Teukolsky functions, Starobinsky transformations are used to generate the $s = +2$ Teukolsky functions.
  - Fine for numerical calculations.
  - Runs into issues with practicality when dealing with very high order analytical expansion.
  - For applications where only $s = 2$ functions are needed, going the $s = -2$ route may take far longer than needed.

Motivation: Goals

- Develop a procedure to efficiently expand the $s = +2$ Teukolsky functions
- Calculate quantities using the $s = +2$ Teukolsky functions
  - Infinity-side and horizon-side fluxes are the easiest quantities to calculate.

Methods: Teukolsky Functions from MST*

- Represented as hypergeometric functions (Mano, Suzuki, Takasugi (1996)):

$$
{}_sR^{+}_{lm\omega} = {}_sC^{+}_{lm\omega}(z) \sum_{k=-\infty}^{\infty} \frac{(\nu+1+s-i\epsilon)_k}{(\nu+1-s+i\epsilon)_k} a_k^{\nu} (2iz)^k U(k+1+s+\nu-i\epsilon, 2k+2+2\nu; -2iz),
$$

$$
{}_sC^{+}_{lm\omega}(z) = 2^{\nu} e^{-\pi\epsilon} e^{-i\pi(\nu+1+s)} e^{iz} z^{\nu-s} \left(1-\frac{\epsilon\kappa}{z}\right)^{-s-i(\epsilon+\tau)/2}.
$$

$$
{}_sR^{-}_{lm\omega} = {}_sC^{-}_{lm\omega}(z) \sum_{k=-\infty}^{\infty} a_k^{\nu}\ {}_2F_1(k+\nu+1-i\tau, -k-\nu-i\tau, 1-s-i\epsilon-i\tau, 1-\frac{z}{\epsilon\kappa}),
$$

$$
{}_sC^{-}_{lm\omega} = e^{-iz+i\epsilon\kappa} \left(\frac{\epsilon\kappa}{z}\right)^{i\tau+s} \left(1-\frac{\epsilon\kappa}{z}\right)^{-s-i(\epsilon+\tau)/2}
$$

$$
z=\omega(r-r_-),\ \epsilon = 2M\omega,\ \kappa=\sqrt{1-a^2},\ \tau=(\epsilon-ma)/\kappa
$$

- Expansion procedure flows as: $\nu \to a_k^{\nu} \to {}_sR^{\pm}_{lm\omega}$.

- Forms as written are not suitable for expansion.

Methods: Recasting the Teukolsky functions

- Recast the Teukolsky functions using hypergeometric identities

$$
{}_2F_1(a,b,c;\xi) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(1-\xi)^{-a}{}_2F_1\left(a, c-b, a-b+1; \frac{1}{1-\xi}\right)
+ \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(b)\Gamma(c-b)}(1-\xi)^{-b}{}_2F_1\left(c-a, b, b-a+1; \frac{1}{1-\xi}\right),
$$

$$
U(a,b;\xi) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)}M(a,b;\xi) + \frac{\Gamma(b-1)}{\Gamma(a)}\xi^{1-b}M(a-b+1, 2-b;\xi)
$$

- The Teukolsky functions are recasted in the following form

$$
{}_sR^{+}_{lm\omega} = {}_sC^{+}_{lm\omega}(z)\frac{\Gamma(\nu+1-s+i\epsilon)}{\Gamma(\nu+1+s-i\epsilon)}\sum_{k=-\infty}^{\infty} a^{\nu}_{k}(2iz)^{k}\left(U^{k}_{1}(z)+U^{k}_{2}(z)\right),
$$

$$
{}_sR^{-}_{lm\omega} = {}_sC^{-}_{lm\omega}(z)\left(\frac{\epsilon\kappa}{z}\right)^{-i\tau}\sum_{k=-\infty}^{\infty} a^{\nu}_{k}\left(F^{k}_{1}(z)+F^{k}_{2}(z)\right).
$$

Methods: Teukolsky Sub-functions

- We then end up expanding the following functions

$$F_1^k = \frac{\Gamma(1-s-i\epsilon-i\tau)\Gamma(-2k-2\nu-1)}{\Gamma(-k-\nu-i\tau)\Gamma(-k-\nu-s-i\epsilon)}\left(\frac{\epsilon\kappa}{z}\right)^{k+\nu+1} {}_2F_1\left(k+1+\nu-i\tau, k+\nu+1-s-i\epsilon, 2k+2\nu+2; \frac{\epsilon\kappa}{z}\right),$$

$$F_2^k = \frac{\Gamma(1-s-i\epsilon-i\tau)\Gamma(2k+2\nu+1)}{\Gamma(k+\nu+1-i\tau)\Gamma(k+\nu+1-s-i\epsilon)}\left(\frac{\epsilon\kappa}{z}\right)^{-k-\nu} {}_2F_1\left(-k-\nu-s-i\epsilon, -k-\nu-i\tau, -2k-2\nu; \frac{\epsilon\kappa}{z}\right),$$

$$U_1^k = \frac{\Gamma(k+\nu+1+s-i\epsilon)\Gamma(-2k-2\nu-1)}{\Gamma(k+\nu+1-s+i\epsilon)\Gamma(-k-\nu+s-i\epsilon)} M(k+\nu+1+s-i\epsilon, 2k+2\nu+2; -2iz),$$

$$U_2^k = \frac{\Gamma(2k+2\nu+1)}{\Gamma(k+\nu+1-s+i\epsilon)}(-2iz)^{-2k-2\nu-1} M(-k-\nu+s-i\epsilon, -2k-2\nu; -2iz).$$

- In an analytic expansion, truncation rules depend on the leading order contribution, a truth table of leading order contribution necessary

Methods: Truth tables for $s = +2$

- Leading-order power of a_ν^k

	$k < 2l - 1$	$-2l \le k \le -l - 3$	$-l - 2 \le k \le -l - 1$	$k = -l$	$k \ge -l + 1$
$[a_k^\nu]_\epsilon^{l=2}$	$\|k\| - 2$	$\|k\| - 2$	$\|k\| - 2$	1	$\|k\|$
$[a_k^\nu]_\epsilon^{l\neq 2}$	$\|k\| - 2$	$\|k\|$	$\|k\| - 2$	$l - 1$	$\|k\|$

- Leading-order power of F_1^k, F_2^k, U_1^k, U_2^k

	$k \ge -l + 2$	$k = -l + 1$	$k = -l$	$k = -l - 1$	$k = -l - 2$	$k \le -l - 3$
$\left[F_1^k\right]_\eta$	$2k + 2l - 1$	1	-1	-1	1	$2k + 2l + 2$
$\left[F_2^k\right]_\eta$	$-2k - 2l$	1	-1	-1	1	$-2k - 2l - 3$
$\left[U_1^k\right]_\eta$	-3	-3	-3	-2	0	0
$\left[U_2^k\right]_\eta$	$-2k - 2l - 1$	-3	-3	-2	0	$-2k - 2l - 4$

Results: Infinity-side Fluxes

- For eccentric-equatorial orbits, the infinity-side fluxes have the following structure

$$\langle \dot{E} \rangle_\infty = \frac{32\mu^2(1-e^2)^{3/2}}{5M^2p^5}\Big(\mathcal{L}_0 + p^{-1}\mathcal{L}_1 + p^{-3/2}\mathcal{L}_{3/2} + p^{-2}\mathcal{L}_2 + p^{-5/2}\mathcal{L}_{5/2} + p^{-3}\left(\mathcal{L}_3 + \log(p)\mathcal{L}_{3L}\right)$$
$$+ p^{-7/2}\left(\mathcal{L}_{7/2} + \log(p)\mathcal{L}_{7/2L}\right) + p^{-4}\left(\mathcal{L}_4 + \log(p)\mathcal{L}_{4L}\right) + p^{-9/2}\left(\mathcal{L}_{9/2} + \log(p)\mathcal{L}_{9/2L}\right)$$
$$+ p^{-5}\left(\mathcal{L}_5 + \log(p)\mathcal{L}_{5L}\right) + p^{-11/2}\left(\mathcal{L}_{11/2} + \log(p)\mathcal{L}_{11/2L}\right) + p^{-6}\left(\mathcal{L}_6 + \log(p)\mathcal{L}_{6L} + \log^2(p)\mathcal{L}_{6L2}\right)$$
$$+ p^{-13/2}\left(\mathcal{L}_{13/2} + \log(p)\mathcal{L}_{13/2L} + \log^2(p)\mathcal{L}_{13/2L2}\right) + \cdots\Big)$$

- In the case of Kerr, the coefficients have additional structure

$$\mathcal{L}_{iLj}(e) \to \mathcal{L}_{iLj}(a,e) = \sum_{k=0}^{k=i} a^k \left(\mathcal{L}^{Sk}_{iLj}(e) + \mathcal{L}^{SkF}_{iLj}(a,e)\right)$$

- Expanded to 7PN and e^{20}.

Results: Sample Coefficients

- We present select coefficients for the infinity-side fluxes

$$\mathcal{L}^{S1}_{9/2} = -\frac{343985009}{498960} + \frac{1369\gamma}{9} - \frac{385\pi^2}{9} + \frac{95723\log(2)}{315} + \left(-\frac{33607234229}{2328480} + \frac{159356\gamma}{63} - \frac{11695\pi^2}{18} + \frac{33316\log(2)}{105} + \frac{188811\log(3)}{40}\right)e^2 + \left(-\frac{361951335499}{8467200} + \frac{3179761\gamma}{420} - \frac{21443\pi^2}{12} + \frac{158977523\log(2)}{1260} - \frac{130364883\log(3)}{2240}\right)e^4 + \left(-\frac{1005841269121}{22353408} + \frac{1719275\gamma}{252} - \frac{115525\pi^2}{72} - \frac{1619131561\log(2)}{1260} + \frac{5626660383\log(3)}{17920} + \frac{4982421875\log(5)}{13824}\right)e^6 + \left(-\frac{1449035920823}{59609088} + \frac{13553357\gamma}{8064} - \frac{487355\pi^2}{1152} + \frac{3086900646601\log(2)}{362880} + \frac{14207516721\log(3)}{16384} - \frac{39395017578125\log(5)}{9289728}\right)e^8 + \left(-\frac{2546262906169}{198696960} + \frac{107343\gamma}{2240} - \frac{3321\pi^2}{256} - \frac{191003235106681\log(2)}{4536000} - \frac{228062402092231\log(3)}{114688000} + \frac{1658023115234375\log(5)}{74317824} + \frac{688360827060361\log(7)}{88473600}\right)e^{10} + \cdots + \alpha_{20}e^{20} + \cdots,$$

$$\mathcal{L}^{S1F}_{13/2} = \left(\frac{256}{15} + \frac{3824}{5}e^2 + \frac{27306}{5}e^4 + \frac{51688}{5}e^6 + \frac{22911}{4}e^8 + \frac{6363}{8}e^{10} + \frac{805}{64}e^{12}\right) \times \left(\log\kappa + \frac{1}{2}\psi^{(0)}\left(\frac{2ia}{\kappa}\right) + \frac{1}{2}\psi^{(0)}\left(-\frac{2ia}{\kappa}\right)\right)$$

Results: Horizon-side Fluxes

- Horizon-side fluxes have their own structure

$$\langle \dot{E} \rangle_{\rm H} = \frac{32\mu^2(1-e^2)^{3/2}}{5M^2p^{15/2}}\Big(\mathcal{B}_{5/2} + p^{-1}\mathcal{B}_{7/2} + p^{-3/2}\mathcal{B}_4 + p^{-2}\mathcal{B}_{9/2} + p^{-5/2}\mathcal{B}_5 + p^{-3}\left(\mathcal{B}_{11/2} + \log(p)\mathcal{B}_{11/2L}\right)$$
$$+ p^{-7/2}\mathcal{B}_6 + p^{-4}\left(\mathcal{B}_{13/2} + \log(p)\mathcal{B}_{13/2L}\right) + p^{-9/2}\left(\mathcal{B}_7 + \log(p)\mathcal{B}_{7L}\right) + p^{-5}\left(\mathcal{B}_{15/2} + \log(p)\mathcal{B}_{15/2L}\right)$$
$$+ p^{-11/2}\left(\mathcal{B}_8 + \log(p)\mathcal{B}_{8L}\right) + p^{-6}\left(\mathcal{B}_{17/2} + \log(p)\mathcal{B}_{17/2L} + \log^2(p)\mathcal{B}_{17/2L2}\right) + \cdots\Big)$$

- Red coefficients survive as $a \to 0$ (Munna, Evans, Forseth (2023))

- Same additional structure present

$$\mathcal{B}_{iLj}(e) \to \mathcal{B}_{iLj}(a,e) = \sum_{k=0} a^k \left(\mathcal{B}^{Sk}_{iLj}(e) + \mathcal{B}^{SkF}_{iLj}(a,e)\right)$$

- Expanded to 6PN and e^{20}

Results: Sample Coefficients

- We present select coefficients for the horizon-side fluxes:

$$\mathcal{B}_{5/2} = -\frac{1}{4}a(1+3a^2)\left(1+\frac{15}{2}e^2+\frac{45}{8}e^4+\frac{5}{16}e^6\right)$$

$$\begin{aligned}\mathcal{B}_4 =& \frac{1}{12}\left(42a^4+12i\left(3a^3+a\right)\psi^{(0)}\left(-\frac{2ia}{\kappa}\right)-12i\left(3a^3+a\right)\psi^{(0)}\left(\frac{2ia}{\kappa}\right)-18a^2\kappa+85a^2-6\kappa+6\right)\\ &+\frac{1}{48}e^2\left(9a^4\kappa+2217a^4+726i\left(3a^3+a\right)\psi^{(0)}\left(-\frac{2ia}{\kappa}\right)-726i\left(3a^3+a\right)\psi^{(0)}\left(\frac{2ia}{\kappa}\right)-1107a^2\kappa\right.\\ &\left.+4991a^2-354\kappa+372\right)+\frac{5}{64}e^4\left(9a^4\kappa+1185a^4+390i\left(3a^3+a\right)\psi^{(0)}\left(-\frac{2ia}{\kappa}\right)\right.\\ &\left.-390i\left(3a^3+a\right)\psi^{(0)}\left(\frac{2ia}{\kappa}\right)-603a^2\kappa+2659a^2-186\kappa+204\right)+\frac{5}{384}e^6\left(27a^4\kappa+2811a^4\right.\\ &\left.+834i\left(3a^3+a\right)\psi^{(0)}\left(-\frac{2ia}{\kappa}\right)-834i\left(3a^3+a\right)\psi^{(0)}\left(\frac{2ia}{\kappa}\right)-1305a^2\kappa+5741a^2-390\kappa+444\right)\\ &+\frac{5}{3072}e^8\left(9a^4\kappa+885a^4+222i\left(3a^3+a\right)\psi^{(0)}\left(-\frac{2ia}{\kappa}\right)-222i\left(3a^3+a\right)\psi^{(0)}\left(\frac{2ia}{\kappa}\right)-351a^2\kappa\right.\\ &\left.+1565a^2-102\kappa+120\right)\end{aligned}$$

Summary

- A procedure for expanding the $s = +2$ Teukolsky functions was developed.
  - Expanded infinity-side fluxes up to 7PN and e^{20}.
  - Expanded horizon-side fluxes up to 6PN and e^{20}.
- Extend calculation to…
  - Conservative quantities
  - Spherical inclined orbits

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