The orbital evolution of EMRI during self force resonance with altanative regularization procedure

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Motivation

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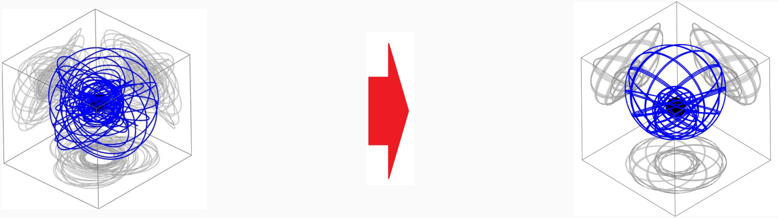
Motivation

- Self-Force Resonance occurs when ω_r and ω_θ are in rational ratio (Flanagan&Hinderer, 2012).
  - $j_r \Omega^r = j_\theta \Omega^\theta = \Omega \quad j_r, j_\theta \in \mathbb{Z}$
- The impact is $\sqrt{\eta^{-1}}$ and can be larger than 1PA
  - $\phi = \eta^{-1} \phi^{\mathrm{Adiabatic}} + \eta^{-\frac{1}{2}} \phi^{\mathrm{Resonance}} + \phi^{\mathrm{1PostAdiabatic}} + \cdots$
- Most EMRI systems will experience the large resonances(Ruangsri&Hughes, 2014).
- Integrablity is initially destroyed at the resonance.

$\rightarrow$ Self-Force Resonance is a very important phenomenon



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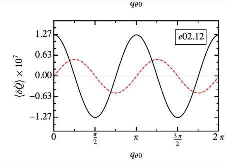
Motivation2 Conservative Part

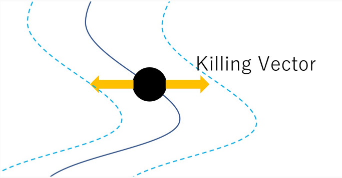
- Separable into dissipative and conservative parts.
  - $G^{\mathrm{sym}}[z, z'] = \frac{1}{2}(G^{\mathrm{ret}}[z, z'] + (G^{\mathrm{adv}}[z, z']))$
- Contribution of the conservative part during resonance is pointed out (Isoyama et al. 2013& 2019).
- Verified by Nasipak 2022 for scalar case.



- GR case??
- Regularization is necessary for the conservative part.
  - $\lim_{z' \to z} G^{\mathrm{sym}}[z, z'] \to \infty$

Motivation3 New Regularization Procedure

- Isoyama2013 proposed an altanative regularization method.
  - very simple
  - reduce computational cost?
- Shift the orbit in the direction of the killing vector
- There have been no implementation of this method.
  - It is a point of concern how well it works...
- important to try independet methods for the same problem



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Resonance and the Significance of This Study

Final Goal

- Calculate conservative part of resonance effect in GR .
- Expecting the altanative regularization method to work well.

This work

- First, try it out with the scalar field.
- Conservative part of Hamiltonian
- Verify how well the new regularization method works.

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Motivation

Method

Result

Hamilton eq

We can take action-angle variable

$$\dot{q^\alpha} = \frac{\partial H}{\partial J_\alpha} = \omega^\alpha + \frac{\partial H_{\rm int}}{\partial J_\alpha}$$
$$\dot{J}_\alpha = -\frac{\partial H}{\partial q_\alpha} = -\frac{\partial H_{\rm int}}{\partial q_\alpha}$$

- GR case

$$H_{\rm int} = \frac{1}{2} h_{\mu\nu} u^\mu u^\nu$$ Metric reconstruction is necessary

- Scalar toy model case

$$H_{\rm int} = \phi = \int d\tau' G(x, x(\tau'))$$

  - G is a Green function for Teukolsky eqation.
  - We can obtain it straightforwardly

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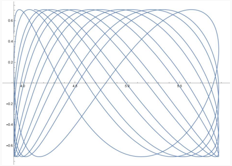
Orbital Averaging

$$\begin{cases} \dot{q}^\alpha = \omega^\alpha \\ \left\langle \dot{J}_\alpha \right\rangle = \left\langle \frac{\partial H_{\rm int}}{\partial q^\alpha} \right\rangle \end{cases}$$

$$\langle A \rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} d\tau A(x(\tau))$$

$$\neq \int_{\rm Region} A dq^\alpha$$ for resonant orbit

Non-resonance orbit

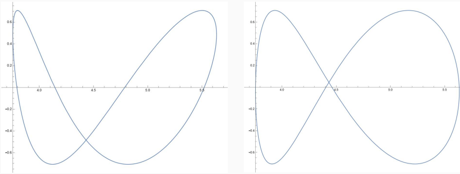


Resonance (&conservative) case

$$\left\langle \frac{\partial H}{\partial q^\alpha} \right\rangle = \frac{1}{2} \frac{\partial \langle H \rangle}{\partial q^\alpha_{\rm ini}}$$

initial phase dependence appears!

resonant orbit



descritization of ω

As the source $z(\tau)$ is periodic, τ integral makes ω descritized.

$$\int d\tau \mathrm{e}^{i\omega\omega_t\tau-(n_r\omega^r+n_\theta\omega^\theta+m\omega^\phi)\tau}$$
$$\to \frac{1}{\omega_t}\delta\left(\omega - n_r\Omega^r - n_\theta\Omega^\theta - m\Omega^\phi\right)$$

$$\boxed{\Omega^\alpha = \frac{\omega^\alpha}{\omega^t}}$$

For non-resonant orbit, ω is charactarized by (n_r, n_θ, m)

For resonant case,

$$nj_r\Omega^{r-1} = nj_\theta\Omega^{\theta-1} = n\Omega^{-1}$$

Then, ω is charactarized by (n, m)

$$\omega_{n,m} = n\Omega + m\Omega_\phi$$

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Point Spliting Regularization (Altanative Regularization)

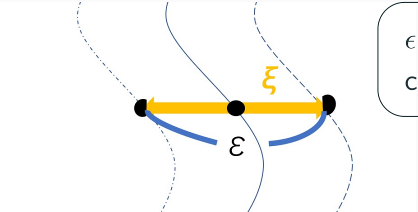
$G^{\mathrm{sym}}[z(\lambda), z(\lambda')]$ and $G^{S}[z(\lambda), z(\lambda')]$ diverge at $z(\lambda) \to z(\lambda')$, so we have to regularize them to calculate separately.

Point Spliting Regularization (Isoyama et al. 2013)

$$z^\mu(\lambda) \to z^\mu_+(\lambda) \equiv z^\mu(\lambda) + \frac{\epsilon}{2}\xi^\mu \ ,$$
$$z^\mu(\lambda) \to z^\mu_-(\lambda) \equiv z^\mu(\lambda) - \frac{\epsilon}{2}\xi^\mu$$

with

$$\xi^\mu(\zeta) \equiv \cos\zeta \ \xi^\mu_{(t)} + (\Omega_\phi \cos\zeta - \Omega \sin\zeta)\xi^\mu_{(\phi)}$$



$\epsilon = 0 \iff$ coincidence limit

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Altanative Regularization Procedure

G^{sym} is given by

$$G^{\mathrm{sym}}_{\omega,l,m}[z,z'] = \mathrm{e}^{-i\omega(t-t')+im(\phi-\phi')} S_{\omega,l,m}(\theta) S_{\omega,l,m}(\theta') \mathrm{Radial}(r,r') \ .$$

due to the killing direction, we can extract the regularized expression very easily,

$$G^{\mathrm{sym}}_{\omega,l,m}[z_+,z_-] = \mathrm{e}^{-i\omega\epsilon\cos\zeta + im\epsilon(\Omega_\phi\cos\zeta - \Omega\sin\zeta)} \times G^{\mathrm{sym}}_{\omega,l,m}[z,z'].$$

ϵ, ζ do not depend on τ. So Hamiltonian is given by

$$H_{\omega,l,m}(\epsilon) = \mathrm{e}^{-i\omega\epsilon\cos\zeta + im\epsilon(\Omega_\phi\cos\zeta - \Omega\sin\zeta)} \times H_{\omega,l,m}(\epsilon = 0)$$

ϵ dependence is calculated very easily!

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Final Expression

Finally

$$H(\epsilon) = \sum_{n,l,m} \mathrm{e}^{in\epsilon\cos\zeta + im\epsilon\sin\zeta} H_{\omega_{nm},l,m}(\epsilon = 0)$$

Explicit Expression

$$\begin{aligned}\langle H \rangle &= \sum_{n,l,m} \langle H \rangle_{n,l,m} \mathrm{e}^{in\epsilon_1 + im\epsilon_2} \\ &= \sum_{n,l,m} \mathrm{e}^{in\epsilon_1 + im\epsilon_2} \int d\tau d\tau' \mathrm{e}^{-i\omega_{n,m}(\Delta t - \Delta t') + im(\Delta\phi - \Delta\phi')} \\ &\quad \times \Theta_{\omega_{nm},l,m}(\theta)\Theta_{\omega_{nm},l,m}(\theta') R_{\omega_{nm},l,m}(r, r')\end{aligned}$$

We need not decompose the Spheroidal Harmonics into the Spherical Harmonics.

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S-part decomposition

S-part is also decomposed into (N, m) modes

$$H^S = \int d\tau' G^S[z^+, z^-(\tau')]$$

Fourier Transformation

$$G^S_{N,m} = \int d\epsilon_1 d\epsilon_2 G^S(\epsilon_1, \epsilon_2) \mathrm{e}^{-iN\Omega\epsilon_1 - im\epsilon_2\Omega_\phi}$$

$$H^S = \sum_{N,m} \mathrm{e}^{iN\epsilon\cos\zeta + im\epsilon\sin\zeta} H^S_{N,m}$$

- $\langle H^{(S)} \rangle$ has no periodicity with ϵ.
- This causes discontinuous at the boundary.

Therefore, apply a window function to smooth at the boundary.

$$W(\epsilon) = \cos^4(\alpha\Omega\epsilon)\cosh^4(\alpha\Omega\epsilon)$$

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Regularization

l mode summation first

$$H_{N,m} = \sum_{l=|m|}^{\infty} H_{N,l,m} \tag{1}$$

S-part subtraction

$$\begin{aligned} H^R &= \lim_{\epsilon \to 0} H(\epsilon) - H^S(\epsilon) \\ &= \lim_{\epsilon \to 0} \mathrm{e}^{iN\epsilon\cos\zeta + im\epsilon\sin\zeta} \left(\sum_{N,m} H_{N,m}(\epsilon = 0) - \sum_{N,m} H^S_{N,m}(\epsilon = 0) \right) \\ &= \sum_{N,m} H_{N,m}(\epsilon = 0) - H^S_{N,m}(\epsilon = 0) \end{aligned} \tag{2}$$

Regularization parameter ϵ does not appear in actual computations.

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Outline

1. Calculating $\langle H \rangle_{n,l,m}$ to integrate the mode functions.
2. Sum up $\langle H \rangle_{n,l,m}$ over l

$$H_{n,m} = \sum_l \langle H \rangle_{n,l,m}$$

3. regularization for (n, m)

$$\langle H^{(R)} \rangle = \sum_{n,m} \langle H \rangle_{n,m} - \langle H^{(S)} \rangle_{n,m}$$

4. Do the same procedure for fifferent initial phase $\Delta\lambda$

set up

a	p	e	x
0.9	4.51	0.2	$\frac{\pi}{4}$

$\Omega_r : \Omega_\theta = 1 : 2$

$\Delta\lambda = 0.3$

$\Lambda = 4.6$

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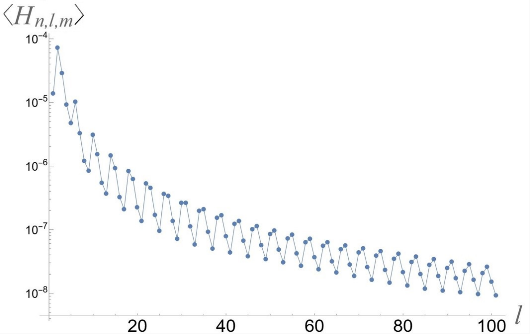
l-sum1

$$\sum_l \langle H \rangle_{n,l,m} \quad (3)$$



- $H \propto \frac{1}{l^2}$ too slow
- Hamiltonian is oscillating for l
- The oscillation makes fitting difficult

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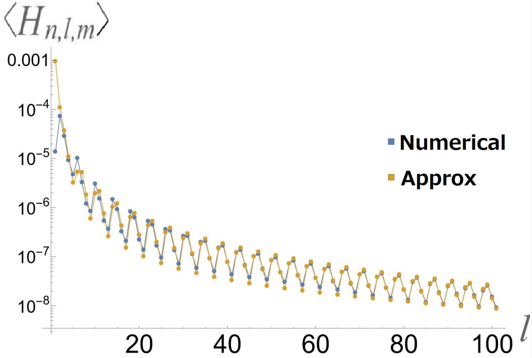
l-sum2: WKB aprox

- WKB-approximation $R_{\omega,l,m}(r)$ and $\Theta_{\omega,l,m}(\theta)$ for$l$
- We derive analytical assymptotic expressions for $\langle H\rangle_{n,l,m}$,$l \to \infty$ (up to subleading order)

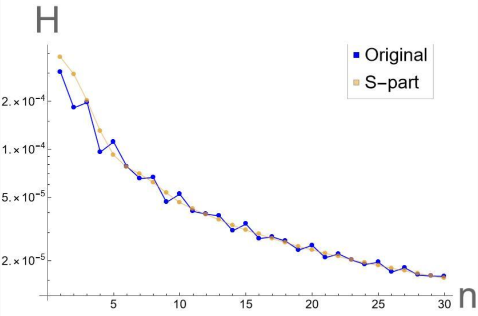


$\to$ we can sum over l mode

Regularization

$$\langle H^R_{n,m}\rangle = \left(\sum_l \langle H^{\text{int}}\rangle_{n,l,m}\right) - \langle H^{(S)}\rangle_{n,m}$$

- Fitting is necessary for computational cost reasons



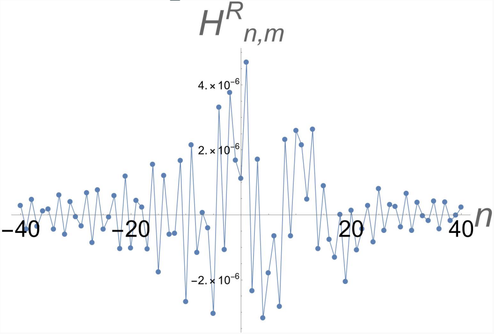
- Cancelating divergent part
- Regularization procedure looks working well...

Regularization

$$\langle H^R_{n,m}\rangle = \left(\sum_l \langle H^{\rm int}\rangle_{n,l,m}\right) - \langle H^{(S)}\rangle_{n,m}$$

- Fitting is necessary for computational cost reasons

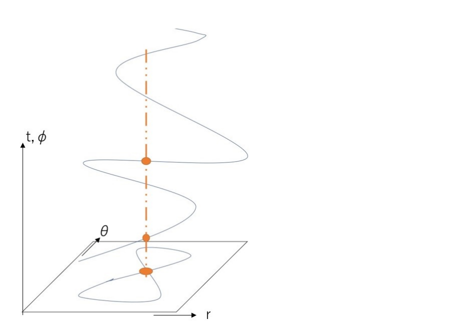
H^R behaving not well...

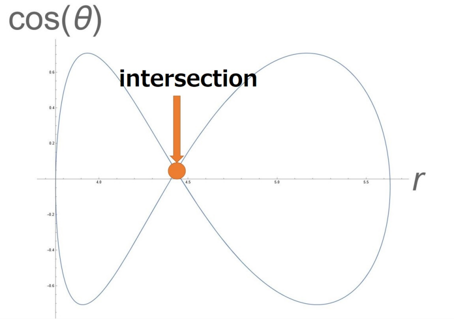


- Strong oscillation requires careful fitting

Cause of the Oscillation (Orbital Crossing)

$H_{n,m}$ is the Fourier transform in t and $\phi \rightarrow$ It can be projected onto the r-θ plane



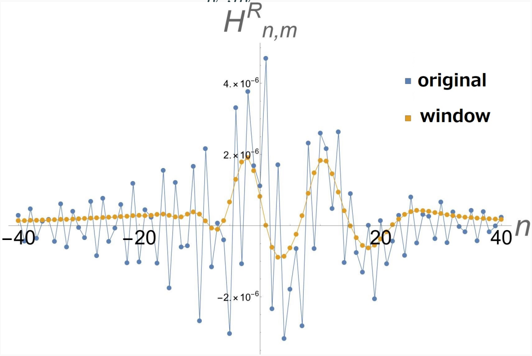


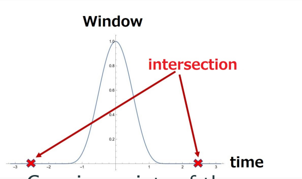
When projected, the singularity(intersection) appears.

Removing the Oscillation

- Identify the cause of oscillation (orbit crossing)
- Apply a physically appropriate window function

$$\langle \tilde{H}^R_{n,m}\rangle = \sum_{n',m'} W_{n-n',m-m'}\langle H^R\rangle_{n',m'}$$





Crossing points of the $r\theta$-projected orbit

succeeding in eliminating oscillatory behavior!

Convergence is not so fast→Higher order S-part

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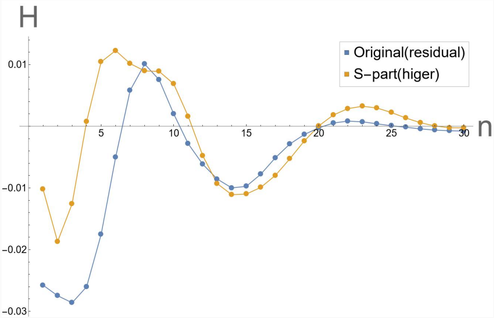
Accelerating convergence

- To accelerate the convergence further, we use the higer order S-part(Heffernan 2022)



- not matching well...

This is the problem we are working on now.

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Summary and Future Work

Summary

- We numerically calculate the conservative part during resonance for scalar field
- We encountered several challenges and addressed them to some extent

Future Work

- Successful cancellation with higher S-part
- Extend the model to include gravitational interactions

Thank you for listening.

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Result

$$H_{n,l,m}(\tau) = \frac{1}{l(l+1)} \int d\tau f(\tau)^2 g(\tau) \cos^2[\sqrt{l(l+1)}\theta(\tau)] \tag{4}$$

$$= f(\tau)^2 g(\tau) \frac{1}{2}(1 + \cos[2\sqrt{l(l+1)}\theta(\tau)]) \tag{5}$$

$$H_{n,l,m}(\tau) = H^{\mathrm{avr}}_{n,l,m}(\tau) + H^{\mathrm{osc}}_{n,l,m}(\tau) \tag{6}$$

- H can be decomposed into not-oscilating avaraged part $H^{\mathrm{avr}}_{n,l,m}(\tau)$ and oscilating part $H^{\mathrm{osc}}_{n,l,m}(\tau)$
- $H^{\mathrm{osc}}_{n,l,m}(\tau)$ is the cause of oscilation.

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4: S-part

Calculating $\langle H^{(S)} \rangle$ corresponding $G^{(S)}$(S-part)

$$\langle H^{(S)} \rangle = \int d\tau \int d\tau' G^{(S)}(x_+(\tau), x_-(\tau')) \tag{7}$$

$$= \frac{\psi_{-1}(\zeta)}{\epsilon} + \psi_1(\zeta)\epsilon + O(\epsilon^3) \tag{8}$$

- $x_\pm$ is shifted ϵ in the Killing direction.
- $\epsilon = \sqrt{\epsilon_1^2 + \epsilon_1^2}\ , \zeta = \arctan\left(\frac{\epsilon_2}{\epsilon_1}\right)$

Fourier transformation for $\epsilon_1, \epsilon_1,$

$$\langle H^{(S)} \rangle_{n,m} = \frac{\Omega^2}{4\pi^2} \int d\epsilon_1 \int d\epsilon_2 \mathrm{e}^{-i\Omega(n\epsilon_1 + m\epsilon_2)} \langle H^{(S)} \rangle \tag{9}$$

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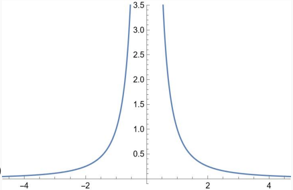
4:S-part(Window function)

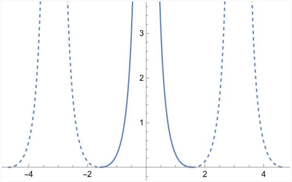
$$\langle H^{(S)} \rangle = \frac{\psi_{-1}(\zeta)}{\epsilon} + \psi_1(\zeta)\epsilon + O(\epsilon^3) \tag{10}$$

- $\langle H^{(S)} \rangle$ has no periodicity with ϵ.
- This causes discontinuous at the boundary.

Therefore, apply a window function to smooth at the boundary.

$$W(\epsilon) = \cos^4 \alpha\Omega\epsilon \cosh^4 \alpha\Omega\epsilon \tag{11}$$





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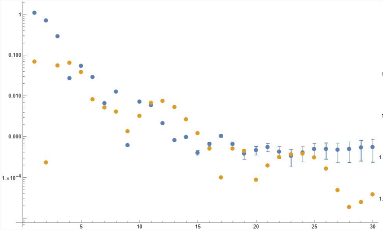
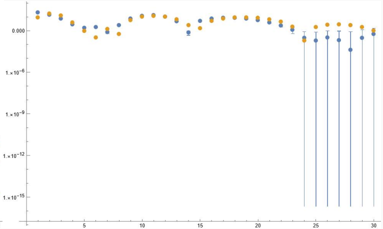

1
0.100
0.010
0.000
1.×10^-4
5
10
15
20
25
30

(a) $m = 1$


0.000
1.×10^-6
1.×10^-9
1.×10^-12
1.×10^-15
5
10
15
20
25
30

(b) $m = 9$

Appendix

In practice, it is convenient to use Mino variable λ

We can show

$$\left\langle \frac{dQ}{d\tau} \right\rangle = -\frac{\partial Q}{\partial J_r} \left\langle \frac{\partial(\Sigma H)}{\partial \Gamma_r} \right\rangle_\lambda - \frac{\partial Q}{\partial J_\theta} \left\langle \frac{\partial(\Sigma H)}{\partial \Gamma_\theta} \right\rangle_\lambda \tag{12}$$

$$= -\frac{\partial Q}{\partial J_r} \frac{\partial \langle \Sigma H \rangle_\lambda}{\partial \Delta\Gamma_r}_\lambda - \frac{\partial Q}{\partial J_\theta} \frac{\partial \langle \Sigma H \rangle}{\partial \Delta\Gamma_\theta} \tag{13}$$

$$\tag{14}$$

$$\Gamma = \Upsilon\lambda + \Delta\Gamma \tag{15}$$

To obtain the evolution of the orbital elements, we have to calculate $\langle \Sigma H^{\mathrm{sym}} \rangle_\lambda$

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実装 5: 正則化 (振動の原因)

- before averaging $H_{n,m} - H^{(S)}_{n,m}$

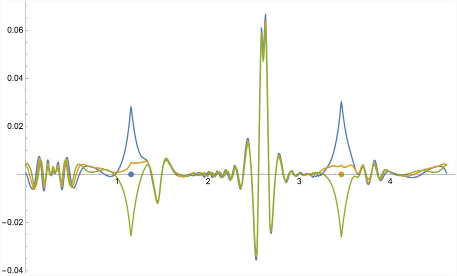

Figure 2: $n = 3$ に固定し、横軸 λ(時間)。青が

- $\lambda = 1.1, 3.5$ に m を変えたときの振動が見られる
- この時刻は $r - \theta$ が交差する時刻に対応

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実装 3: detail of WKB

mode funcion $R(r)$ follows the equation below.

$$-\mathcal{R}'' + \left[\frac{\lambda}{\Delta} - U(r) + \frac{1}{4}\left(\frac{\Delta'}{\Delta}\right)^2 + \frac{1}{2}\left(\frac{\Delta'}{\Delta}\right)'\right]\mathcal{R} = 0 \quad (16)$$

$$\mathcal{R} = R\sqrt{\Delta} \quad (17)$$

$$U(r) = \frac{1}{\Delta}\frac{\omega^2(r^2+a^2)^2 - 4ram\omega + m^2a^2}{\Delta} - \omega^2a^2 \quad (18)$$

$$\lambda = l(l+1) + \ldots \quad (19)$$

l,λ dominates for large l

R の WKB 近似 (リーディング)

$$R_{\mathrm{in/up}} \approx \left(\frac{1}{l(l+1)\Delta}\right)^{\frac{1}{4}} \exp\left(\pm\sqrt{l(l+1)}\int^r \sqrt{\frac{1}{\Delta}}dr\right),$$

radial part of green function

radial part is

$$R(r,r') = \frac{1}{W}\left(R_{\rm in}(r)R_{\rm up}(r')U(r'-r) + R_{\rm in}(r')R_{\rm up}(r)U(r-r')\right) \tag{22}$$

W is Wronskian.

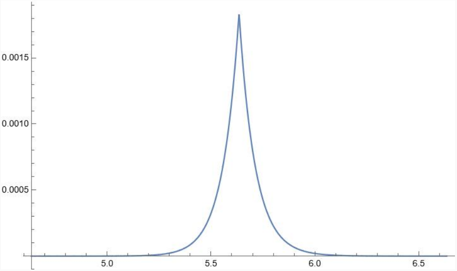


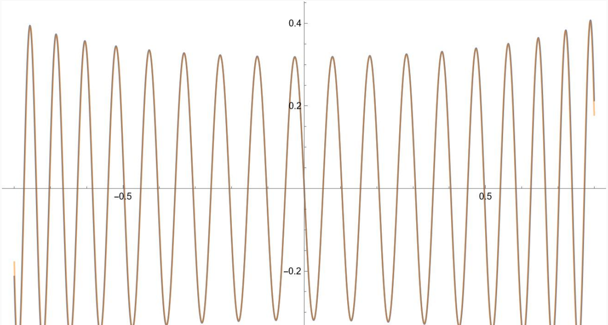
Figure 3: y-axis Radial,x-axis r',$r = 5.64$

実装 3:　angle WKB

同様に θ のモード関数 $S_{\omega,l,m}$ も WKB 近似して

$$S(\theta) \approx \begin{cases} \dfrac{k}{\sqrt[4]{1-\cos^2\theta}} \cos\left[\sqrt{l(l+1)}\,\arcsin[\cos\theta]\right], & (l+m = \mathrm{ev} \\ \dfrac{k}{\sqrt[4]{1-\cos^2\theta}} \sin\left[\sqrt{l(l+1)}\,\arcsin[\cos\theta]\right], & (l+m = \mathrm{oc} \end{cases} \tag{23}$$

k は規格化定数



To combine these WKB-approximated mode functions,

$$G(x(\tau), x(\tau')) = f(x(\tau)) f(x(\tau')) \cos(\sqrt{l(l+1)\theta}) \cos(\sqrt{l(l+1)\theta'} \\ \exp\left[-\sqrt{l(l+1)}\left|\int_{r(\tau)}^{r(\tau')} \frac{1}{\sqrt{\Delta}} dr\right|\right]$$

Another Approximation

For large l, large $r - r'$is decaying $\rightarrow \tau \sim \tau'$

$\delta\tau = \tau' - \tau << \tau$

$$\int_{r(\tau)}^{r(\tau')} \frac{1}{\sqrt{\Delta}} dr \sim \frac{\dot{r}}{\sqrt{\Delta}} \delta\tau \tag{24}$$

$$G(\tau, \tau + \delta\tau) \sim \left[f(x(\tau))^2 + f'\delta\tau\right] \mathrm{e}^{-\sqrt{l(l+1)}\frac{\dot{r}}{\sqrt{\Delta}}\delta\tau} \tag{25}$$

From this, we can integrate over τ'=$\delta\tau$ explicitly.

実装 3:Result

$$H_{n,l,m}(\tau) = \frac{1}{l(l+1)} \int d\tau f(\tau)^2 g(\tau) \cos^2[\sqrt{l(l+1)}\theta(\tau)] \tag{26}$$

$$= f(\tau)^2 g(\tau) \frac{1}{2}(1 + \cos[2\sqrt{l(l+1)}\theta(\tau)]) \tag{27}$$

$$H_{n,l,m}(\tau) = H^{\mathrm{avr}}_{n,l,m}(\tau) + H^{\mathrm{osc}}_{n,l,m}(\tau) \tag{28}$$

- H can be decomposed into not-oscilating avaraged part $H^{\mathrm{avr}}_{n,l,m}(\tau)$ and oscilating part $H^{\mathrm{osc}}_{n,l,m}(\tau)$
- $H^{\mathrm{osc}}_{n,l,m}(\tau)$ is the cause of oscilation.

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実装 3: Result(numerical)

$$H(x(\tau)) = \int d\delta\tau G(x(\tau), x(\tau + \delta\tau)) \tag{29}$$

Is the approximation correct?

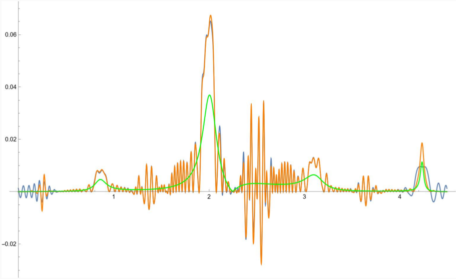

Figure 5: y-axis$\langle H \rangle_{n,l,m}$, x-axis λ、Blue：numerical、Orange：WKB

mostly correct

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実装 3: cause of l-oscilation

$H^{\rm osc}_{n,l,m}(x(\tau))$ is the cause of l-oscilation

stationary approximation

$$
\begin{aligned}
H^{\rm osc}_{n,l,m}(x(\tau)) =& (x(\tau))\cos[\sqrt{l(l+1)}\theta(\tau)] \\
\simeq& f(x(\tau))\cos\left[\sqrt{l(l+1)}\left(\theta(\tau_0)+\ddot{\theta}(\tau_0)\frac{\tau^2}{2}\right)\right]
\end{aligned}
\tag{30}
$$

- τ_0 is the stationary point.

$$
\langle H^{\rm osc}_{n,l,m}\rangle \simeq F(x(\tau))\cos\left[2\sqrt{l(l+1)}\theta(\tau_0)-\frac{\pi}{4}\right] \tag{31}
$$

This is the cause of l-oscilation

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