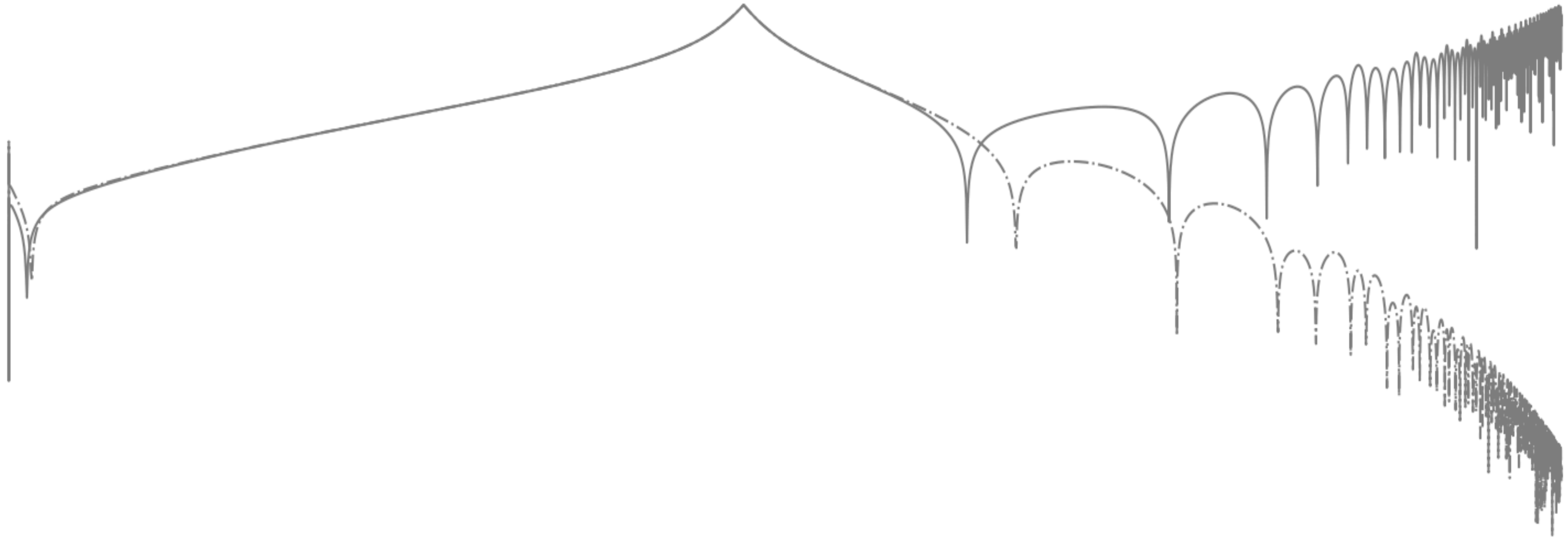
Comparing different metric reconstruction procedures in Kerr spacetime



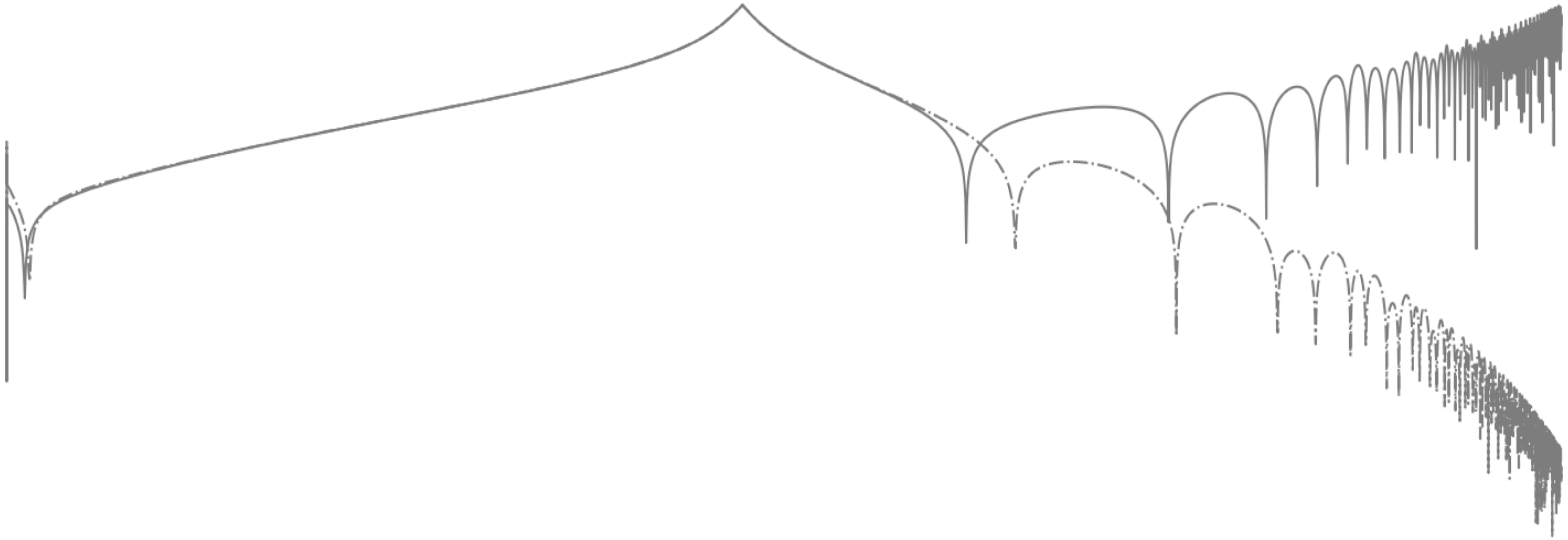
Zachary Nasipak
NASA Postdoctoral Fellow
Goddard Space Flight Center

03 July 2023

26th Capra Meeting 2023
Niels Bohr Institute
Copenhagen, Denmark

Tools for

^Comparing different metric reconstruction procedures in Kerr spacetime



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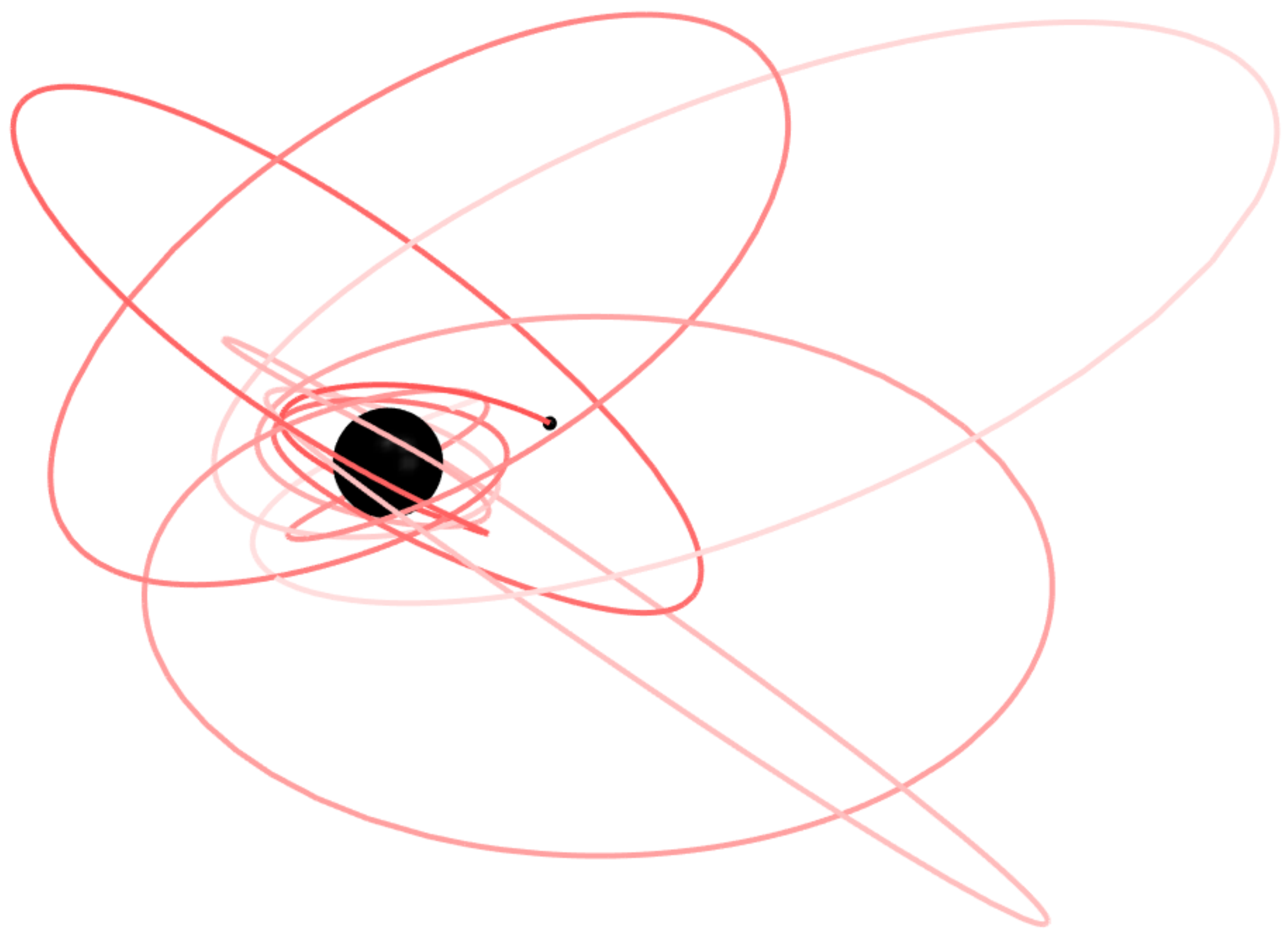
NPP NASA Postdoctoral Program

Motivation



- **Solve for perturbations of Kerr**

$$g_{\mu\nu} = g_{\mu\nu}^{\text{Kerr}} + \epsilon h_{\mu\nu}^{(1)} + \epsilon^2 h_{\mu\nu}^{(2)} + O(\epsilon^3)$$

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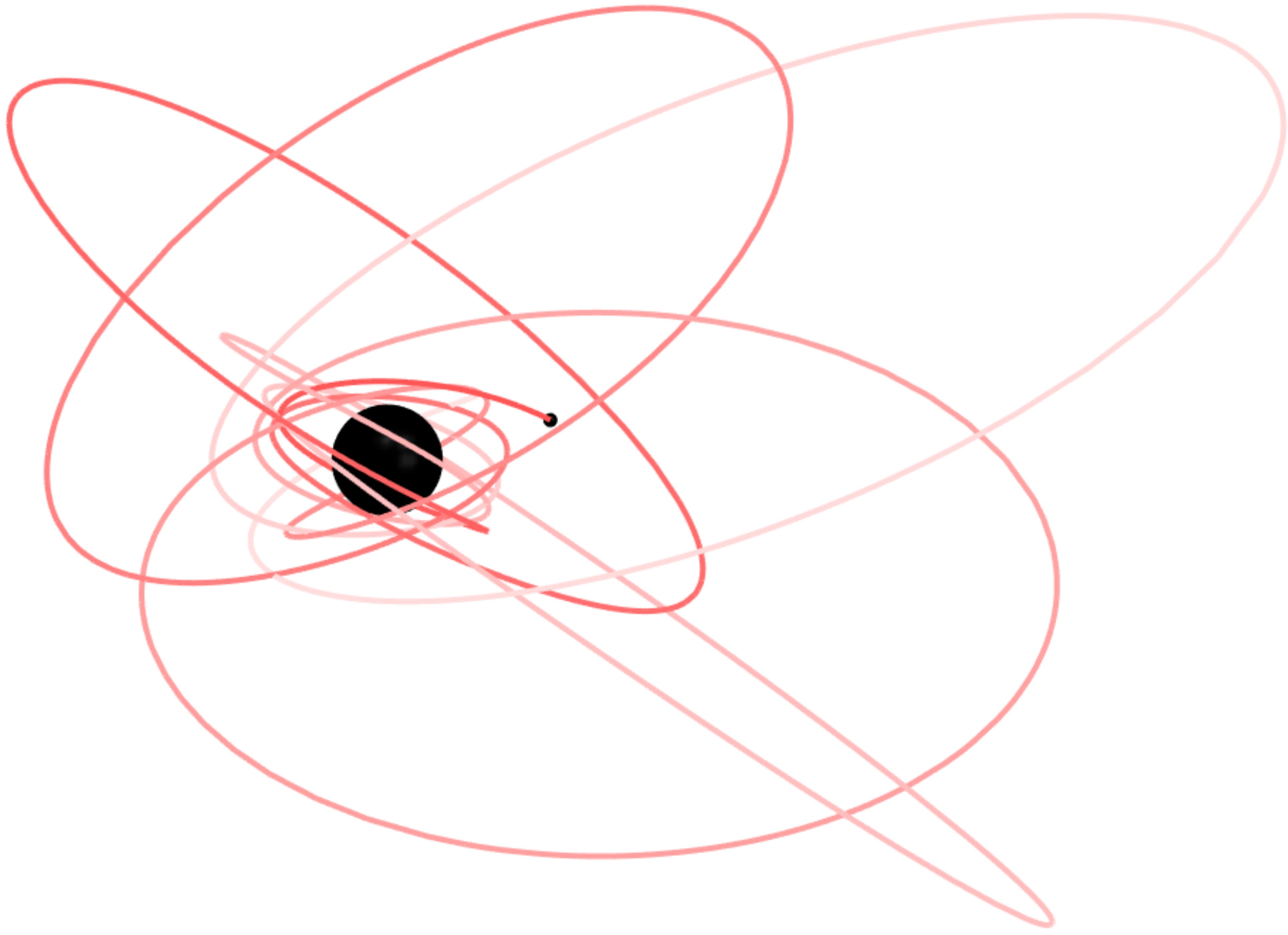


- **Solve for perturbations of Kerr**

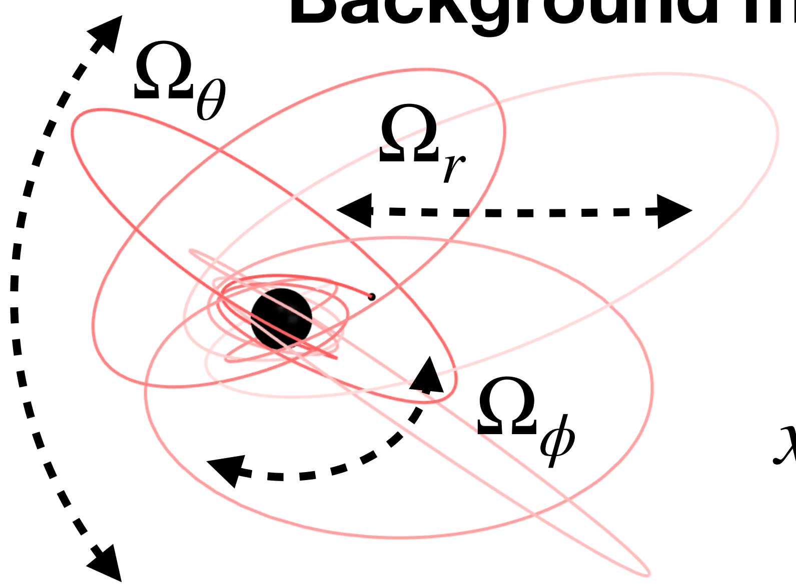
$$g_{\mu\nu} = g_{\mu\nu}^{\text{Kerr}} + \epsilon h_{\mu\nu}^{(1)} + \epsilon^2 h_{\mu\nu}^{(2)} + O(\epsilon^3)$$

- **Considerations for 1st-order**
  - Dealing w/ lack of spherical symmetry
    - Teukolsky Eqs vs Einstein Eqs
    - Frequency vs time domain
  - Gauge(s)
    - Lorenz, radiation, Bondi-Sachs, etc
  - Covering 4D parameter space
  - Sufficiently regular data for 2nd-order
    - Puncture schemes, regularisation
  - Accessible, open-source codes

A roadmap for Kerr*

A roadmap for Kerr*

Background motion



(a, p, e, x)

↓

$x_p^\mu(\lambda); (E, L_z, Q)$

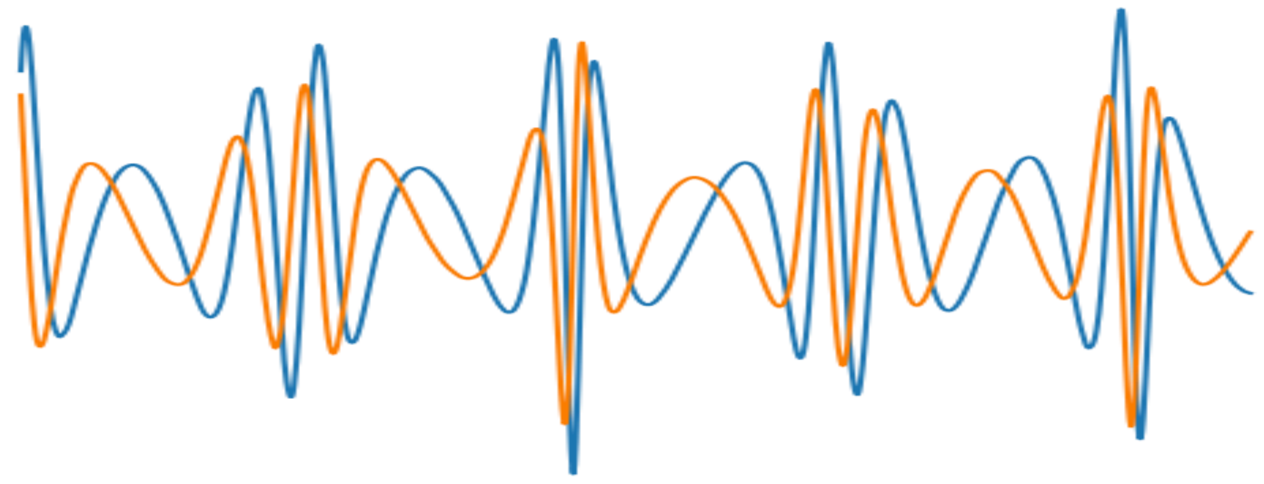
Fluxes

$\langle \dot{E} \rangle$

$\langle \dot{L}_z \rangle$

$\langle \dot{Q} \rangle$

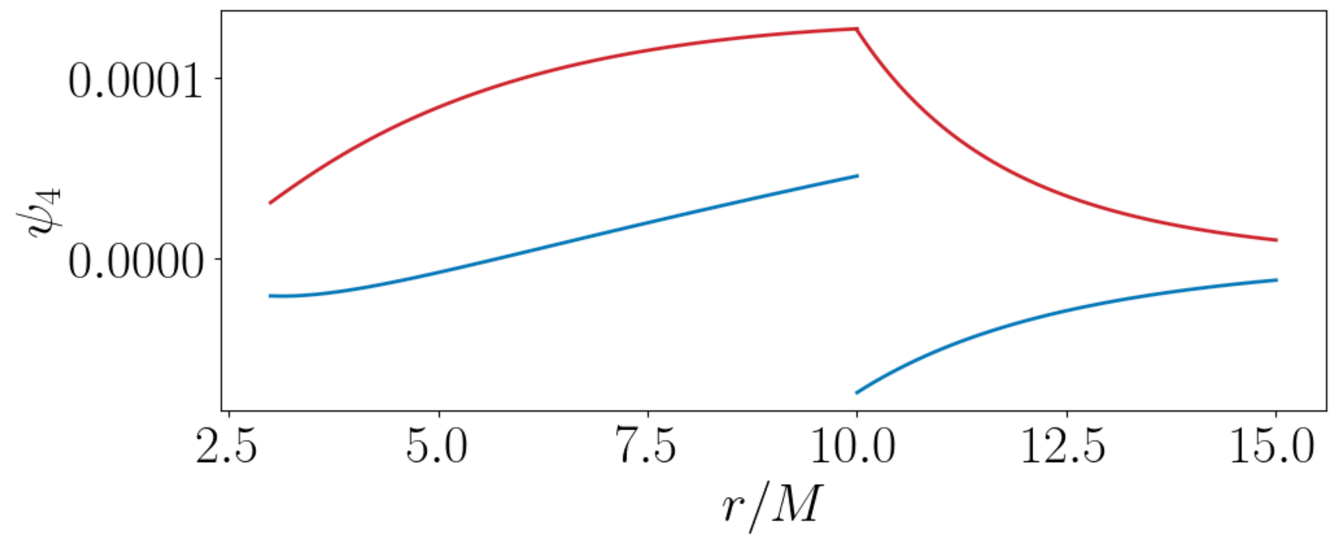
Inspirals + Waveforms



Teukolsky equation

$$\mathcal{T}_0 h = \psi_0 \qquad \mathcal{O}\psi_0^{(1)} = 8\pi T_0^{(1)}[x_p]$$

$$\mathcal{T}_4 h = \psi_4 \qquad \mathcal{O}'\psi_4^{(1)} = 8\pi T_4^{(1)}[x_p]$$



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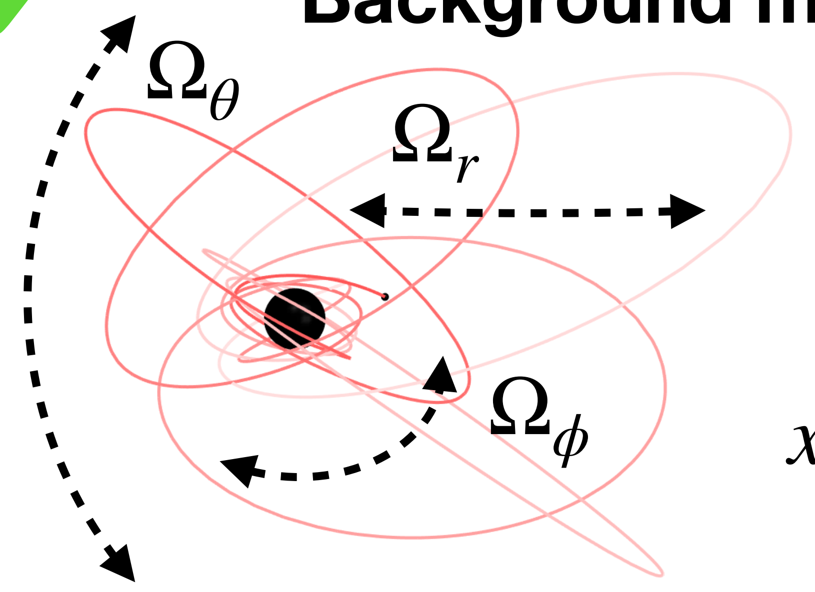

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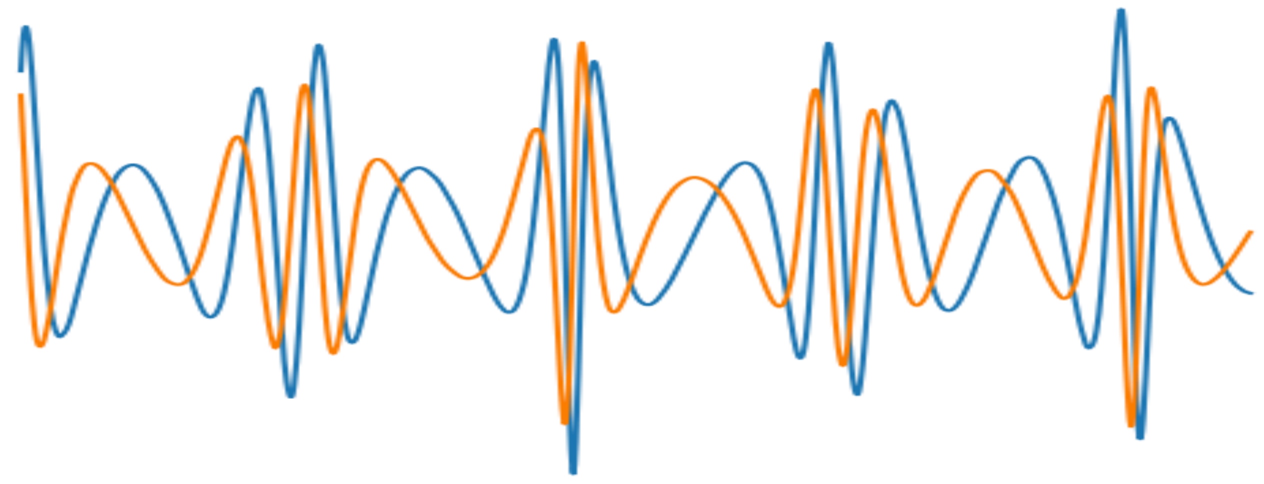

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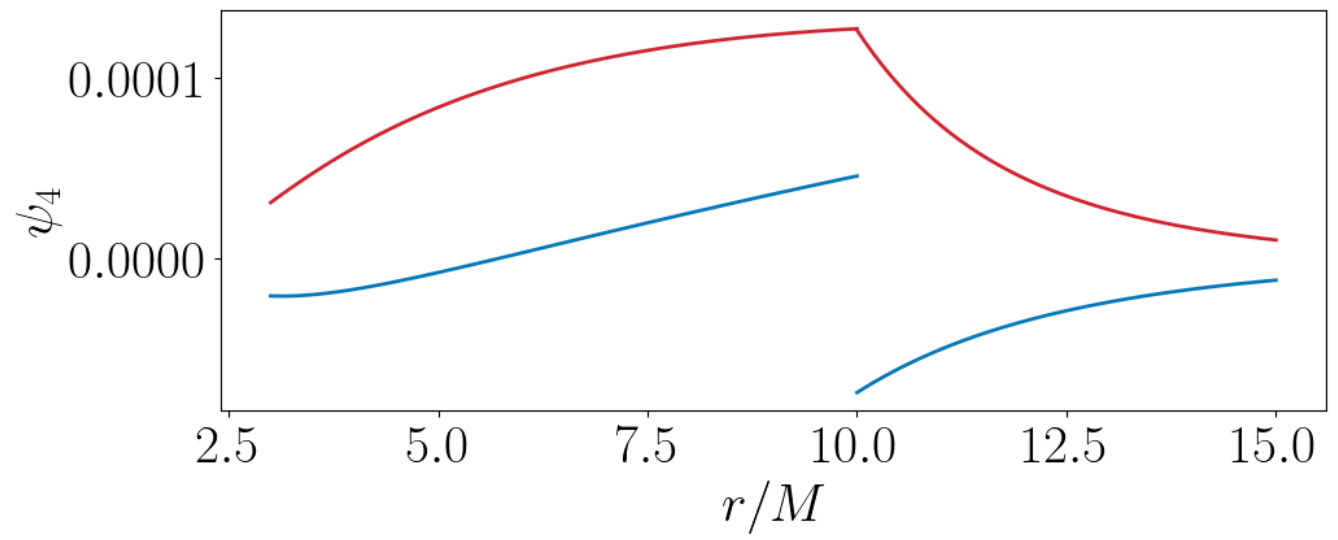
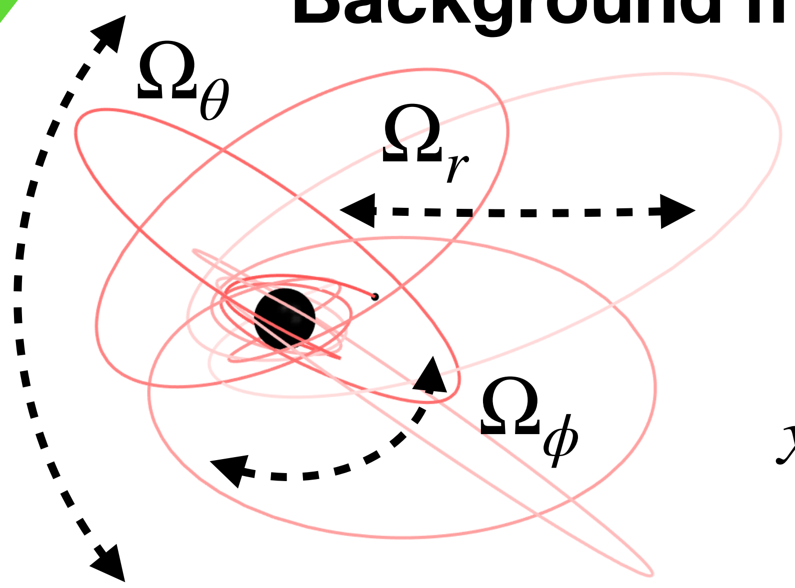


A roadmap for Kerr*

pybhpt

Background motion



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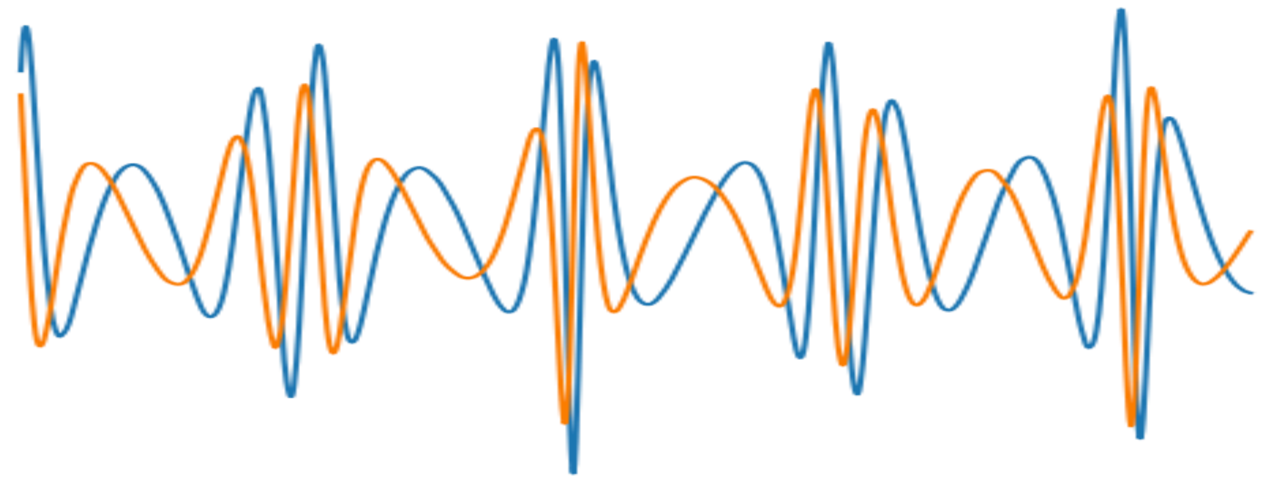

1st-order GSF

$$u^\alpha F^{(1)}_\alpha = 0$$

$$F^{(1)}_\alpha = \nabla_{\alpha\mu\nu} h^{\mu\nu}_{\mathrm{R}(1)}$$

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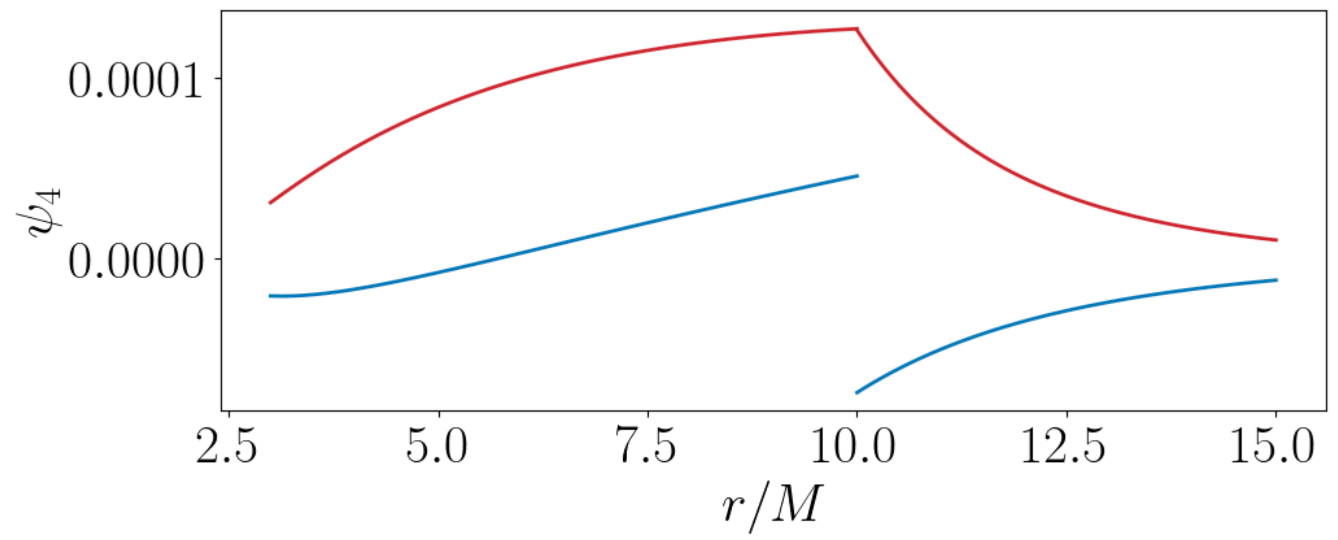


Metric reconstruction

1st-order metric perturbation

$h^{(1)}_{\mu\nu}$, $h^{\mathrm{R}(1)}_{\mu\nu}$

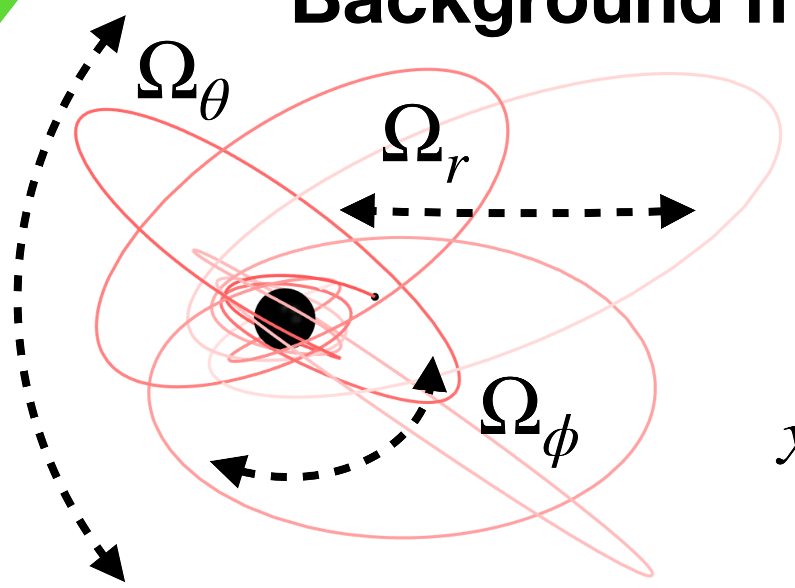
Generalized Detweiler redshift

2nd-order

A roadmap for Kerr*

pybhpt

Background motion



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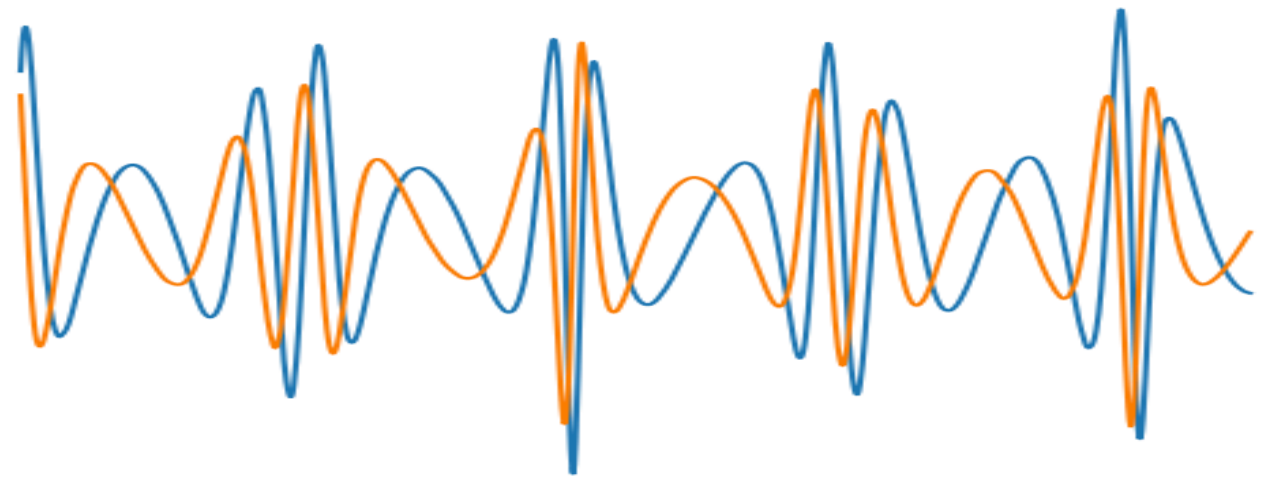

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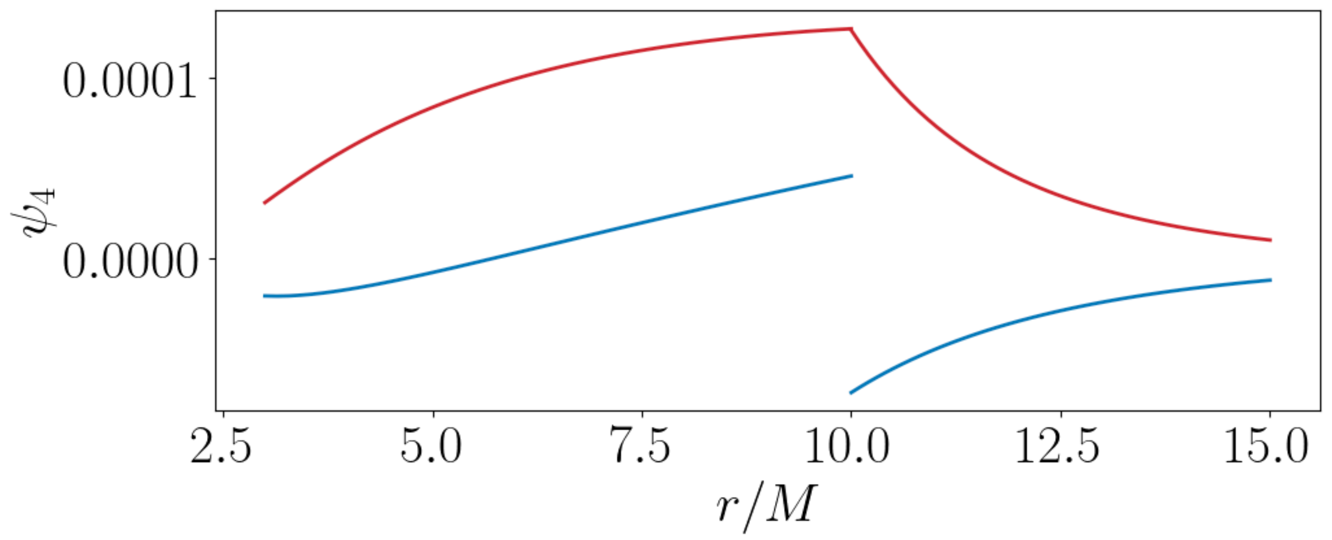

Metric reconstruction

1st-order metric perturbation

$$h_{\mu\nu}^{(1)},\ h_{\mu\nu}^{\mathrm{R}(1)}$$

Generalized Detweiler redshift

2nd-order

Metric reconstruction

$$h_{\mu\nu} = \sum_X [\mathrm{Re}(S_4^\dagger \Phi_{+2}^X)_{\mu\nu} + \mathrm{Re}(S_0^\dagger \Phi_{-2}^X)_{\mu\nu}] + \dot{\tilde{g}}_{\mu\nu} + x_{\mu\nu} + \mathscr{L}_\xi g_{\mu\nu}$$

Metric reconstruction

Completion $h_{\mu\nu}^{\ell=0,1}$

$$h_{\mu\nu} = \sum_X [\mathrm{Re}(S_4^\dagger \Phi_{+2}^X)_{\mu\nu} + \mathrm{Re}(S_0^\dagger \Phi_{-2}^X)_{\mu\nu}] + \dot{\tilde{g}}_{\mu\nu} + x_{\mu\nu} + \mathscr{L}_\xi g_{\mu\nu}$$

Gauge

Corrector tensor

Metric reconstruction

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Gauge

Hertz potentials

$$\mathscr{O}\Phi_{+2,i} = \eta_{+2,i} \qquad \mathscr{O}'\Phi_{-2} = \eta_{-2,i}$$

$$\psi_0^X, \psi_4^X \to \Phi_{-2}^X, \Phi_{+2}^X$$

Corrector tensor

Metric reconstruction

Completion $h_{\mu\nu}^{\ell=0,1}$

$$h_{\mu\nu} = \sum_X [\mathrm{Re}(S_4^\dagger \Phi^X_{+2})_{\mu\nu} + \mathrm{Re}(S_0^\dagger \Phi^X_{-2})_{\mu\nu}] + \dot{\tilde{g}}_{\mu\nu} + x_{\mu\nu} + \mathscr{L}_\xi g_{\mu\nu}$$

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$$\psi_0^X, \psi_4^X \to \Phi^X_{-2}, \Phi^X_{+2}$$

Corrector tensor

$$\mathscr{O}'\psi_4^{(-)} = 0 \qquad \mathscr{O}\psi_0^{(-)} = 0$$

$$\mathscr{O}'\psi_4^{\mathrm{M}} = 8\pi T_4^M \qquad \mathscr{O}\psi_0^{\mathrm{M}} = 8\pi T_0^M$$

$$\mathscr{O}'\psi_4^{(+)} = 0 \qquad \mathscr{O}\psi_0^{(+)} = 0$$

$r_+ \leftarrow r$ r_{min} r_{max} $r \to \infty$

Metric reconstruction

Completion $h_{\mu\nu}^{\ell=0,1}$

$$h_{\mu\nu} = \sum_X [\mathrm{Re}(S_4^\dagger \Phi_{+2}^X)_{\mu\nu} + \mathrm{Re}(S_0^\dagger \Phi_{-2}^X)_{\mu\nu}] + \dot{\tilde{g}}_{\mu\nu} + x_{\mu\nu} + \mathscr{L}_\xi g_{\mu\nu}$$

Gauge

Hertz potentials

$$\mathcal{O}\Phi_{+2,i} = \eta_{+2,i} \qquad \mathcal{O}'\Phi_{-2} = \eta_{-2,i}$$

$$\psi_0^X, \psi_4^X \rightarrow \Phi_{-2}^X, \Phi_{+2}^X$$

Corrector tensor

Focus on vacuum

$$\mathcal{O}'\psi_4^{(-)} = 0 \qquad \mathcal{O}\psi_0^{(-)} = 0$$

$$\mathcal{O}'\psi_4^{\mathrm{M}} = 8\pi T_4^M \qquad \mathcal{O}\psi_0^{\mathrm{M}} = 8\pi T_0^M$$

$$\mathcal{O}'\psi_4^{(+)} = 0 \qquad \mathcal{O}\psi_0^{(+)} = 0$$

$r_+ \leftarrow r$ | $r_{\min}$ | $r_{\max}$ | $r \rightarrow \infty$

Reconstruction procedures

- **GHZ(+) method**
  - Green, Hollands, Zimmerman (2020)
  - Toomani, Zimmerman, Spiers, Hollands, Pound, Green (2021)
- **AAB(+) method**
  - Dolan, Kavanagh, Wardell (2022)
  - Dolan, Durkan, Kavanagh, Wardell (2023)

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  - *Vacuum regions:* construct "shadowless" Hertz potential via CCK+Ori procedure

Outgoing radiation gauge

$$h_{\mu\nu}^{\mathrm{ORG}} = 2\mathrm{Re}(S_4^\dagger \Phi_{+2}^{\mathrm{ORG}})_{\mu\nu}$$

$$\partial_r^4 \bar{\Phi}^{\mathrm{ORG}} \sim \psi_4$$

Ingoing radiation gauge

$$h_{\mu\nu}^{\mathrm{IRG}} = 2\mathrm{Re}(S_0^\dagger \Phi_{-2}^{\mathrm{IRG}})_{\mu\nu}$$

$$\partial_r^4 \bar{\Phi}^{\mathrm{IRG}} \sim \psi_0$$

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Reconstruction procedures

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Outgoing radiation gauge

$$h^{\mathrm{ORG}}_{\mu\nu} = 2\mathrm{Re}(S_4^\dagger \Phi^{\mathrm{ORG}}_{+2})_{\mu\nu}$$

$$\partial_r^4 \bar{\Phi}^{\mathrm{ORG}} \sim \psi_4$$

Ingoing radiation gauge

$$h^{\mathrm{IRG}}_{\mu\nu} = 2\mathrm{Re}(S_0^\dagger \Phi^{\mathrm{IRG}}_{-2})_{\mu\nu}$$

$$\partial_r^4 \bar{\Phi}^{\mathrm{IRG}} \sim \psi_0$$

- **AAB(+) method**
  - Dolan, Kavanagh, Wardell (2022)
  - Dolan, Durkan, Kavanagh, Wardell (2023)
  - *Vacuum regions:* CCK+Ori-like procedure w/ linear combination of spin-weights

Anti-symmetric gauge

$$\hat{h}^{\mathrm{aAAB}}_{\mu\nu} = \mathrm{Re}(S_0^\dagger \Phi^{\mathrm{aAAB}}_{-2})_{\mu\nu} - \mathrm{Re}(S_4^\dagger \Phi^{\mathrm{aAAB}}_{+2})_{\mu\nu}$$

$$\dot{\Phi}^{\mathrm{aAAB}}_0 \sim \psi_0 \qquad \dot{\Phi}^{\mathrm{aAAB}}_4 \sim \psi_4$$

Symmetric gauge

$$\hat{h}^{\mathrm{sAAB}}_{\mu\nu} = \mathrm{Re}(S_0^\dagger \Phi^{\mathrm{sAAB}}_{-2})_{\mu\nu} + \mathrm{Re}(S_4^\dagger \Phi^{\mathrm{sAAB}}_{+2})_{\mu\nu}$$

$$\eth^4 \bar{\Phi}^{\mathrm{sAAB}}_0 \sim \psi_0 \qquad \eth'^4 \bar{\Phi}^{\mathrm{sAAB}}_4 \sim \psi_4$$

Hertz potentials w/ pybhpt

Load pybhpt

```
from pybhpt.geo import KerrGeodesic
from pybhpt.teuk import TeukolskyMode
from pybhpt.hertz import HertzMode
from pybhpt.hertz import available_gauges
import numpy as np
print(available_gauges)
```

[1] ✓ 0.2s Python

```
['IRG', 'ORG', 'SAAB0', 'SAAB4', 'ASAAB0', 'ASAAB4']
```

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```
['IRG', 'ORG', 'SAAB0', 'SAAB4', 'ASAAB0', 'ASAAB4']
```

Calculate background geodesic

```
a, p, e, x, nsamples = (0.9, 8., 0.2, 0.9, 2**9)
geo = KerrGeodesic(a, p, e, x, nsamples)
```

[2] ✓ 0.3s Python

Construct ψ_4

```
s, j, m, k, n = (-2, 2, 2, 1, 3)
teuk = TeukolskyMode(-2, j, m, k, n, geo)
teuk.solve(geo)
```

[3] ✓ 0.1s Python

Hertz potentials w/ pybhpt

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```

[3] ✓ 0.1s Python

Produce Hertz potentials Φ from ψ_4

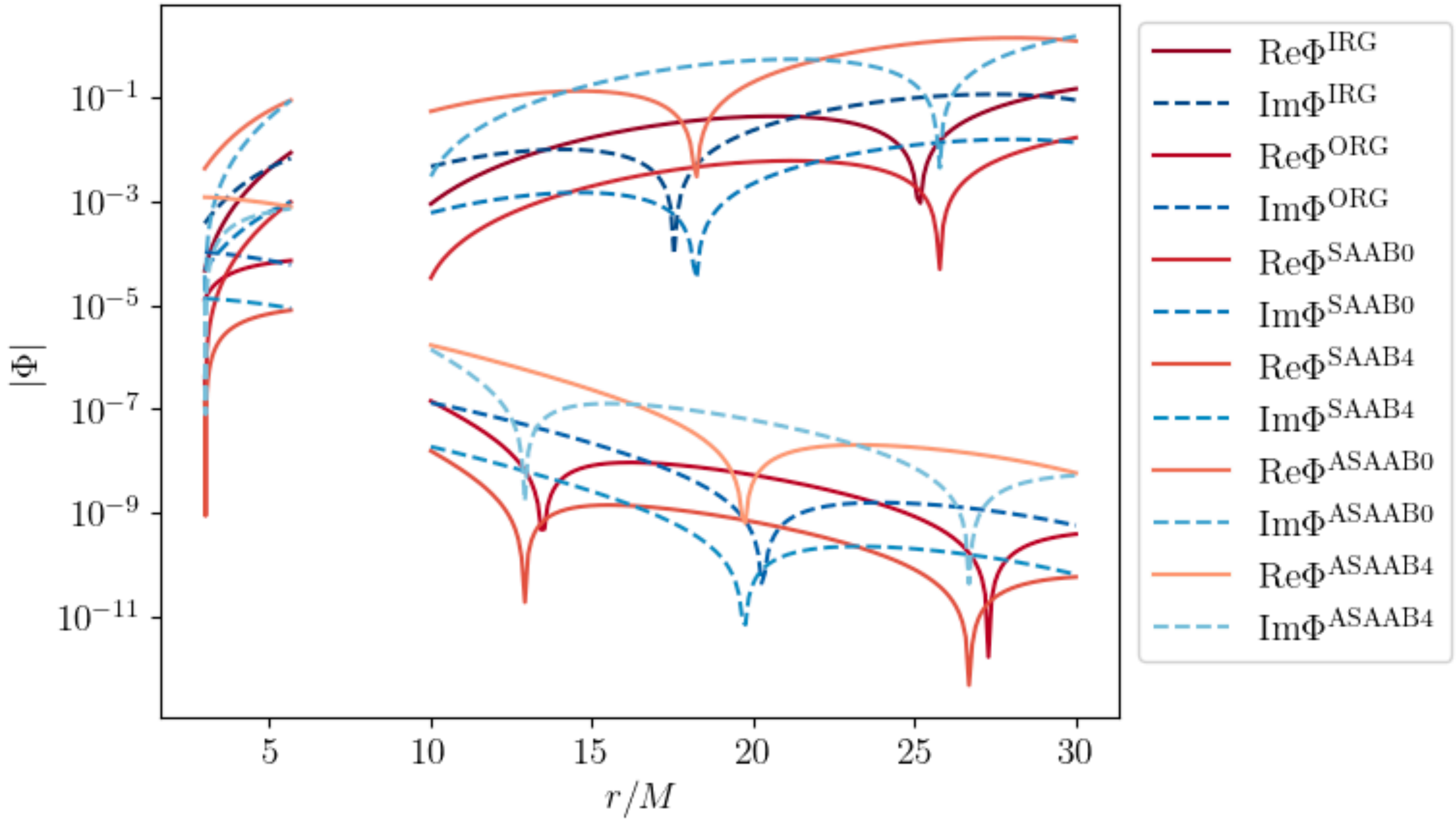
[4]

```
rmin, rmax = geo.radialpoints[[0, -1]]
rinner = np.linspace(3., rmin - 0.001, 200)
rupper = np.linspace(rmax + 0.001, 30, 200)
r = np.concatenate((rinner, rupper))
```

[4]

```
phi = {}; PhiOfR = {}
for gauge in available_gauges:
    phi[gauge] = HertzMode(teuk, gauge)
    phi[gauge].solve()
    PhiOfR[gauge] = phi[gauge](r)
```

✓ 0.2s Python Python



Metric perturbations w/ pybhpt

$$h_{ab} = \sum_{n_i} \sum_{s=\pm 2} \tilde{h}^{(n_t,n_r,n_s,n_\phi)}_{ab,s}(r,\theta)\ \partial_t^{n_t}\ \partial_r^{n_z}\ \eth_\pm^{n_s}\ \partial_\phi^{n_\phi}\ \Phi_s(t,r,\theta,\phi)$$

Metric perturbations w/ pybhpt

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Evaluate metric coefficients

```
from pybhpt.metric import MetricCoefficients

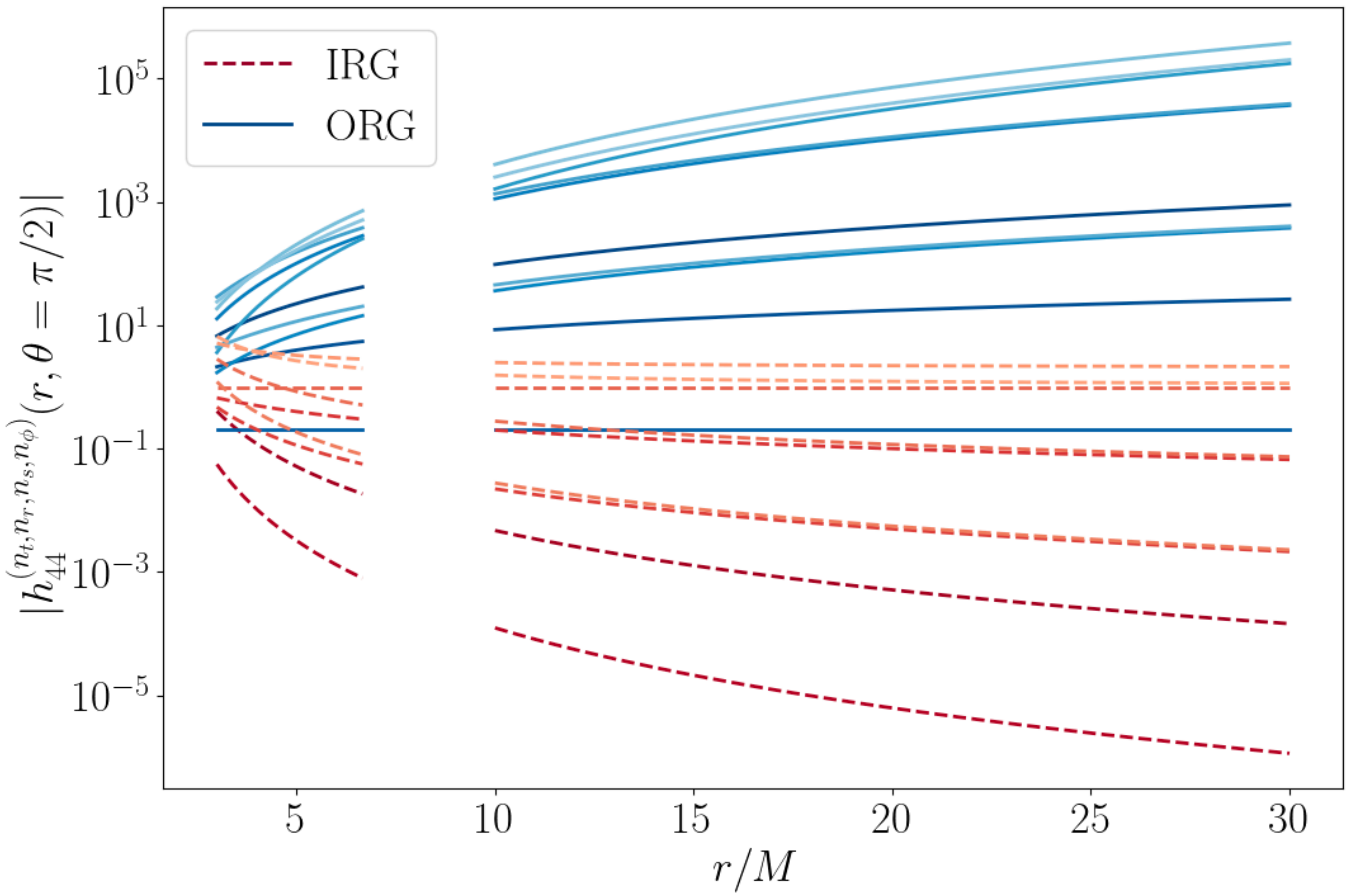
th = np.array([0.5*np.pi])
habIRG = MetricCoefficients("IRG", a, r, th)
habORG = MetricCoefficients("ORG", a, r, th)
```

[7] ✓ 0.8s Python

```
a, b = (2, 2)
nt, nr, ns, nphi = (0, 0, 2, 0)
print(np.max(np.abs(habIRG(a, b, nt, nr, ns, nphi))))
print(habIRG(a, b, nt, nr, ns, nphi).shape)
```

[8] ✓ 0.1s Python

```
... 0.11111111111111109
(400, 1)
```



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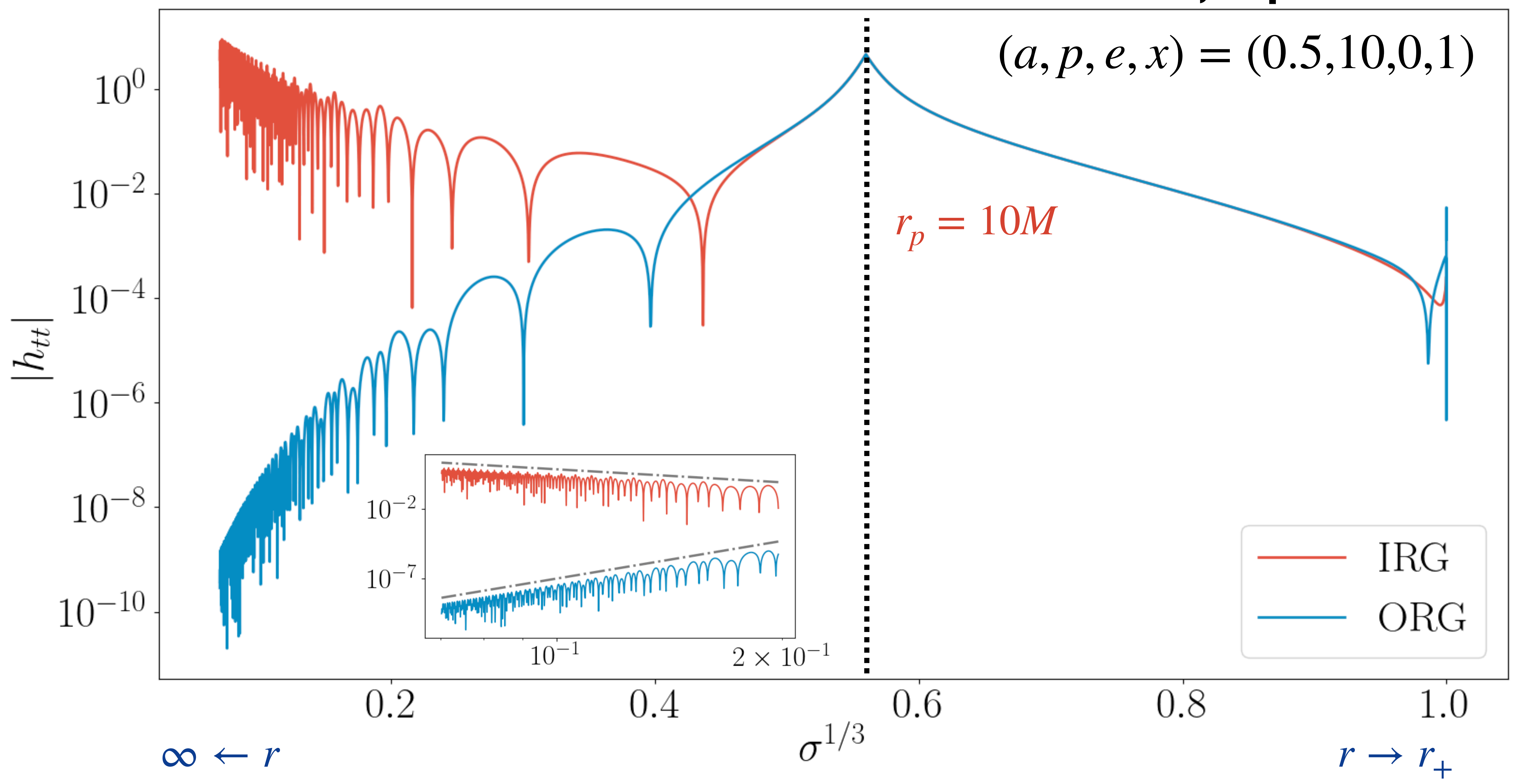
Circular, equatorial



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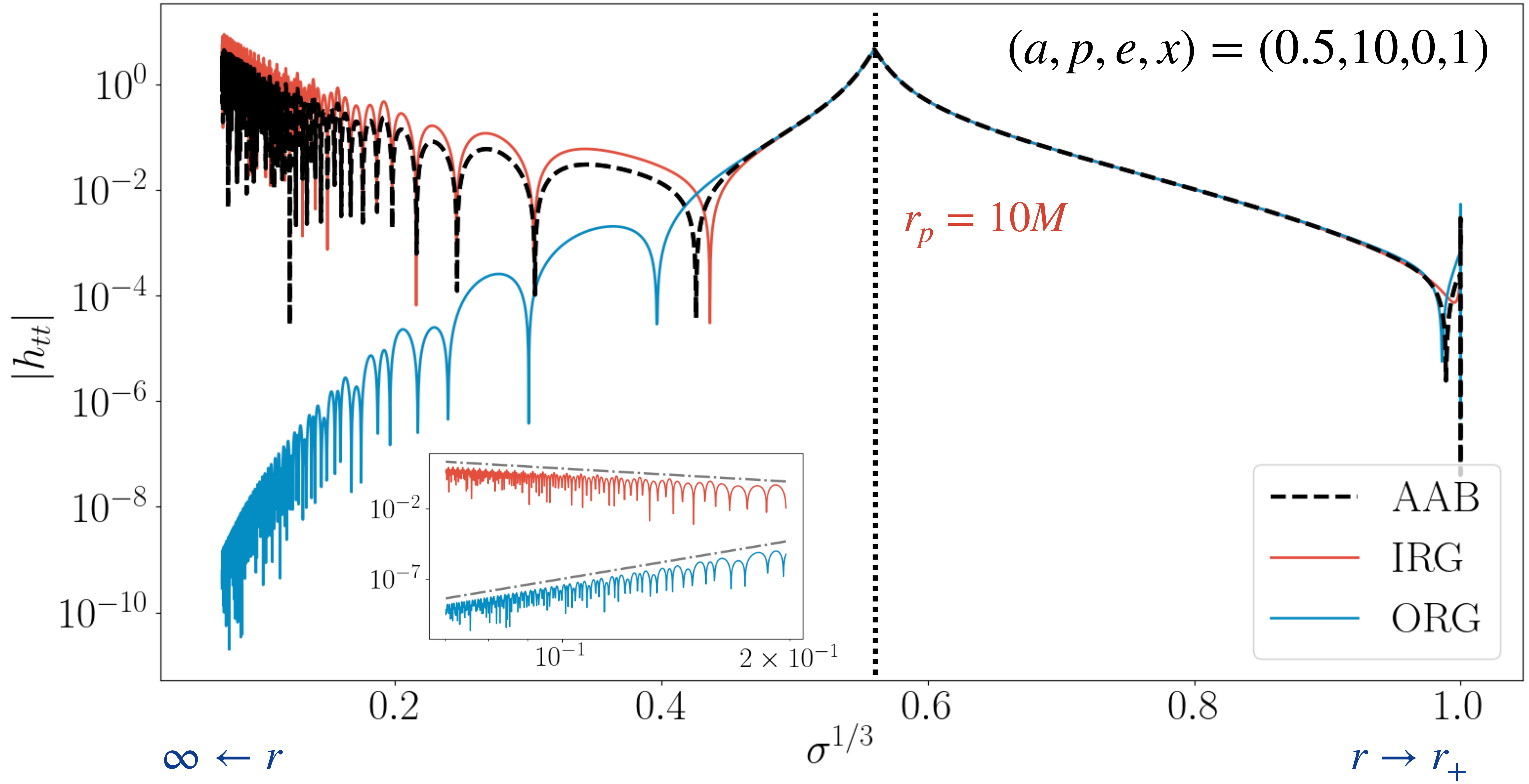

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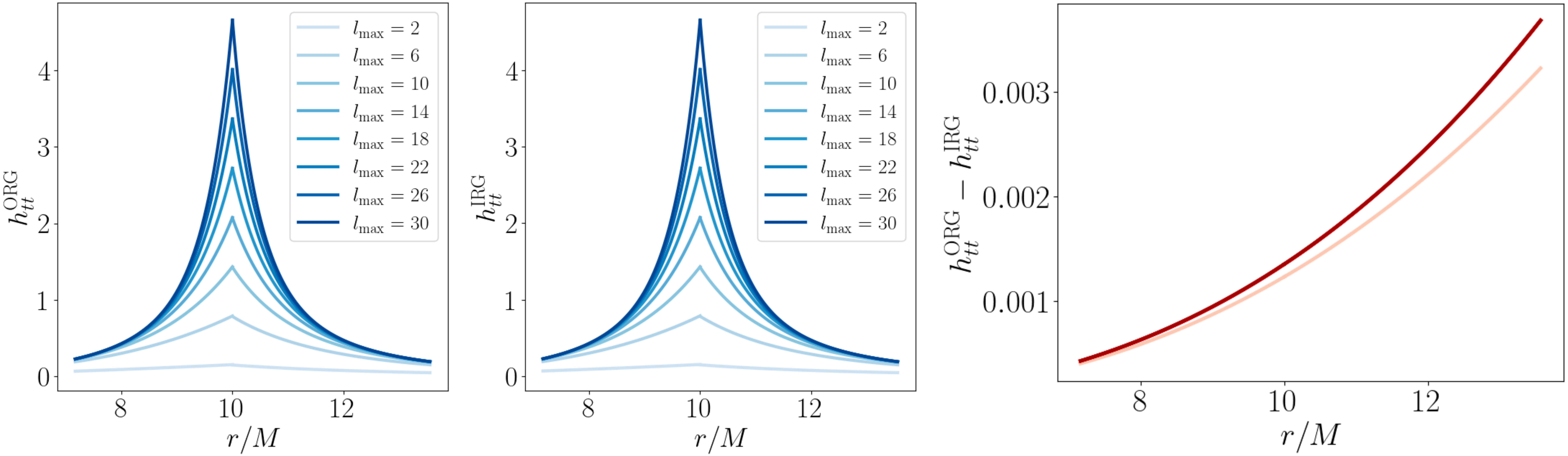

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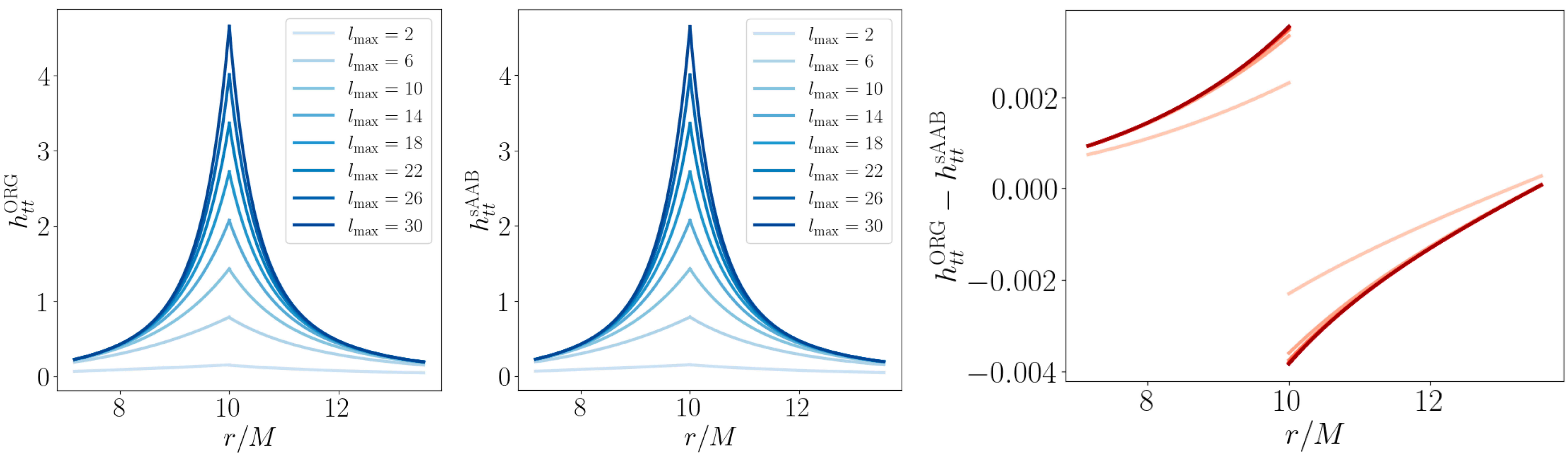


Validation w/ Detweiler redshift

- **Generalized Detweiler redshift invariant**
  - Define redshift along orbit

$$U = \frac{dt}{d\tau}$$

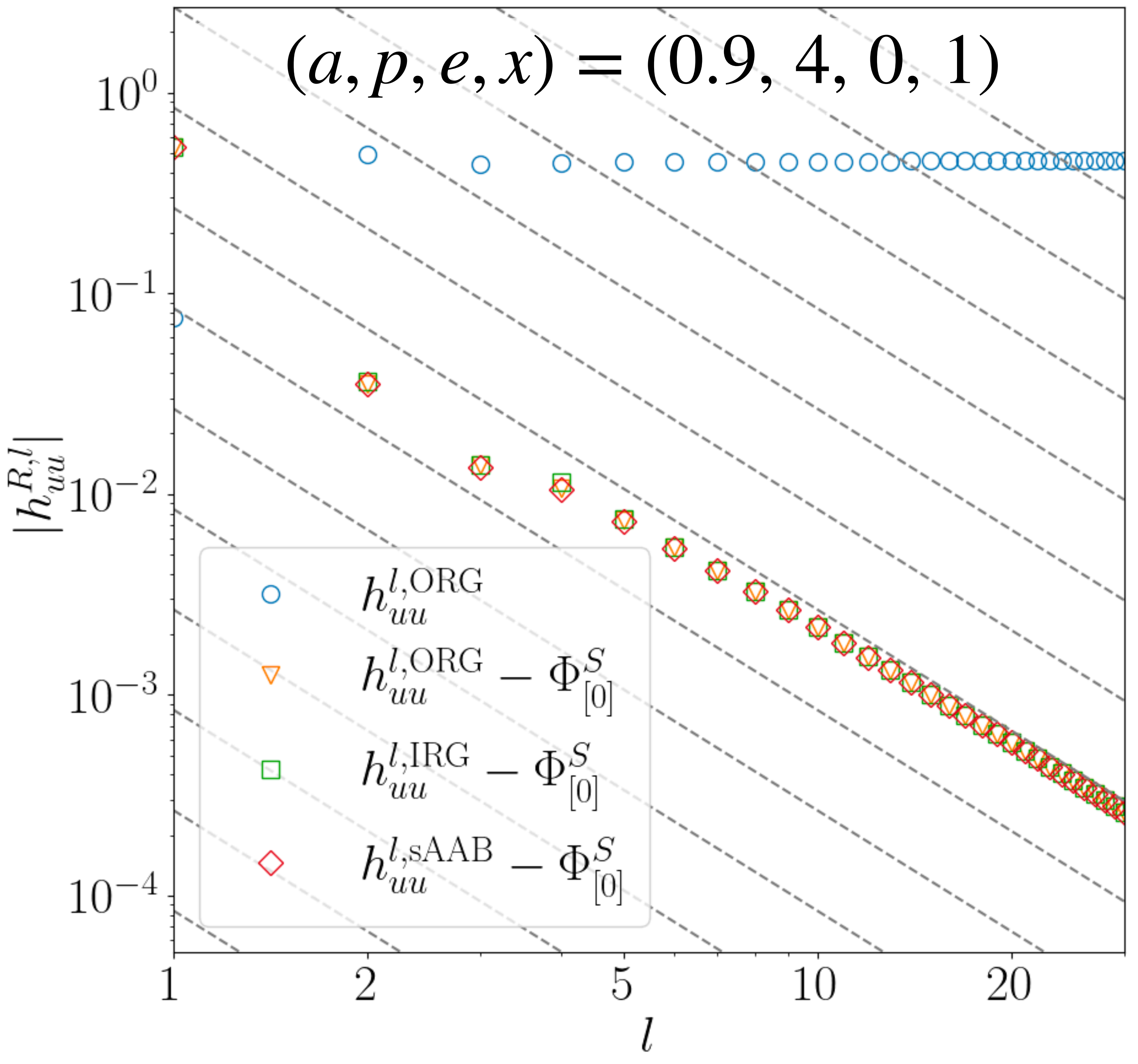
  - Look at average difference between background and perturbed spacetime $\tilde{g}_{\mu\nu} = g_{\mu\nu} + h^{\mathrm{R}}_{\mu\nu}$

$$\langle \Delta U \rangle = \langle \tilde{U} \rangle - \langle U \rangle = \frac{1}{2}\langle U \rangle \langle h^{\mathrm{R}}_{uu} \rangle \qquad \langle f \rangle = \frac{1}{T}\int_0^T f(\tau)d\tau$$

  - “Invariant” measure of conservative perturbations

Validation w/ Detweiler redshift

Circular, equatorial



$$\Delta U^{\mathrm{sAAB}} = -0.325708(3)$$
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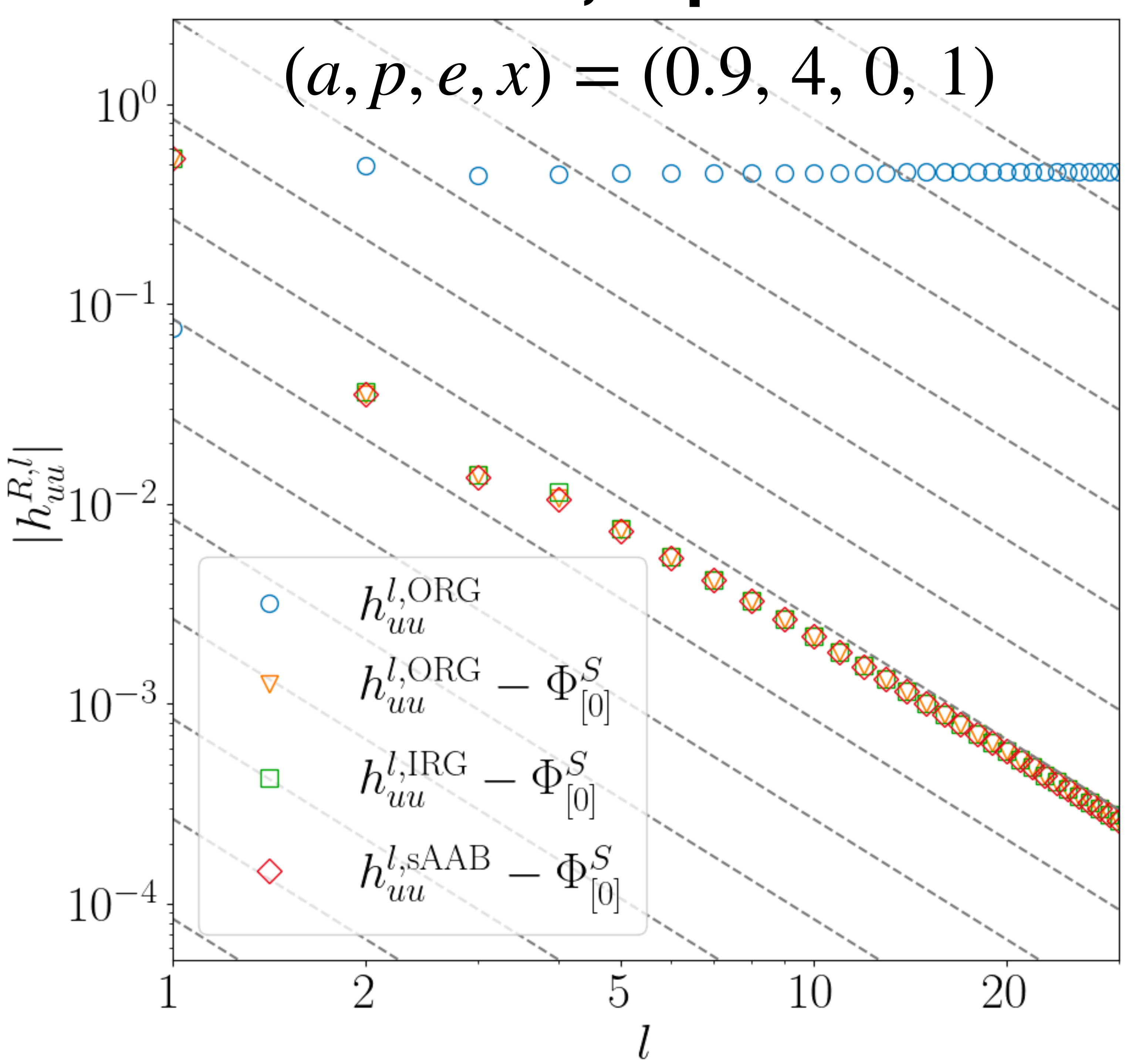
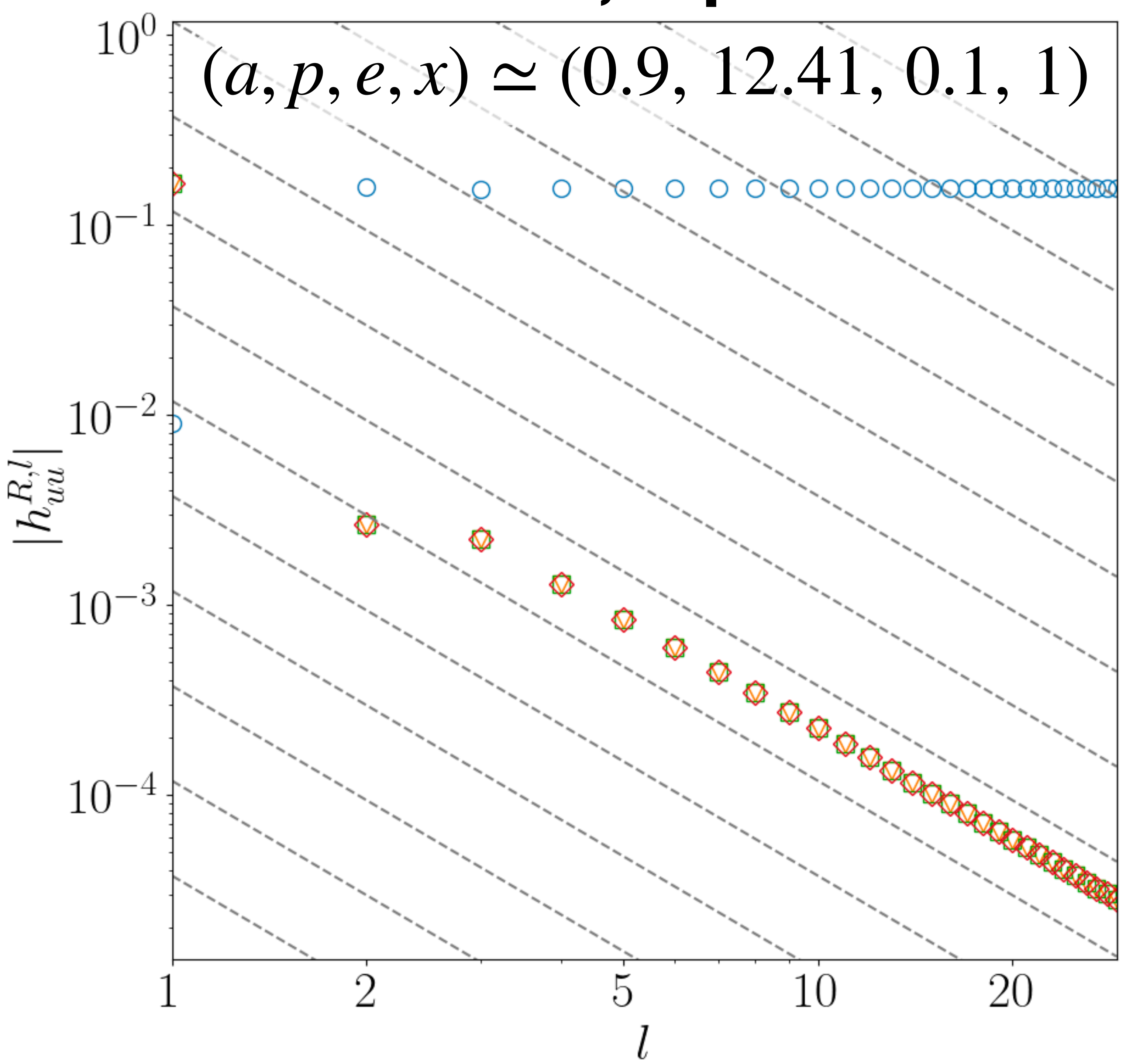
$$\Delta U^{\mathrm{IRG}} = -0.325705(1)$$

Validation w/ Detweiler redshift

Circular, equatorial



Eccentric, equatorial



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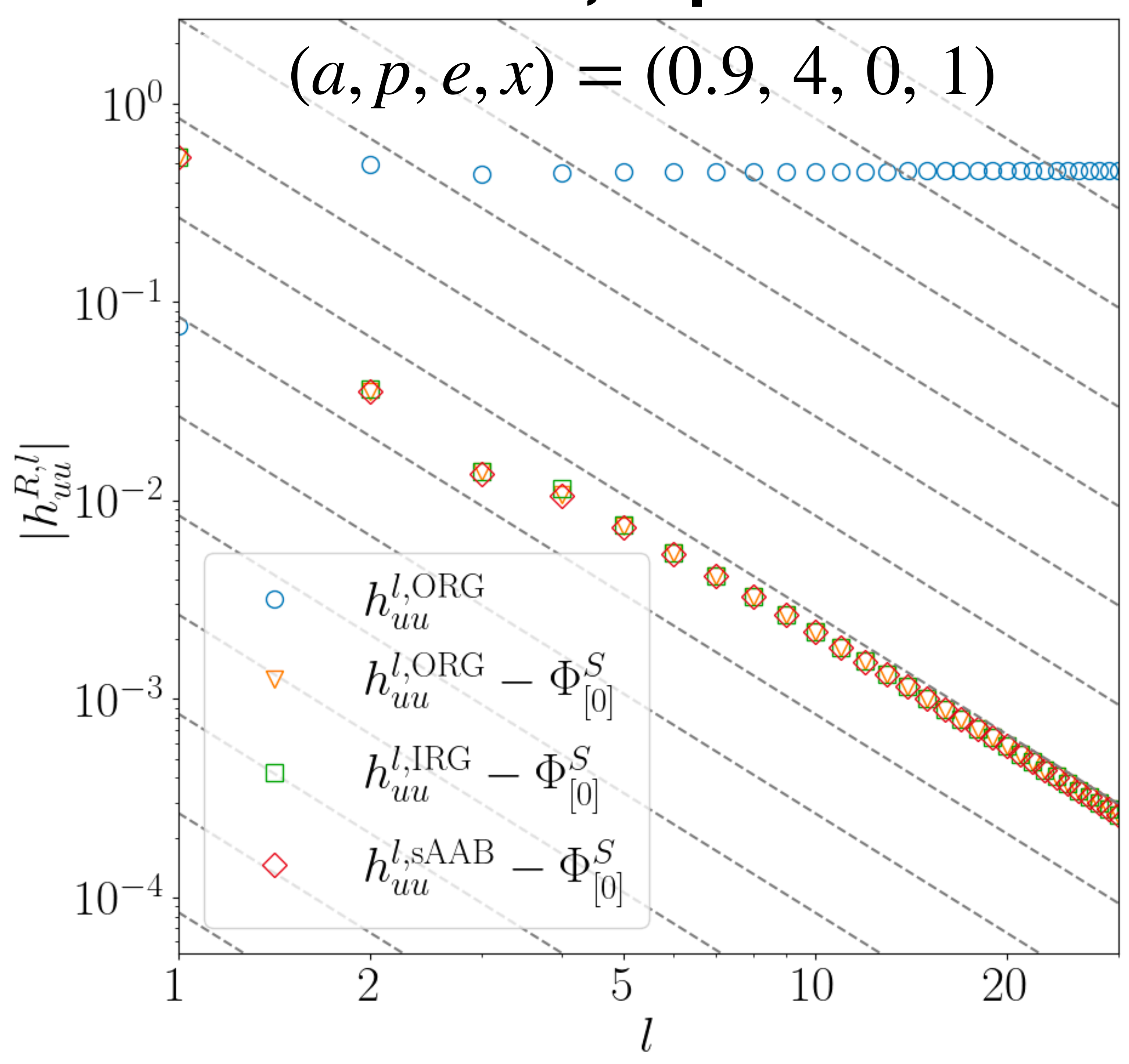
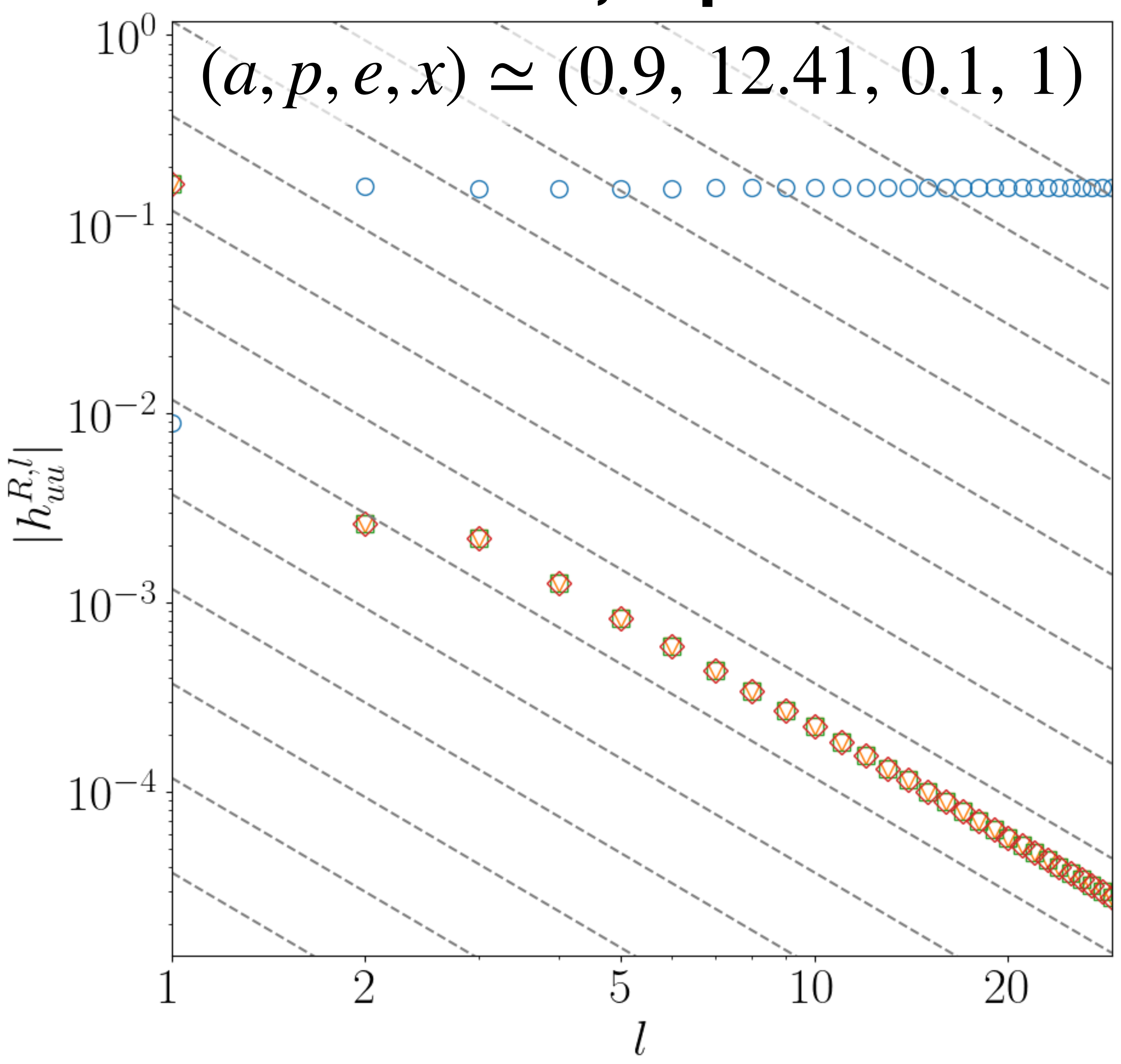
$$\langle \Delta U^{\mathrm{sAAB}} \rangle = -0.08921847(1)$$
$$\langle \Delta U^{\mathrm{ORG}} \rangle = -0.089218463(7)$$
$$\langle \Delta U^{\mathrm{IRG}} \rangle = -0.089218463(7)$$

Validation w/ Detweiler redshift

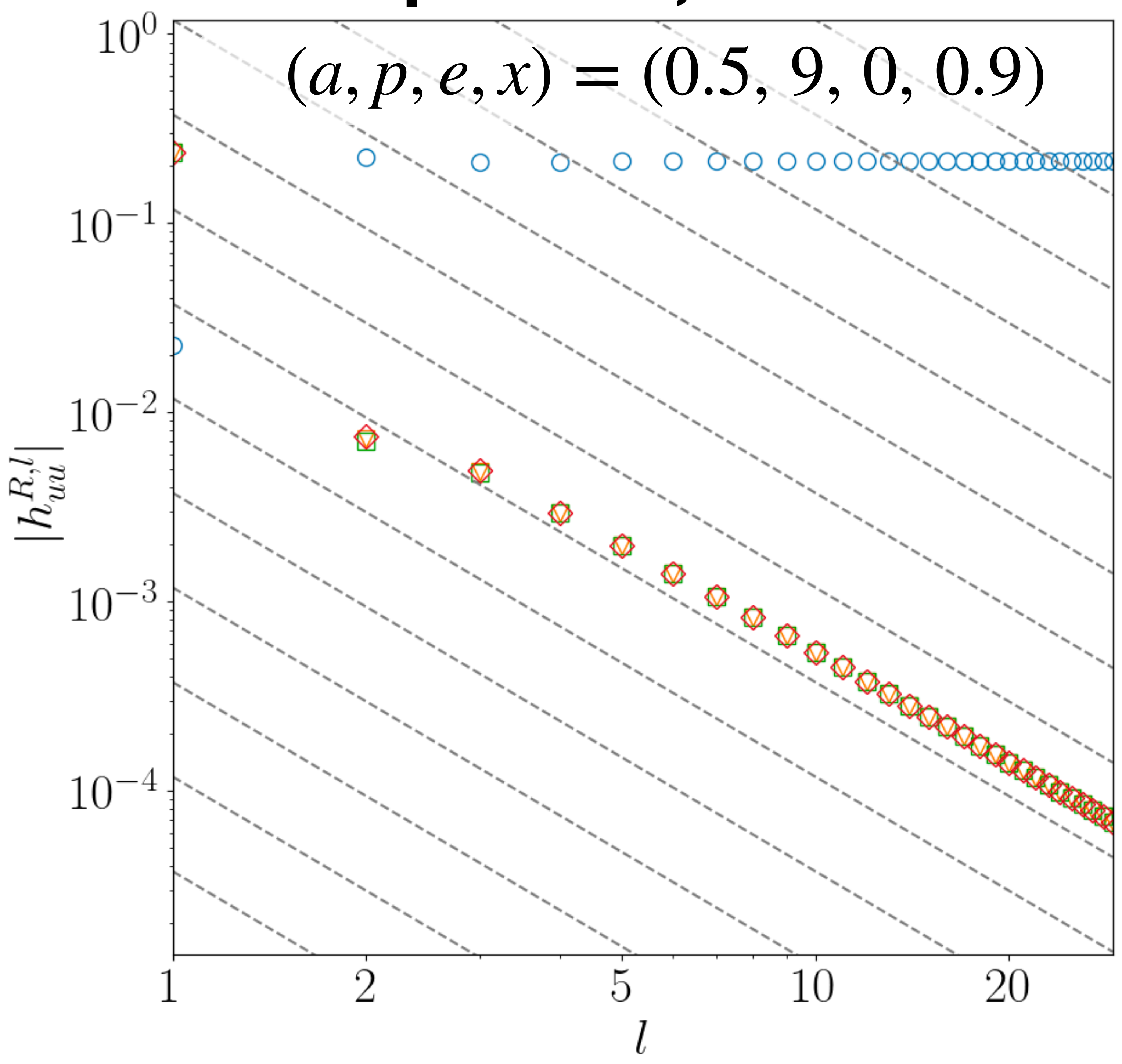
Circular, equatorial



Eccentric, equatorial



Spherical, inclined



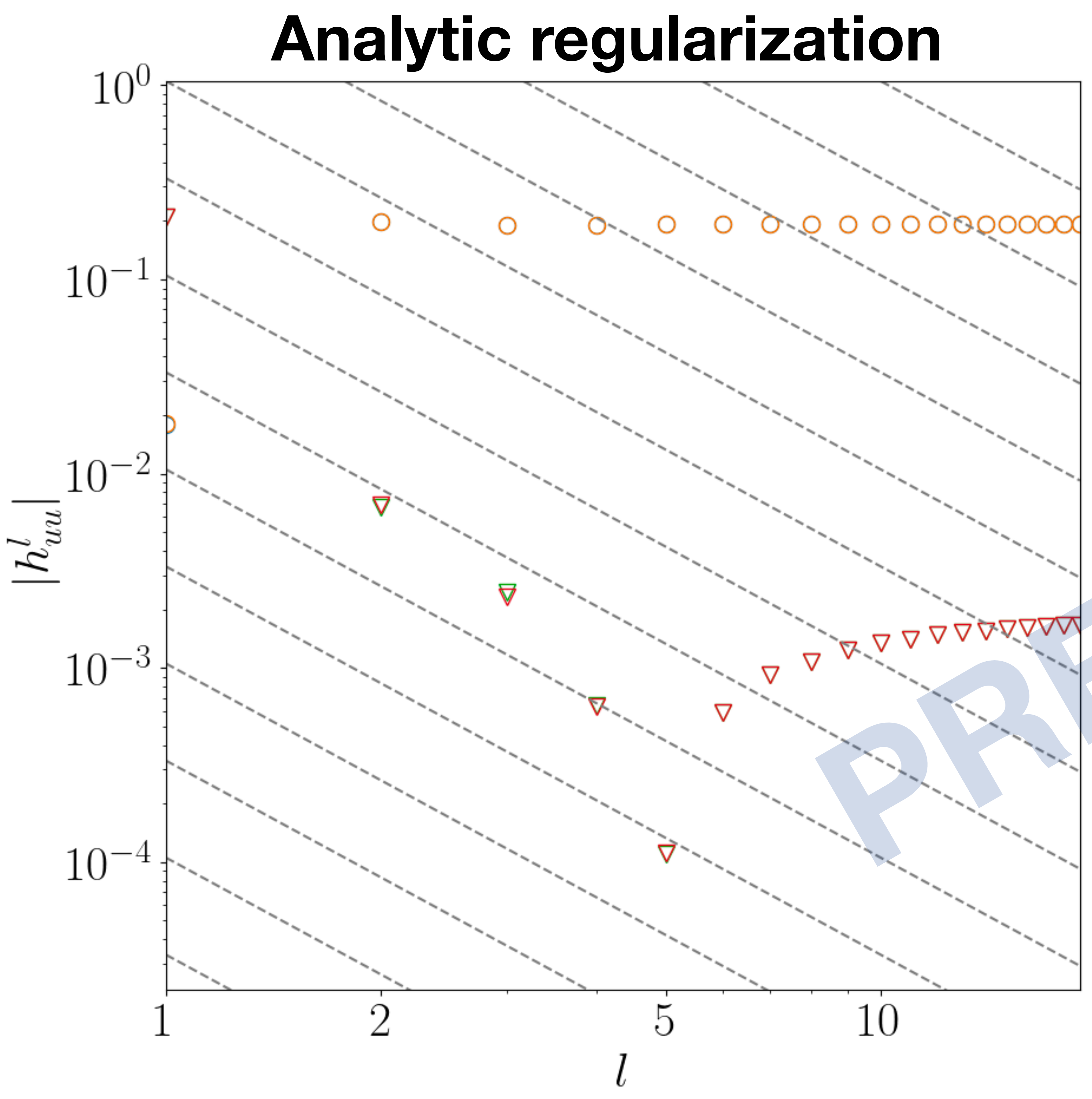
$$\Delta U^{\mathrm{sAAB}} = -0.325708(3)$$
$$\Delta U^{\mathrm{ORG}} = -0.325705(1)$$
$$\Delta U^{\mathrm{IRG}} = -0.325705(1)$$

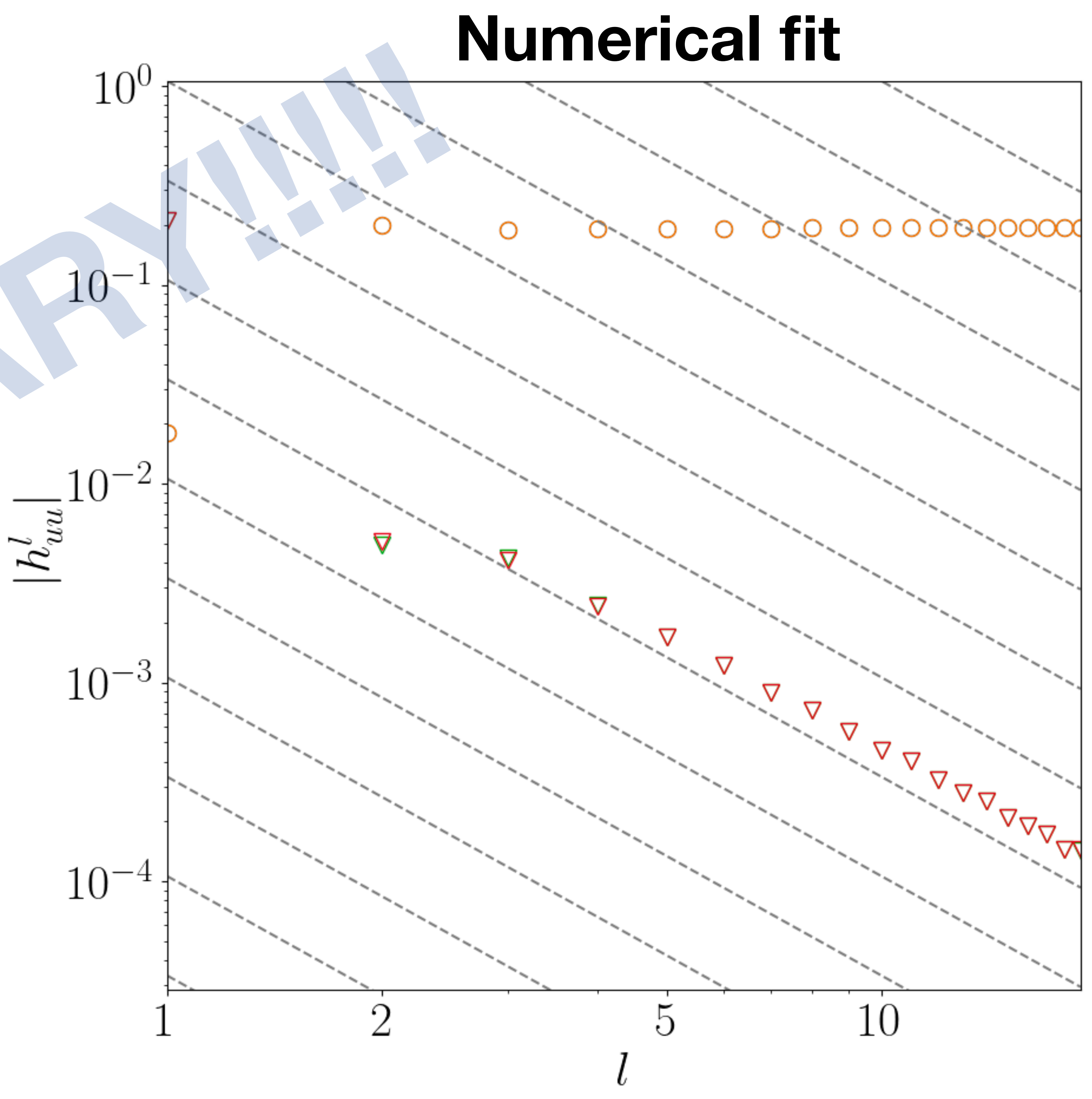
$$\langle \Delta U^{\mathrm{sAAB}} \rangle = -0.08921847(1)$$
$$\langle \Delta U^{\mathrm{ORG}} \rangle = -0.089218463(7)$$
$$\langle \Delta U^{\mathrm{IRG}} \rangle = -0.089218463(7)$$

$$\langle \Delta U^{\mathrm{sAAB}} \rangle = -0.261178(9)$$
$$\langle \Delta U^{\mathrm{ORG}} \rangle = -0.261178(9)$$
$$\langle \Delta U^{\mathrm{IRG}} \rangle = -0.261178(9)$$

Generic Detweiler redshift?

$$(a, p, e, x) = (0.5,\ 10.,\ 0.1,\ 0.9)$$





Conclusions

- Multiple metric reconstruction methods in vacuum: **GHZ**+ and **AAB**+
- Publicly-available open-source Python code **pybhpt**
- New generalized Detweiler redshift results for inclined orbits
- Future/ongoing work:
  - Incorporate hyperboloidal slicing + spatial compactification
  - Add reconstructed pieces in the source region
  - Generalize code to produce 1st-order GSF
  - Use 1st-order metric perturbation to construct 2nd-order source