An on-shell approach to radiation reaction

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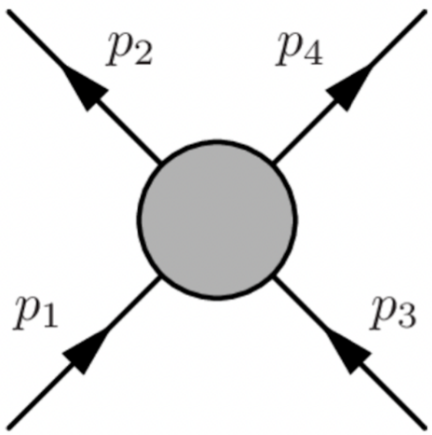
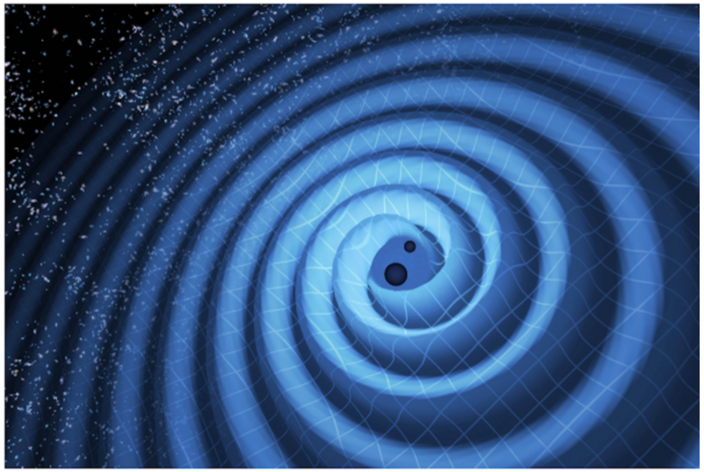
The two-body problem

- 2019: State-of-the-art calculation in the post-Minkowskian approximation using only the classical limit of scattering amplitudes (Bern, Cheung, Roiban, Shen, Solon, Zeng)



$\Rightarrow$



Credit: Tim Pyle

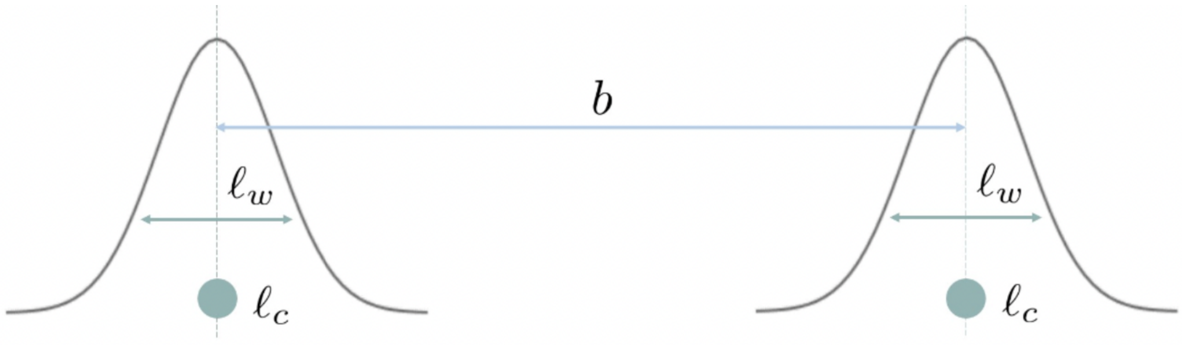
Main message of this talk

We can use on-shell amplitudes to revisit the classical two-body problem

The KMOC formalism

- Binary system as superposition of single particle states

$$|\psi\rangle = \int_{p_1,p_2} \phi_1(p_1)\,\phi_2(p_2)\,e^{\frac{ib\cdot p_1}{\hbar}}\,|p_1 p_2\rangle$$



Credit: Ben Maybee, 2105.10268

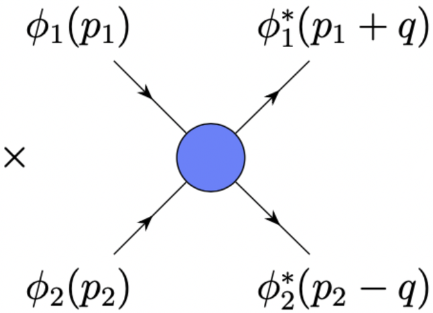
- Interactions are expressed in terms of scattering amplitudes

$$S|\psi\rangle = \int_{p_1,p_1',p_2,p_2'} \phi_1(p_1)\,\phi_2(p_2)\,e^{\frac{ib\cdot p_1}{\hbar}}\,\underbrace{\langle p_1', p_2'|\,S\,|p_1 p_2\rangle}_{S=1+iT}\,|p_1', p_2'\rangle + \ldots$$

- The change in momentum $\langle\psi| S^\dagger \mathbb{P}^\mu S |\psi\rangle - \langle\psi| \mathbb{P}^\mu |\psi\rangle$ is the sum of two contributions. One is linear in the T-matrix

$$I^\mu_{(1)} = \int d\Phi(p_1) d\Phi(p_2) \hat{d}^4 q\, \hat{\delta}(2p_1 \cdot q + q^2) \hat{\delta}(2p_2 \cdot q - q^2) \Theta(p_1^0 + q^0) \Theta(p_2^0 - q^0)$$

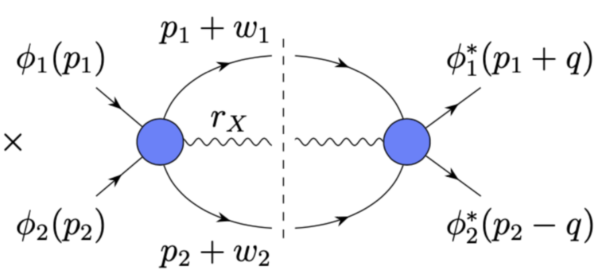
$$\times\, e^{-ib\cdot q}\, iq^\mu \,\times$$



- The remaining one is quadratic in the T-matrix

$$I^\mu_{(2)} = \sum_X \int \prod_{i=1,2} d\Phi(p_i) \hat{d}^4 w_i \hat{d}^4 q\, \hat{\delta}(2p_i \cdot w_i + w_i^2) \Theta(p_i^0 + w_i^0)$$
$$\times\, \hat{\delta}(2p_1 \cdot q + q^2) \hat{\delta}(2p_2 \cdot q - q^2) \Theta(p_1^0 + q^0) \Theta(p_2^0 - q^0)$$
$$\times\, e^{-ib\cdot q/\hbar}\, w_1^\mu\, \hat{\delta}^{(4)}(w_1 + w_2 + r_X)$$

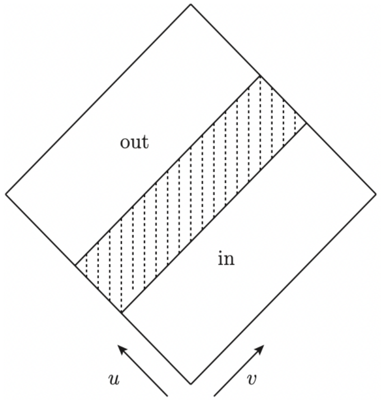
$\times$



Credit: Kosower, Maybee and O'Connell, 1811.10950

KMOC on curved backgrounds

- The same logic applies for QFT on a gravitational plane wave



$$\mathrm{d}s^2 = 2\,\mathrm{d}u\mathrm{d}v - H_{ab}(u)\,x^a x^b\,\mathrm{d}u^2 - \mathrm{d}x^\perp \mathrm{d}x^\perp$$

- The geodesic motion is captured by a 2-point amplitude (memory effects related to $E_{ai}(u) \sim_{u\to+\infty} \delta_{ia} + u c_{ai}$)

$$\underbrace{\mathcal{S}_\alpha}_{S \text{ on } \bar{g}} |\psi\rangle = \int d\Phi\left(p,p'\right)\phi\left(p\right)\underbrace{\langle p'|\,\mathcal{S}_\alpha\,|p\rangle}_{2-point}|p'\rangle + \ldots$$

$$\langle p'|\,\mathcal{S}_\alpha\,|p\rangle = \hat{\delta}^+(p'-p)\frac{4\pi e^{-\frac{i}{2\sqrt{G}p_+}(p'-p)_\perp c^{-1}(p'-p)_\perp}}{\sqrt{\det(G\;c)}\,\hbar}$$

- 2-points on the background $\Rightarrow$ geodesic motion

$$\langle\psi|\,\mathcal{S}_\alpha^\dagger \mathbb{P}^\mu \mathcal{S}_\alpha\,|\psi\rangle = p^\mu + \sqrt{G}p_+\,\delta_a^\mu\,c_i^a\,b^i - n^\mu\left(\frac{2\sqrt{G}p_+\,p_a c_i^a b^i - Gp_+^2 c_i^a b^i c_j^b b^j \delta_{ab}}{2p_+}\right)$$

- 3-points on the background $\Rightarrow$ radiative observables.
Deviations from geodesic motion appear in powers of m/Λ

$$\langle\psi|\,\mathcal{S}_\alpha^\dagger \mathbb{W}_{\mu\nu\sigma\rho}(u,\hat{x})\mathcal{S}_\alpha\,|\psi\rangle = \frac{i\kappa}{2\pi\hbar^{\frac{1}{2}}}\int_0^\infty \hat{\mathrm{d}}\omega\,\mathrm{e}^{-i\omega u} k_{[\mu}\varepsilon_{\nu]}^{-\eta} k_{[\sigma}\varepsilon_{\rho]}^{-\eta}$$

$$\times\int \mathrm{d}\Phi(p)\,\underbrace{\langle\psi|\,\mathcal{S}^\dagger\,|p\rangle\,\langle p,k^\eta|\,\mathcal{S}\,|\psi\rangle}_{2-point\,\times\,3-point}|_{k=\hbar\omega\hat{x}} + \text{ c.c.}$$

All order waveforms from amplitudes (Adamo, Cristofoli, Klisch, Ilderton)

SF approximation on a plane wave from amplitudes on a background

Example: electromagnetism

- 3-points define scattering waveforms and radiation reaction

$$\underbrace{\mathrm{A}_\mu(x)}_{\textit{Background}} = -x^{\perp} E_{\perp}\left(x^{-}\right) n_\mu \quad , \qquad \underbrace{a_{\perp}(u)}_{U(1)\ \textit{memory}} := \int_{-\infty}^{u} ds\, E_{\perp}(s)$$

$$\underbrace{\langle p', k^\eta |\, S\, | p\rangle}_{3-\textit{point}} = \delta^{+,\perp}(q-k-ea(\infty)) \int_y \varepsilon^\eta \cdot P(y) e^{i\int_{-\infty}^{y} \mathrm{d}z\, k\cdot P(z)}$$

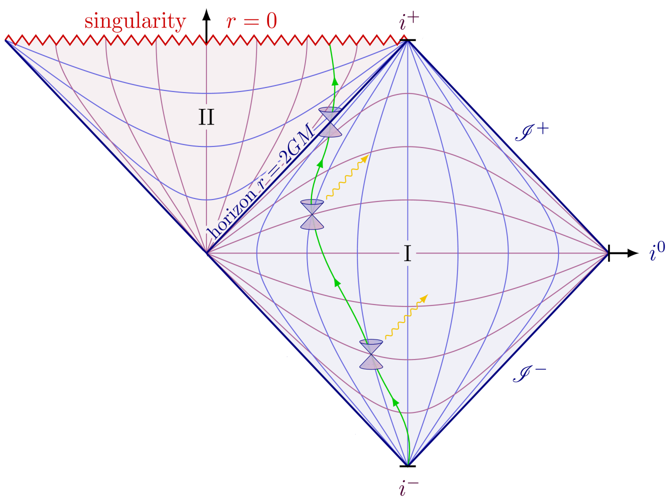
- Scattering waveform from a frequency integral over a 3-point

$$W_{\mu\nu}(u,\hat{x}) = -\frac{ie}{4\pi^2 p_+} \int_{y,\omega} \omega \mathrm{e}^{-i\omega(u-\hat{x}\cdot X(y))} \hat{x}_{[\mu} P_{\nu]}(y)$$

- Radiation reaction from the square of a 3-point

$$\Delta p^\mu = -\frac{2}{3}\frac{e^4 p_+}{4\pi m^4} \int_{-\infty}^{+\infty} du\, E_{\perp}^2(u)\, P^\mu(u) - n^\mu \frac{P(u)\cdot P(+\infty)}{p_+}$$

- Consider QFT on a static background (e.g. Schwarzschild)



$$\mathrm{d}s^2 = \left(\eta_{\mu\nu} + k_\mu k_\nu \frac{2GM}{r}\right) dx^\mu dx^\nu$$

- The geodesic motion is captured by the classical limit of a 2-point (equivalent to an eikonal amplitude in vacuum)

$$\underbrace{\mathcal{S}_\alpha}_{S\ on\ \bar{g}} |\psi\rangle = \int d\Phi\left(p, p'\right) \phi\left(p\right) \underbrace{\langle p'| \mathcal{S}_\alpha |p\rangle}_{2-point} |p'\rangle + \dots$$

$$\langle p'| \mathcal{S}_\alpha |p\rangle = \delta(E' - E) \sum_{\ell=0}^{\infty} (2\ell + 1) P_\ell\left(\hat{p} \cdot \hat{p}'\right) e^{2\mathrm{i}\, \overbrace{I_\ell(r = \infty)}^{}_{radial\ action} - \mathrm{i}\pi\ell}$$

- 2-points on Schwarzschild are function of the radial action $I(r)$

$$\underbrace{\langle p'|\,\mathcal{S}_\alpha\,|p\rangle}_{\textit{geodesic motion}} = \delta(E'-E)N\int d^2x^\perp\, e^{ix^\perp\cdot(\hat{p}-\hat{p}\,')}\left(2I(|x^\perp|)-\pi\,|\vec{p}|\,|x^\perp|\right)$$

$$I(r) = \int_{r_{\text{turn}}}^{r} ds\,\sqrt{\frac{p^2\,s^2+2GMm\,s-\frac{s-2GM}{s}\ell^2}{(s-2GM)^2}}$$

- The classical limit of a 3-point, on the other hand, is determined by Hamilton's principal function $S(t,r,\phi,\theta)$

$$\underbrace{\langle p',k^\eta|\,\mathcal{S}_\alpha\,|p\rangle}_{\textit{radiation}} = -\kappa\int_{\mathbb{R}^{1,3}\setminus\overline{B(r_S)}} d^4x\,\sqrt{-|g|}\int_{l,l',k'}\Lambda^{*,p}(l)\,\Lambda^{p'}(l')$$

$$\times\Lambda^{k}_{\mu\nu}{}^{\rho\sigma}(k')\,\mathcal{E}^{\eta}_{\rho\sigma}\left[2\partial^\mu S_{l'}\,\partial^\nu S_l - g^{\mu\nu}\left(\partial S_{l'}\cdot\partial S_l+\frac{m^2}{2}\right)\right]e^{i(S_{k'}+S_{l'}-S_l)}$$

- We can take the weak field limit by considering a linearized black hole (e.g. Schwarschild in de Donder gauge at order G)

$$\Lambda^{p}(l) = \delta(p_0 - l_0) \underbrace{\int d^2x^{\perp}\, e^{ix^{\perp}\cdot(\hat{p}-\hat{l})}\left(2I_{lin.}(|x^{\perp}|)-\pi\,|\vec{p}|\,|x^{\perp}|\right)}_{f^{eik.}(p,l)}$$

$$\Lambda^{k'}_{\mu\nu}{}^{\rho\sigma}(k)\Big|_{\text{lin. Schw.}} = -2\,\delta^{\rho}_{(\mu}\,\delta^{\sigma}_{\nu)}\,\delta(k'_0 - k_0)\,f^{eik.}(k,k')$$

- The weak field limit of a 3-point on Schwarzschild reproduces the probe limit of a 5-point amplitude on a flat background

$$\underbrace{\langle p', k|\,\mathcal{S}_{\alpha}\,|p\rangle}_{S-matrix\ on\ \bar{g}} = \frac{\hat{\delta}(U\cdot(q+k))}{2M}\underbrace{\mathcal{A}_5(p, P \to p+q, P-q-k, k)}_{S-matrix\ on\ \eta} + \ldots$$

Main message (Adamo, Cristofoli, Klisch, Ilderton)

On-shell amplitudes on a background are the building blocks for self-force