

# Slow-rotation black hole perturbation theory

Nicola Franchini

*Laboratoire Astroparticule & Cosmologie  
Université Paris Cité*

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# BH perturbation in spherical symmetry

$$g_{ab} = g_{ab}^{(0)} + \varepsilon g_{ab}^{(1)}$$



$$R_{ab} = R_{ab}^{(0)} + \varepsilon R_{ab}^{(1)}$$

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Let's assume vacuum:

$$R_{ab}^{(0)} = 0 \longrightarrow g_{ab}^{(0)} = g_{ab}^{\text{Sch}} \quad \text{is the Schwarzschild metric}$$

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$$G=c=2M=1$$

Let's assume vacuum:

$$R_{ab}^{(0)} = 0 \longrightarrow g_{ab}^{(0)} = g_{ab}^{\text{Sch}} \begin{pmatrix} -1 + \frac{1}{r} & 0 & 0 & 0 \\ 0 & \left(1 - \frac{1}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{pmatrix}$$

# BH perturbation in spherical symmetry

For the perturbation we solve  $R_{ab}^{(1)} = 0$

with ansatz  $g_{ab}^{(1)} = g_{ab}^{(1,+)} + g_{ab}^{(1,-)}$

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Polar component

$$(-1)^\ell$$

Parity  
transformation

$\theta \rightarrow \pi - \theta$
$\varphi \rightarrow \varphi + \pi$

Axial component

$$(-1)^{\ell+1}$$

# BH perturbation in spherical symmetry

$$g_{ab}^{(1,+)} = \begin{pmatrix} H_0^\ell(r) & H_1^\ell(r) & 0 & 0 \\ H_1^\ell(r) & H_2^\ell(r) & 0 & 0 \\ 0 & 0 & r^2 K^\ell(r) & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta K^\ell(r) \end{pmatrix} Y^\ell(\theta, \varphi) e^{-i\omega t}$$

Regge-Wheeler gauge

$$g_{ab}^{(1,-)} = \begin{pmatrix} 0 & 0 & h_0^\ell(r) S_\theta^\ell(\theta, \varphi) & h_0^\ell(r) S_\varphi^\ell(\theta, \varphi) \\ 0 & 0 & h_1^\ell(r) S_\theta^\ell(\theta, \varphi) & h_1^\ell(r) S_\varphi^\ell(\theta, \varphi) \\ * & * & 0 & 0 \\ * & * & 0 & 0 \end{pmatrix} e^{-i\omega t}$$

$$(S_\theta^{\ell m}, S_\varphi^{\ell m}) = \left( -\frac{Y_{,\varphi}^{\ell m}}{\sin \theta}, \sin \theta Y_{,\theta}^{\ell m} \right)$$

# BH perturbation in spherical symmetry

**Black box** - manipulate the equations

- 7x polar eqs. - 3x axial eqs.
- 2x 1st order polar eqs. - 2x 1st order axial eqs.
- Definition of master variables



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$$\frac{d^2 \Psi_{\pm}^{\ell}}{dr_*^2} + \left( \omega^2 - V_{\pm}^{\ell} \right) \Psi_{\pm}^{\ell} = 0$$

With tortoise  
coordinate defined as  $dr_* = \frac{dr}{1 - 1/r}$

# BH perturbation in spherical symmetry

Regge-Wheeler potential for axial tensor perturbation

$$V_{-}^{\ell} = \left(1 - \frac{1}{r}\right) \left[ \frac{\ell(\ell + 1)}{r^2} - \frac{3}{r^3} \right]$$

Zerilli potential for polar tensor perturbation,  $\lambda = \ell(\ell + 1) - 2$

$$V_{+}^{\ell} = \left(1 - \frac{1}{r}\right) \left[ \frac{\ell(\ell + 1)}{r^2} - \frac{3}{r^3} \frac{r^2 \lambda(\lambda + 4) + 6r - 3}{(3 + r\lambda)^2} \right]$$

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Let's assume vacuum **with slow rotation**:

$$R_{ab}^{(0)} = 0 \longrightarrow g_{ab}^{(0)} = g_{ab}^{\text{Sch}} + a g_{ab}^{\text{SR},1} + a^2 g_{ab}^{\text{SR},2}$$

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# BH perturbation in slow rotation

$$g_{ab} = g_{ab}^{(0)} + \varepsilon g_{ab}^{(1)}$$

Same ansatz for perturbations!

$$R_{ab} = R_{ab}^{(0)} + \varepsilon R_{ab}^{(1)}$$

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# BH perturbation in slow rotation

**Black box** - manipulate the equations

- 10 mixed equations
- Formalism to decouple
  - Project onto proper angular basis
  - Introduce couplings with opposite parity
- 7x polar-**led** eqs. - 3x axial-**led** eqs.
- 2x 1st order polar-**led** eqs. - 2x 1st order axial-**led** eqs.
- Definition of master variables **that diagonalize the system**

$$\mathcal{P}^l + a \overline{\mathcal{P}}^{l \pm 1}$$

# BH perturbation in slow rotation

$$\frac{d^2 \Psi_{\pm}^{\ell}}{dr_*^2} + \left( \omega^2 - V_{\pm}^{\ell} \right) \Psi_{\pm}^{\ell} = 0$$

$$dr_* = \frac{dr}{(1 - 1/r)(1 + af_1)}$$

$$V_{\pm}^{\ell} = V_{\pm,0}^{\ell} + aV_{\pm,1}^{\ell}$$

In the end, there is just a spin modification



# BH perturbation in slow rotation

$$\frac{d^2 \Psi_{\pm}^{\ell}}{dr_*^2} + \left( \omega^2 - V_{\pm}^{\ell} \right) \Psi_{\pm}^{\ell} = 0$$

$$dr_* = \frac{dr}{(1 - 1/r) (1 + a f_1 + a^2 f_2)}$$

$$V_{\pm}^{\ell} = V_{\pm,0}^{\ell} + a V_{\pm,1}^{\ell} + a^2 V_{\pm,2}^{\ell}$$

In the end, there is just a spin modification also valid at second order in the spin

# Relation between potentials

Chandrasekhar found the *super-potential* that generates the RW/Z potential

$$V_{\pm,0}^{\ell} = \beta_0^2 W_0^{\ell 2} + \beta_0 \frac{dW_0^{\ell}}{dr_*} + \kappa_0 W_0^{\ell}$$

with

$$W_0^{\ell} = \frac{1 - 1/r}{r(3 + \lambda r)}$$

$$\beta_0 = \pm 3, \quad \kappa_0 = \lambda(\lambda + 2)$$

# Relation between potentials

The superpotential still exists at least up to second order in the spin!

$$V_{(\pm)}^{\ell} = \beta_0^2 W^{\ell 2} + \beta_0 \frac{dW^{\ell}}{dr_*} + \kappa_0 W^{\ell} + \kappa_0 \kappa,$$

with

$$W^{\ell} = W_0^{\ell} + aW_1^{\ell} + a^2W_2^{\ell}$$

$$\beta_0 = \pm 3, \quad \kappa = a\kappa_1 + a^2\kappa_2$$

# Relation between potentials

Chandra's superpotential

Regge-Wheeler

Zerilli



# Relation between potentials

Chandra's superpotential  
generalized to 1st and 2nd order in spin

Regge-Wheeler

Zerilli

The transformation proves *isospectrality* between the two equations

# Relation between RW/Z and Teukolsky

Chandra's superpotential  
generalized to 1st and 2nd order in spin

Regge-Wheeler

Zerilli

Chandra's transformation  
theory

Teukolsky

# Relation between RW/Z and Teukolsky

Chandra's superpotential  
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Regge-Wheeler

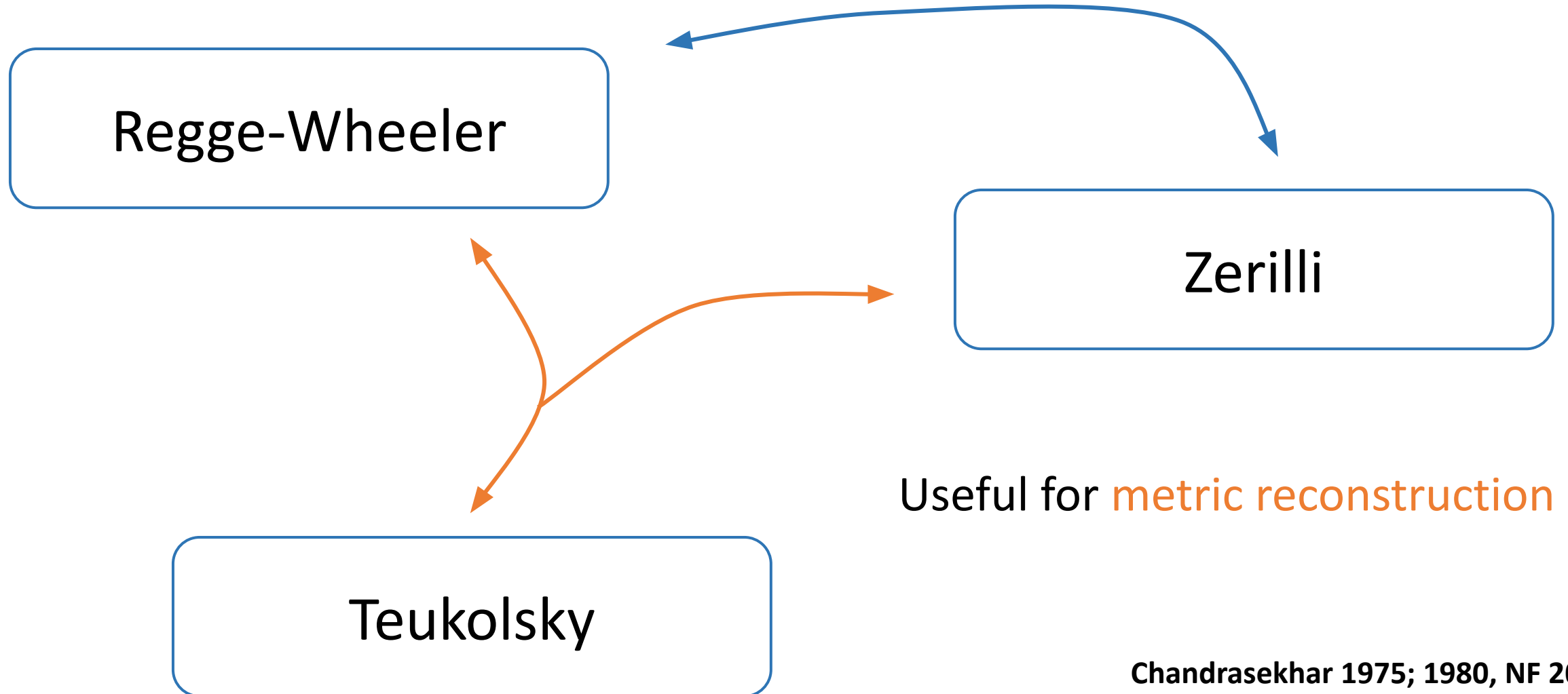
Zerilli

Chandra's transformation  
theory

Teukolsky

Also valid up to 2nd order in  
the spin!

# Relation between RW/Z and Teukolsky





# Conclusions

- We showed how to generalize the Regge-Wheeler and the Zerilli equation in a **spin expansion**
- **Chandra's transformations** between RWZ are still applicable, as well as those to slow-rotating Teukolsky
- Useful for “*perturbation of perturbation*” problem (second order perturbations, beyond-GR BHPT, etc...)

# Open problems

## Kerr metric perturbation conjecture

Is it possible to generalize the Regge-Wheeler and the Zerilli equations at arbitrary spin, without passing through the Teukolsky formalism?

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Thanks for the attention =D

