Lionel ***London*** *& Michelle* ***Gurevich***

An exact tri-diagonalization of Teukolsky's radial equation for QNMs

UK Royal Society
KING'S College LONDON
MIT KAVLI INSTITUTE

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Some open questions

- Why does the merger signal look so simple?
- What are the imprints of nonlinearities? (Nonlinear relative to what?) (Lagos, Sberna, and many others)
- Is it possible to rigorously understand connections between geometric, linear, nonlinear and non-stationary effects?
- Are QNMs useful as a spatial basis in non-QNM scenarios? (i.e. are QNMs broadly useful as special functions?)

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My premise

- Our ability to answer these questions is limited by gaps in our understanding of single BH perturbation theory.
- We can do better:
  - We can better understand the spheroidal nature of gravitational waves during ringdown, and perhaps beyond (completeness and bi-orthogonality)
  - We can better understand the polynomial nature of "overtones" (focus this talk)

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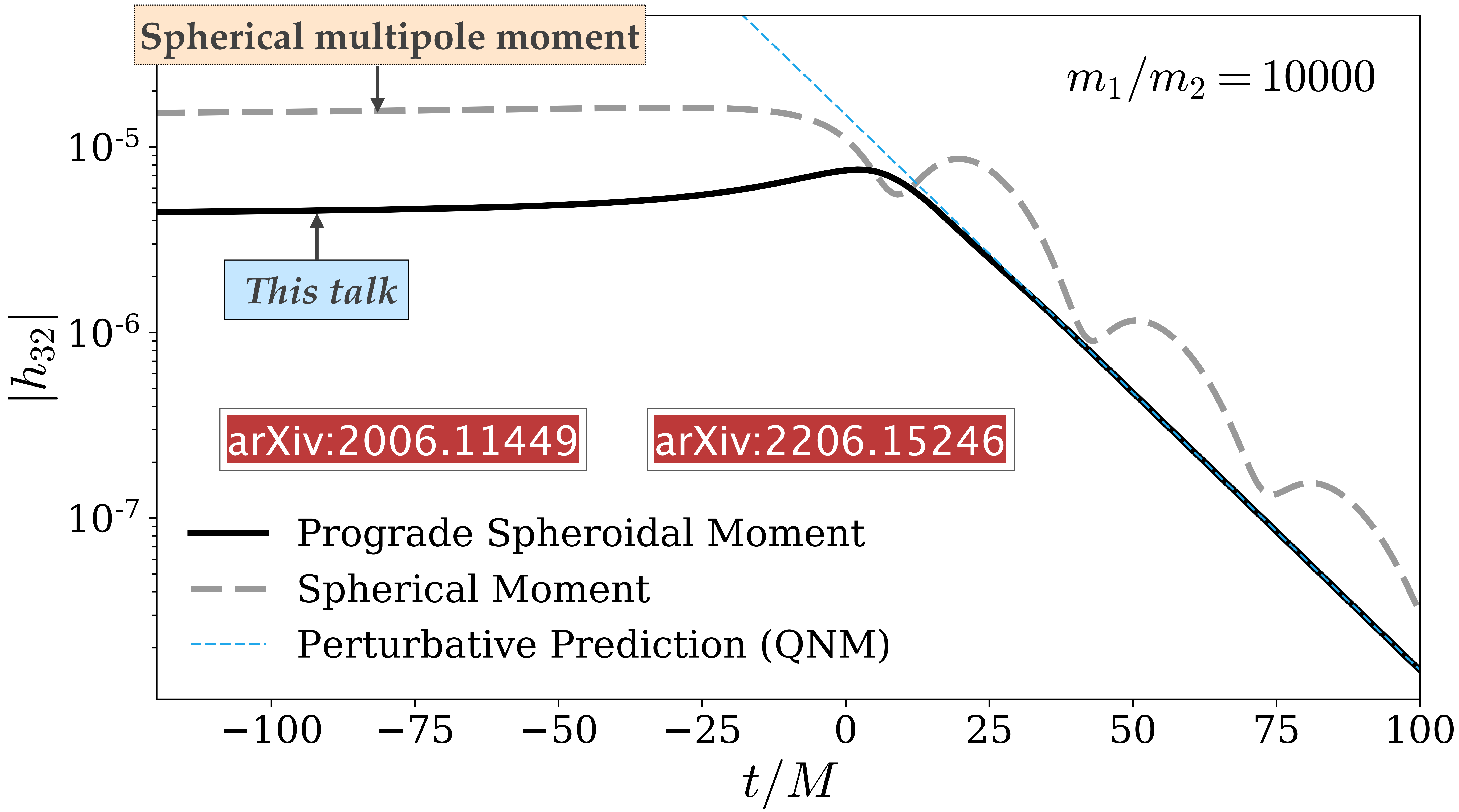
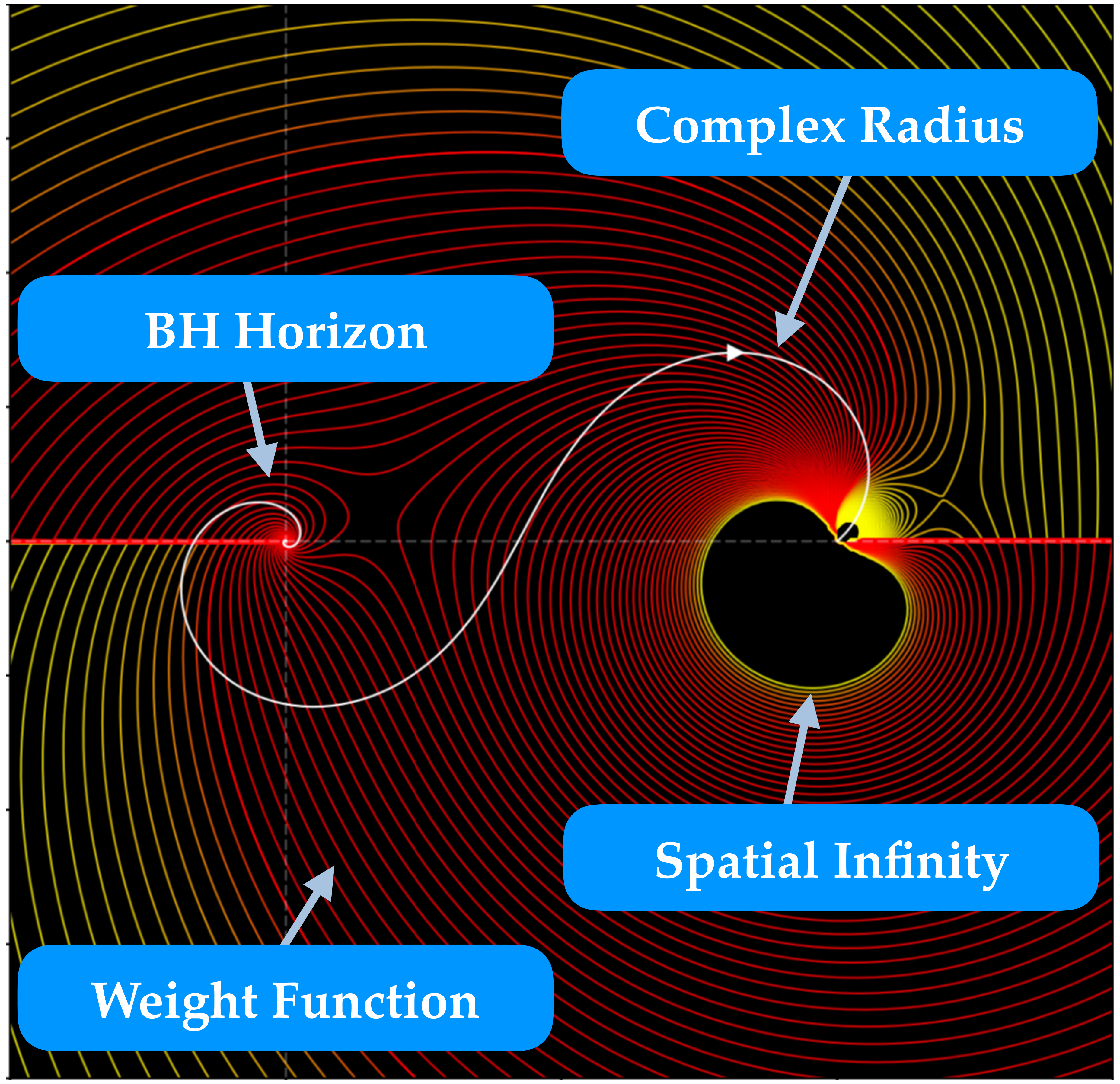
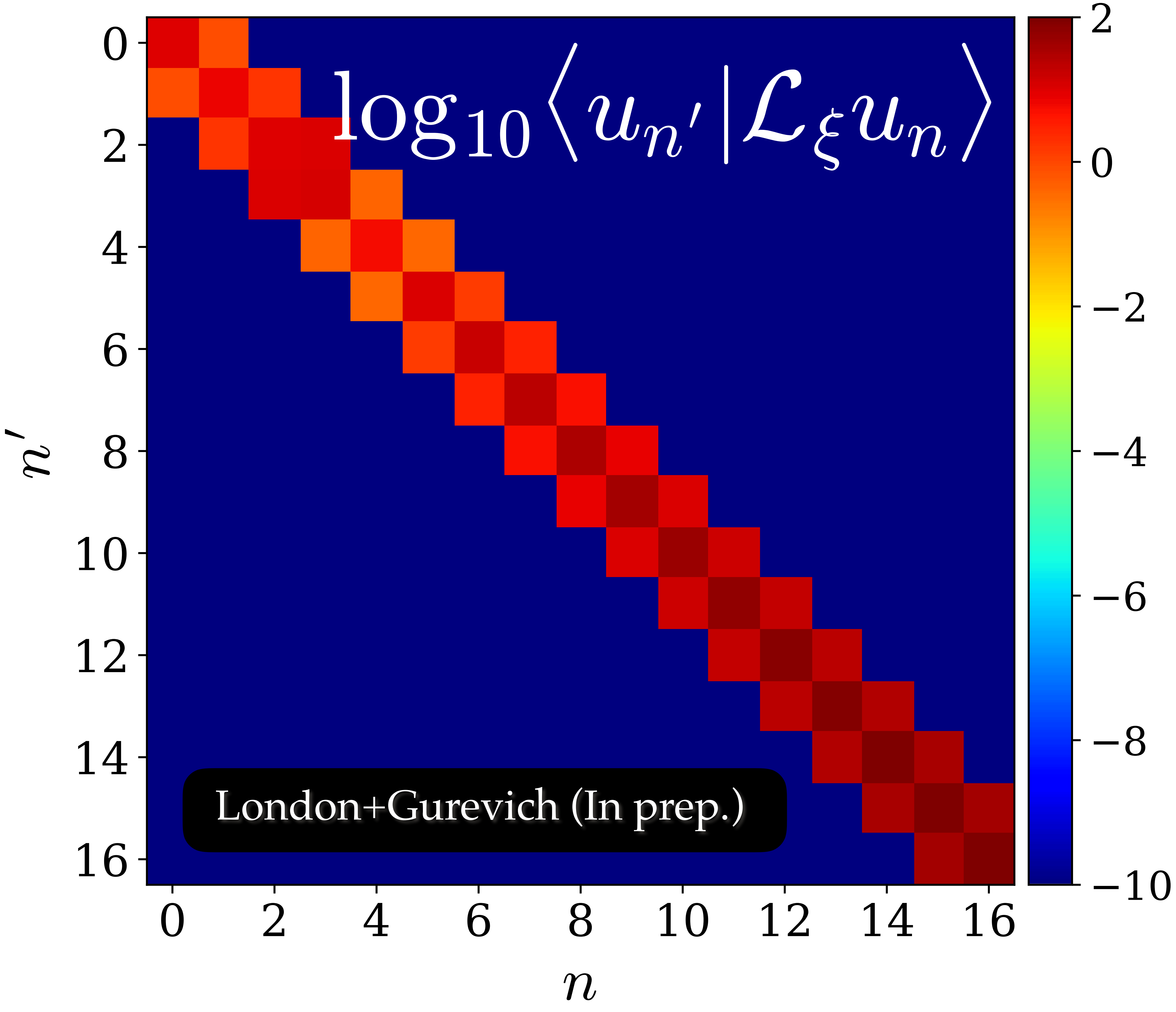
This talk: we can better represent numerical GWs



This talk: we can more deeply understand QNM "overtones"

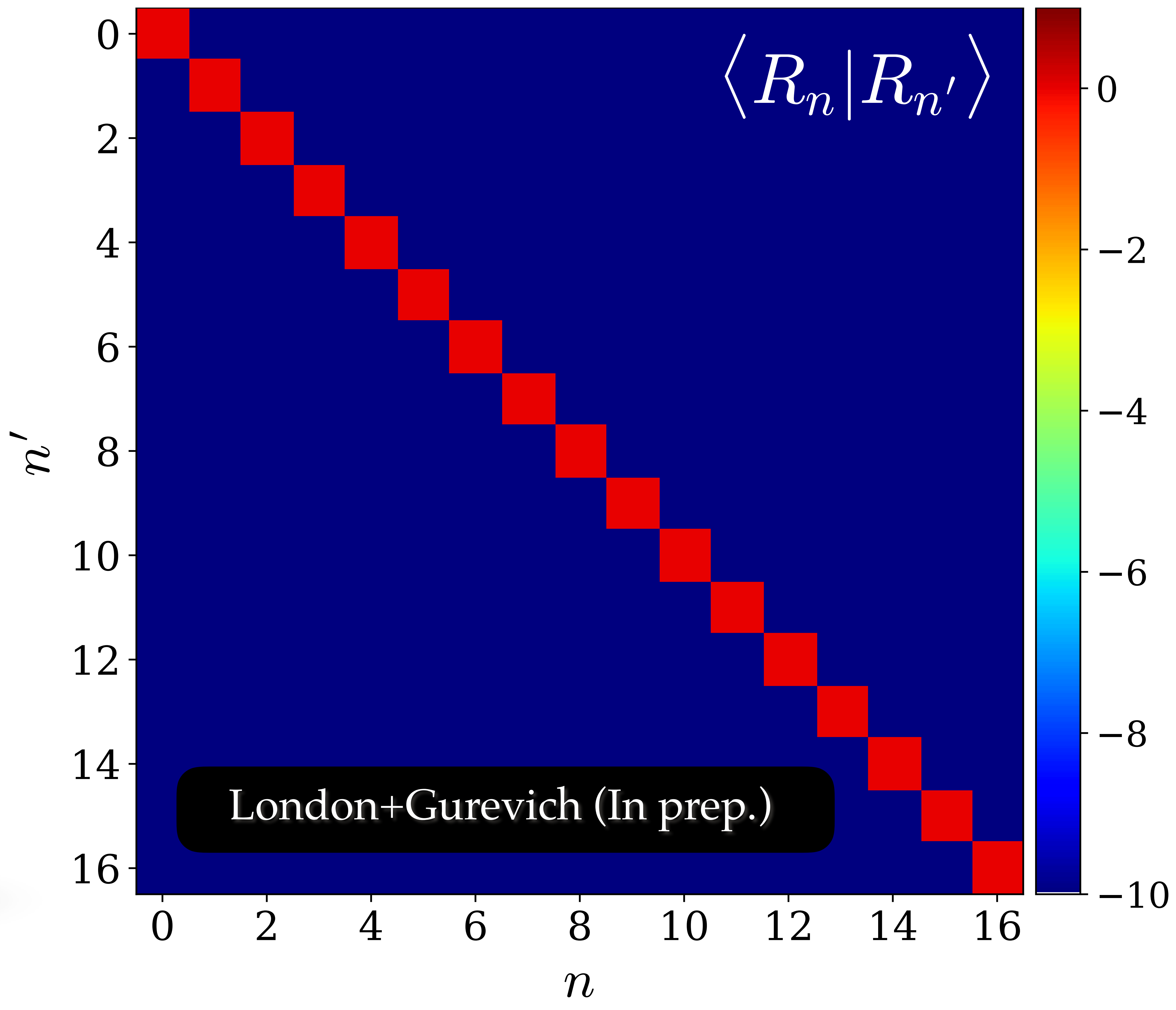


Cartoon of a Kerr inner-product interval



This talk: we can more deeply understand QNM “overtones”

- Development of “new” class of polynomials suited to the QNM problem
- Explicit orthogonality of the homogeneous solutions (for fixed frequency)
- much fun for the future …



A spheroidal picture for GWs from arbitrary sources

arXiv:2006.11449 arXiv:2206.15246

- Are the spheroidal harmonics orthogonal in some way? Yes
- Are the spheroidal harmonics for QNMs complete? Yes
- Can spheroidal harmonics be thought of as useful special functions for GWs? Probably

The hard parts and their solutions

The hard parts

- $$\mathcal{D}_u(\omega_{\ell mn}) = (ua\omega_{\ell mn} - 2s)ua\omega_{\ell mn} - \frac{(m+su)^2}{1-u^2} + \partial_u(1-u^2)\partial_u$$

- $$\mathcal{D}_u(\omega_{\ell mn})^{\dagger} \;=\; \mathcal{D}_u(\omega_{\ell mn})^{*}$$

Their solutions

- $$\mathcal{T} \;=\; \sum_{\ell=2}^{\infty}\sum_{\ell'=2}^{\infty} |Y_{\ell m}\rangle\langle Y_{\ell m}|S_{\ell' m}\rangle\langle Y_{\ell' m}|$$

- $$|S_\ell\rangle \;=\; \mathcal{T}\,|Y_{\ell m}\rangle\,, \quad \text{and} \quad |\tilde{S}_\ell\rangle \;=\; {\mathcal{T}^{\dagger}}^{-1}\,|Y_\ell\rangle$$

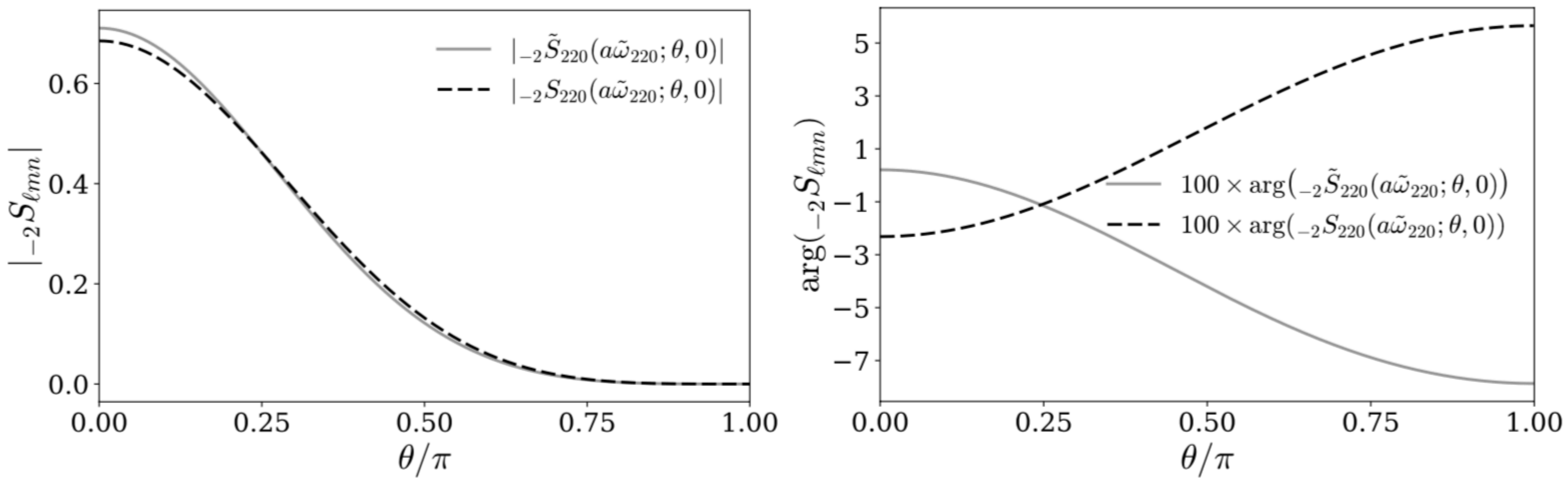
new "adjont-spheroidals"

Some intuition about Adjoint-Spheroidal Harmonics

Example “small spin” expansion for term-by-term comparison:

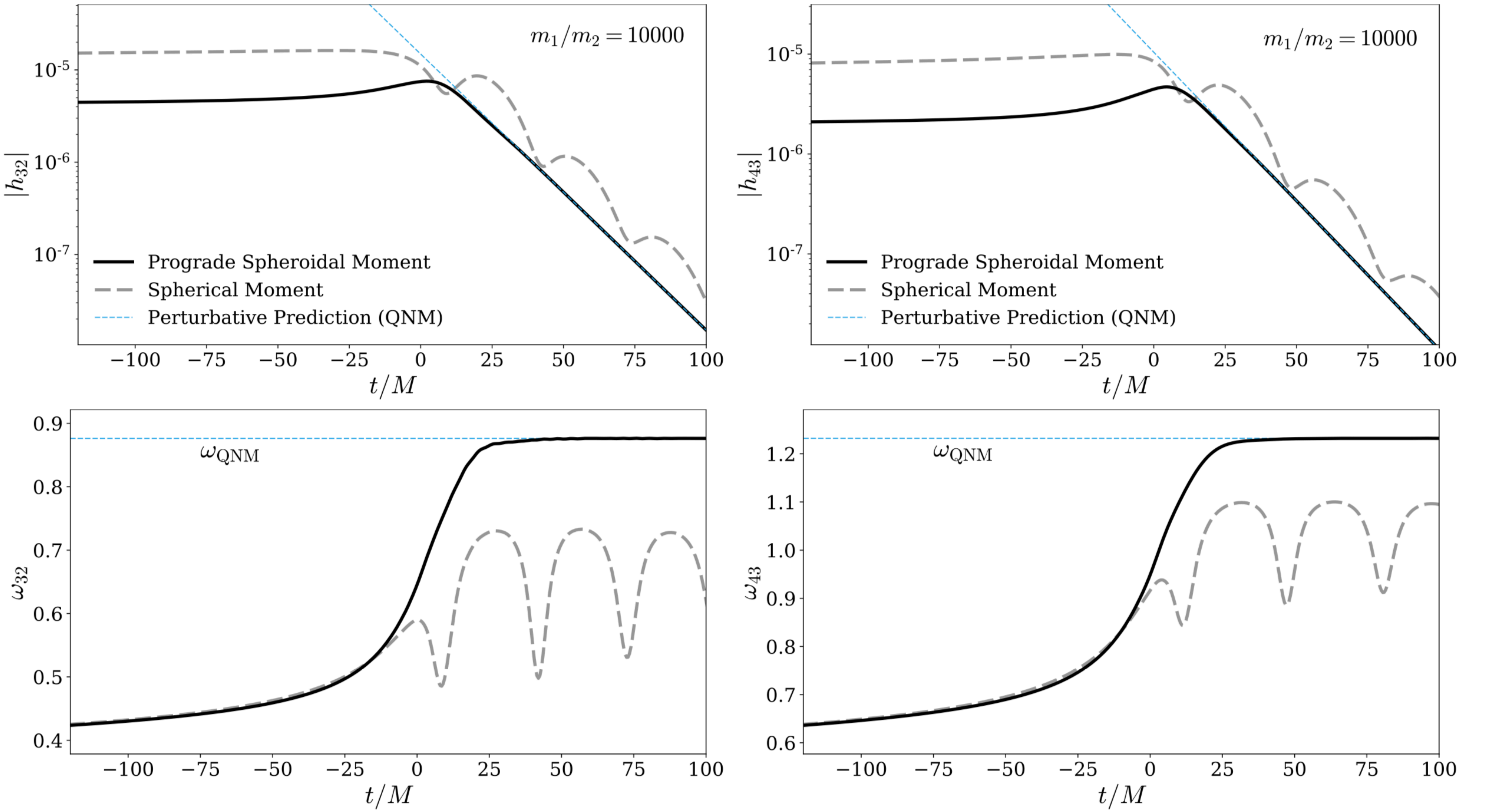
$$S_{\ell mn} \approx Y_{\ell m} + a\tilde{\omega}_{\ell mn} c_{\ell}^{\ell-1} Y_{\ell-1,m} + a\tilde{\omega}_{\ell mn} c_{\ell}^{\ell+1} Y_{\ell+1,m}$$

$$\tilde{S}_{\ell mn} \approx -Y_{\ell m} + a\tilde{\omega}^{*}_{\ell-1,m,n} c^{\ell}_{\ell-1} Y_{\ell-1,m} + a\tilde{\omega}^{*}_{\ell+1,m,n} c^{\ell}_{\ell+1} Y_{\ell+1,m}$$



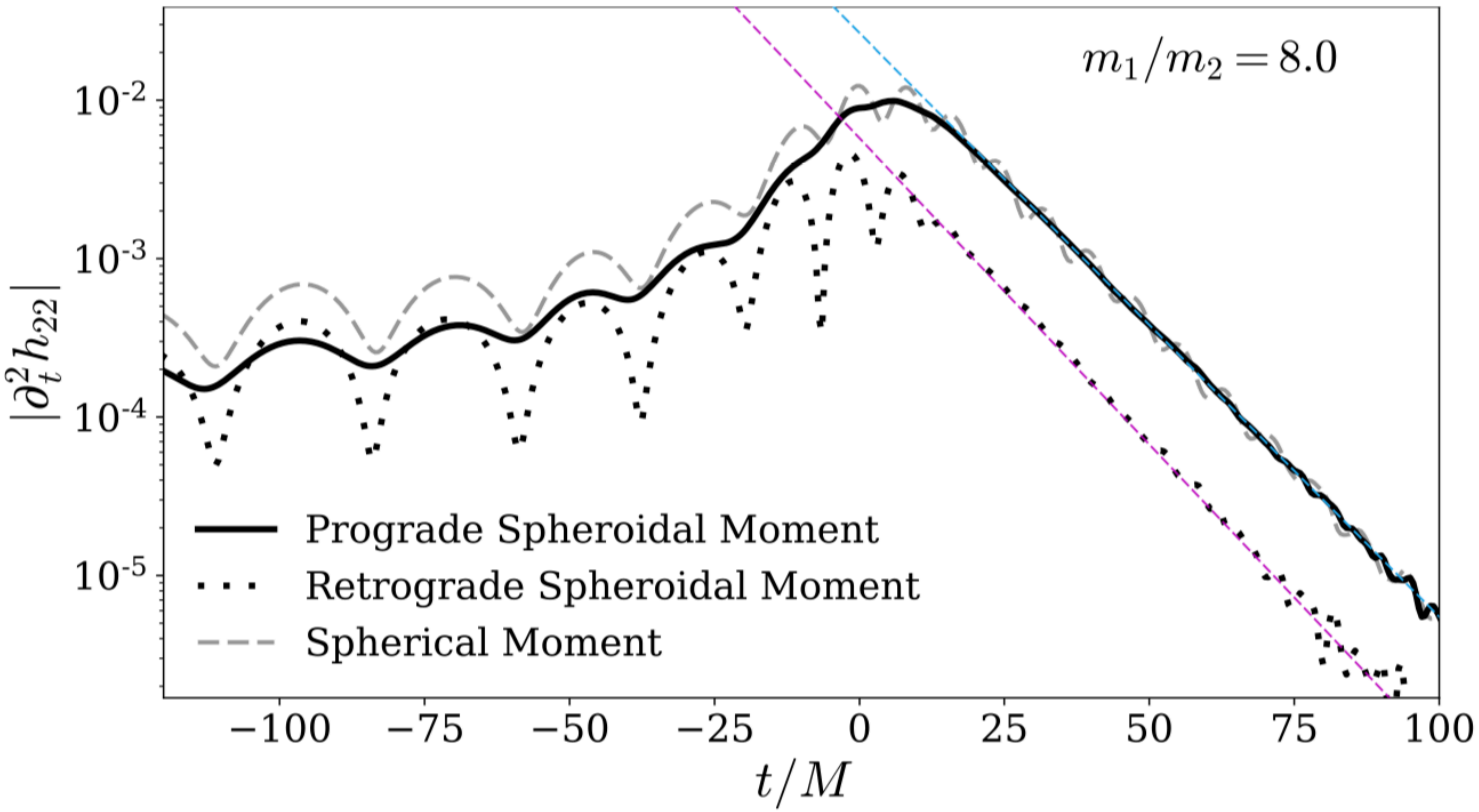
London+Hughes, arXiv:2206.15246

Example applications: Extreme Mass-Ratio Binary



Example applications: Comparable Mass-Ratio Binary

Mass ratio 8:1, **precessing** BBH simulation (BAM, Cardiff University)



An exact tri-diadonalization of Teukolsky's radial equation for QNMs

papers in progress

A purely "spectral" approach to Teukolsky's radial equation for QNMs

papers in progress

Some quirks of the radial problem

- Many studies apply special functions from other problems. Here we focus on and develop special functions motivated directly by Teukolsky's radial problem
- We will draw as much as possible from Sturm-Liouville theory. For this, a suitable coordinate choice is essential
- For QNMs, we are interested in the "exterior problem" — i.e. spacetime between the event horizon and spatial infinity

$$\mathcal{L}_r \, R(r) \;=\; A \, R(r)$$

For QNMs, we seek to solve an eigenvalue problem, where the eigenvalue is the separation constant.

Teukolsky's radial equation

Use fractional radial coord

$$\mathcal{L}_r = \left(\mathrm{A}_0 + r\mathrm{A}_1 + (\mathrm{A}_2\, r)^2 + \frac{\mathrm{A}_3}{r - r_-} + \frac{\mathrm{A}_4}{r - r_+} \right) + (\mathrm{A}_5 + \mathrm{A}_6 r)\,\partial_r + (r - r_-)(r - r_+)\,\partial_r^2$$

The related differential operator is not well formatted for the exterior problem ….

Teukolsky's radial equation

Use fractional radial coord

$$\xi = \frac{r - r_+}{r - r_-}$$

$$\mathcal{L}_\xi = \left(\mathrm{B0} + \frac{\mathrm{B1}}{\xi} + \frac{\mathrm{B2}}{1-\xi} + \frac{\mathrm{B3}^2}{(1-\xi)^2} + \mathrm{B4}(1-\xi) \right) + \left(\mathrm{B5}(1-\xi) + \mathrm{B6}(1-\xi)^2\right) \partial_\xi + \xi(1-\xi)^2 \partial_\xi^2$$

One approach taken by Leaver in 1985 was to use what I'll call a "**fractional radial coordinate**"

ukolsky's radial equation | Use fractional radial coordinates | Apply QNM boundary co

$$R(r(\xi)) = \mu(\xi) f(\xi)$$

$$\mu(\xi) = e^{\frac{2i\delta\tilde{\omega}}{1-\xi}} (1-\xi)^{1+2(s-iM\tilde{\omega})} \times \xi^{-(iM\tilde{\omega}+s)+\frac{i\left(am-2M^2\tilde{\omega}\right)}{2\delta}}$$

From here we can apply the QNM boundary conditions, which amount to a **similarity transformation** of the problem …

$$\left[\mu(\xi)^{-1}\,\mathcal{L}_r\,\mu(\xi(r))\right] f(\xi) \;=\; \left[\mu(\xi)^{-1} A\,\mu(\xi)\right] f(\xi)$$

$$\mathcal{L}_\xi\, f(\xi) \;=\; A\, f(\xi)$$

From here we can apply the QNM boundary conditions, which amount to a **similarity transformation** of the problem …

actional radial coordinates | Apply QNM boundary conditions | Study differential pa

$$\mathcal{L}_\xi = \left(\mathrm{C}_0 + \mathrm{C}_1(1-\xi)\right) + \left(\mathrm{C}_2 + \mathrm{C}_3(1-\xi) + \mathrm{C}_4(1-\xi)^2\right)\partial_\xi + \xi(\xi-1)^2\partial_\xi^2$$

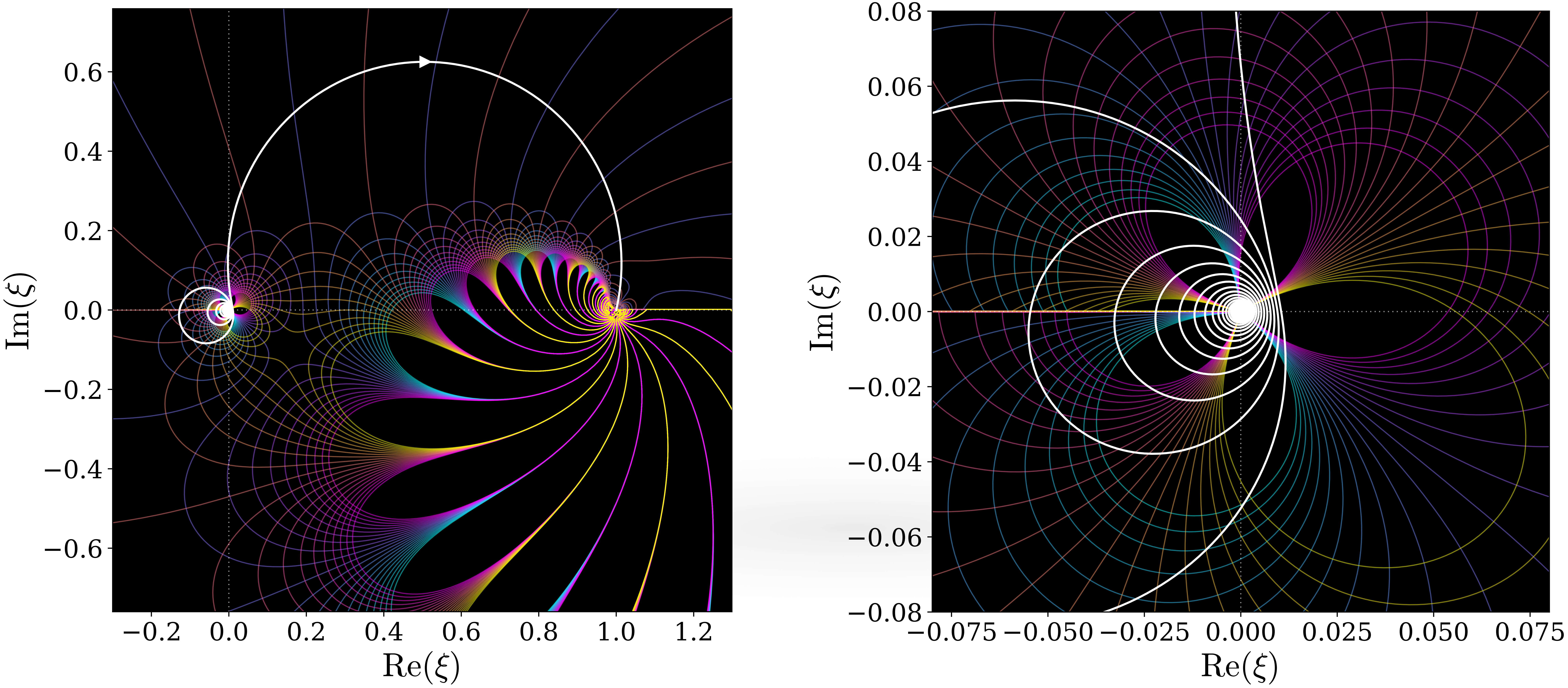
The resulting differential operator has a regular potential, and its second derivative coefficient **simplifies boundary condition requirements** of Sturm-Liouville theory

actional radial coordinates | Apply QNM boundary conditions | Study differential pa

$$\langle \mathrm{a} \,|\, \mathrm{b} \rangle \;=\; \int_0^1 \mathrm{a}(\xi)\, \mathrm{b}(\xi)\;\; \xi^{\mathrm{B}_0} (1-\xi)^{\mathrm{B}_1} e^{\frac{\mathrm{B}_2}{1-\xi}}\, d\xi$$

Further, we can use the transformed differential operator to **construct a scalar product** (symmetric bilinear form) (Green+) **Evaluation scalar products is possible** …

actional radial coordinates | **Apply QNM boundary conditions** | Study differential pa



actional radial coordinates | **Apply QNM boundary conditions** | Study differential pa

$$\mathcal{L}_\xi = (\mathrm{C}_0 + \mathrm{C}_1(1-\xi)) + \mathcal{D}_\xi$$

$$\mathcal{D}_\xi = \left(\mathrm{C}_2 + \mathrm{C}_3(1-\xi) + \mathrm{C}_4(1-\xi)^2\right)\partial_\xi + \xi(\xi-1)^2\partial_\xi^2$$

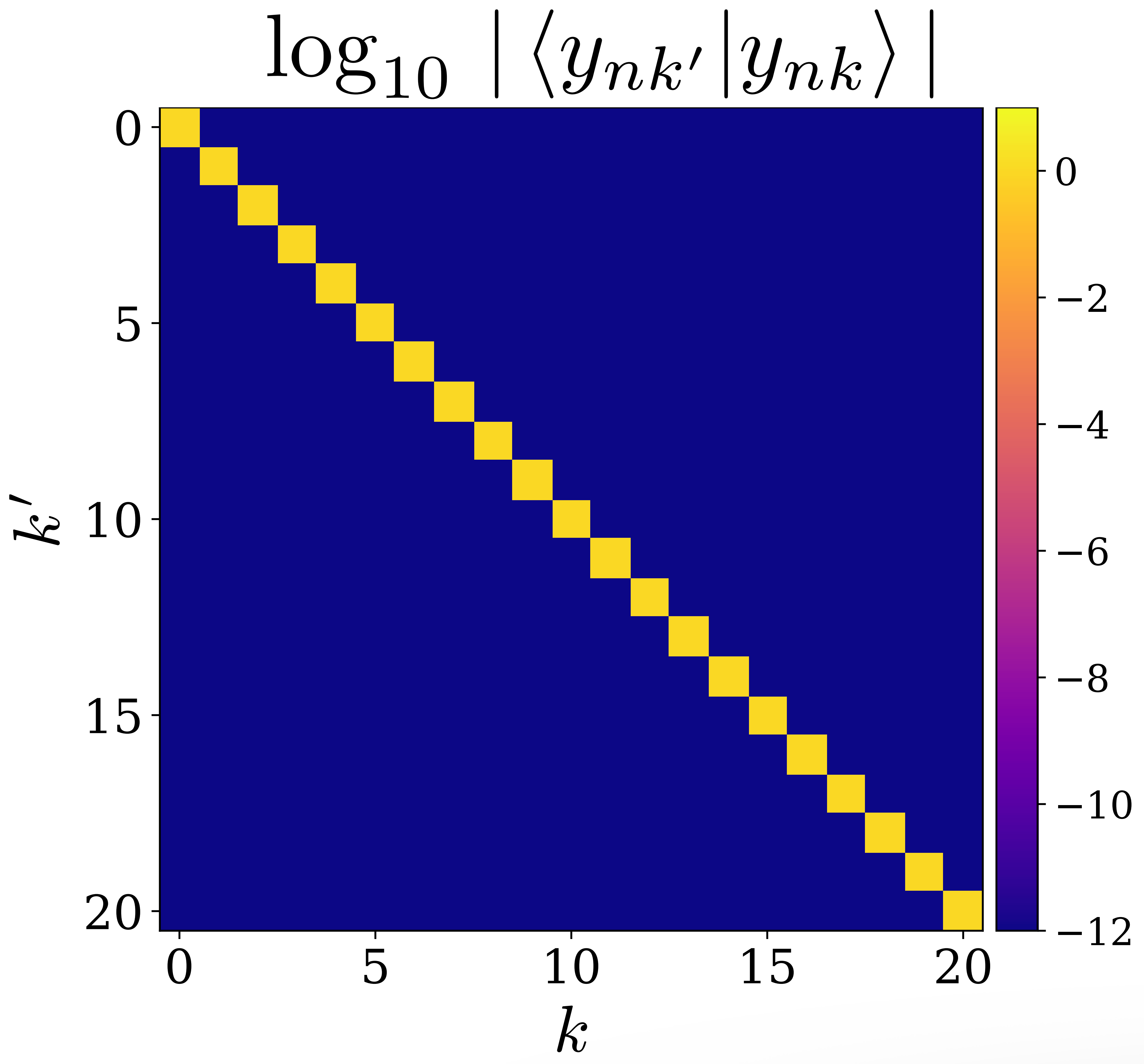
Key idea: If there exist a class of polynomials are eigenfunctions of $\mathcal{D}_\xi$ then they would be extremely well positioned to simplify the determination of solutions to the physical problem

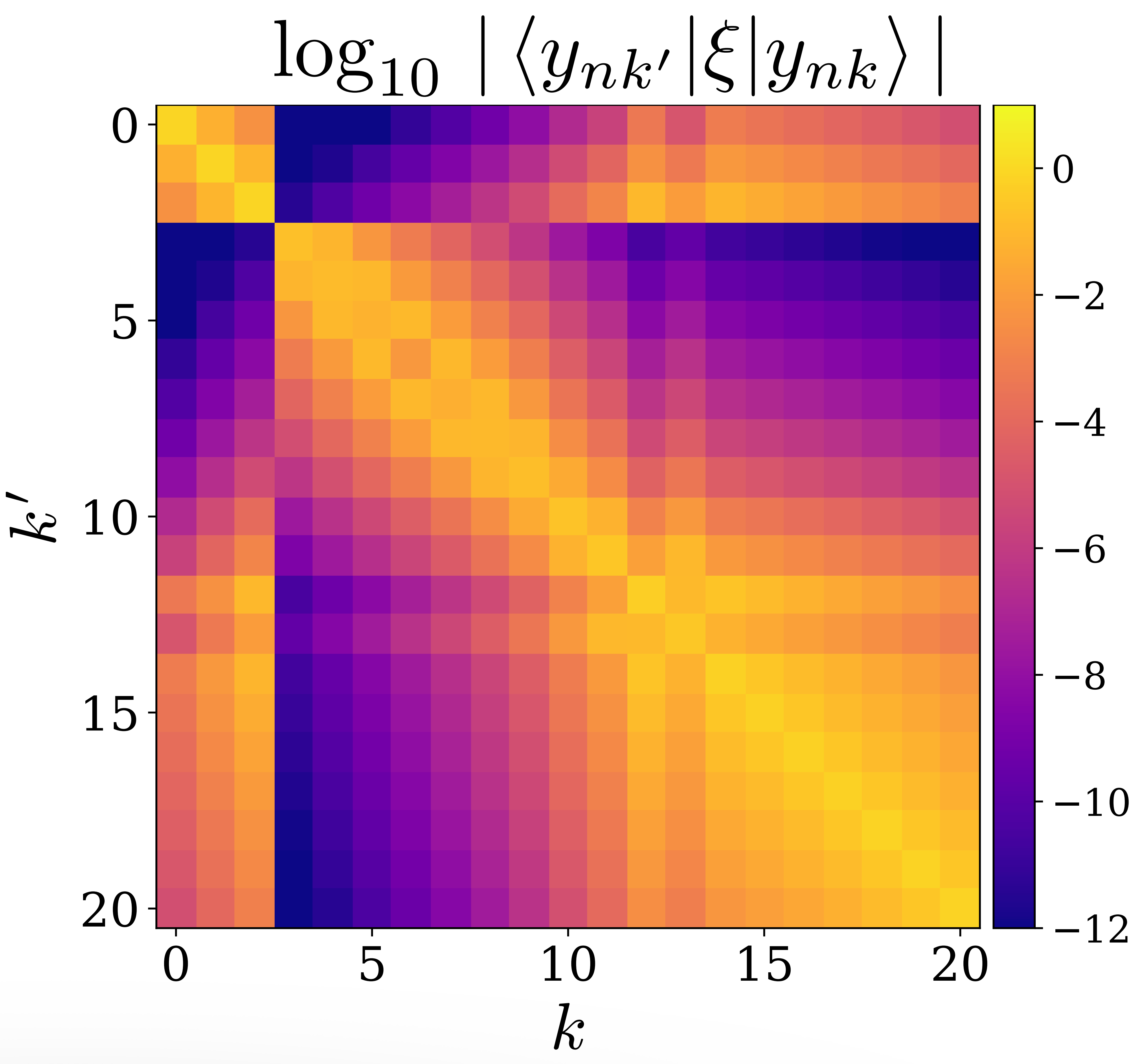
QNM boundary conditions | **Study the differential part** | Confluent Heun polyno

$$y_{nk} = \sum_{j}^{n} a_{jkn}\xi^{j}$$

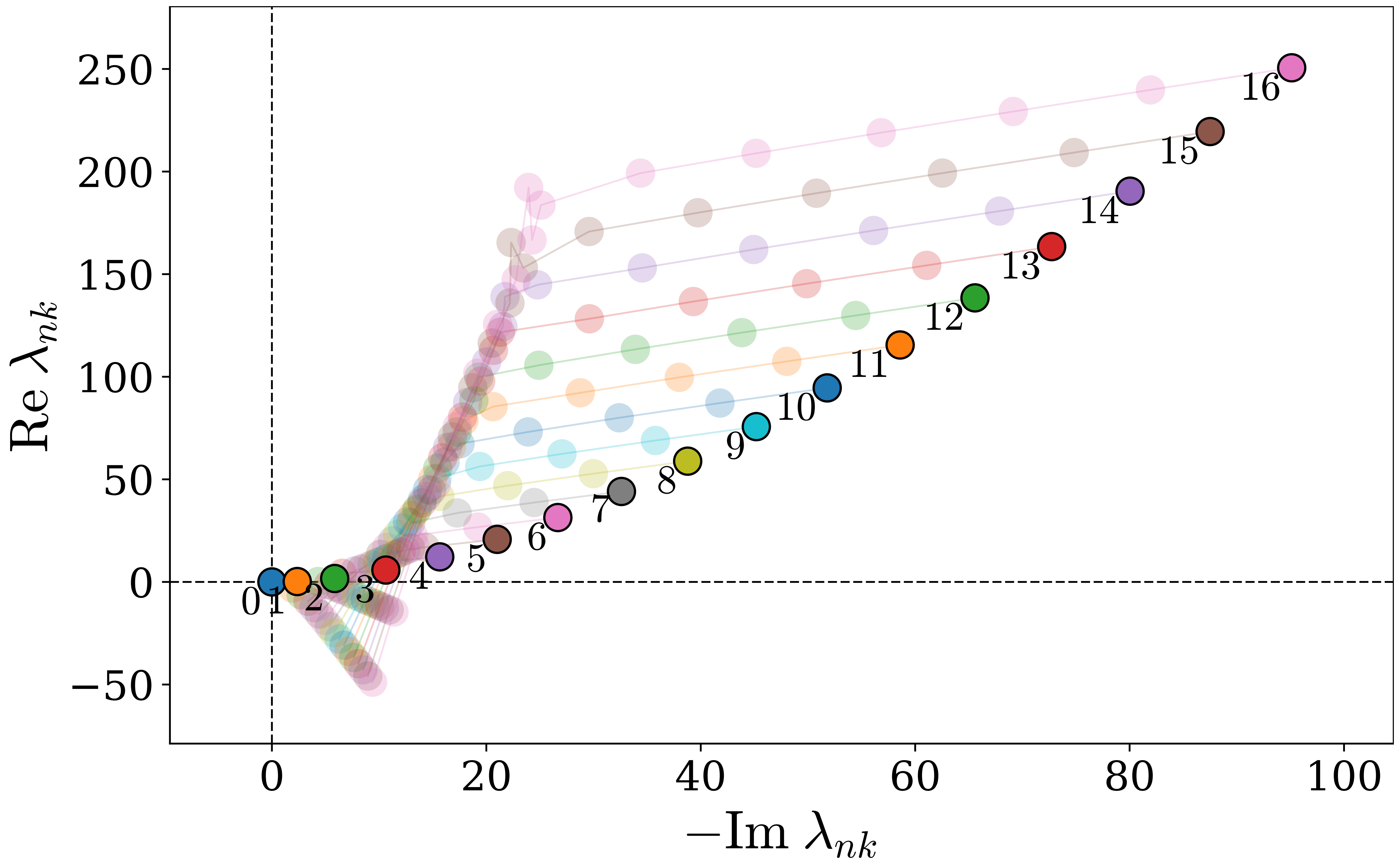
$$\mathcal{D}_\xi \, y_{nk}(\xi) = [\, \lambda_{nk} - n(n-1+\mathrm{C}_4)(1-\xi) \,] \; y_{nk}(\xi)$$

It turns that that polynomial solutions require a somewhat generalized eigenvalue relationship. **Confluent Heun polynomials** are the result …





Using the scalar product, one can show orthogonally between polynomials of fixed order

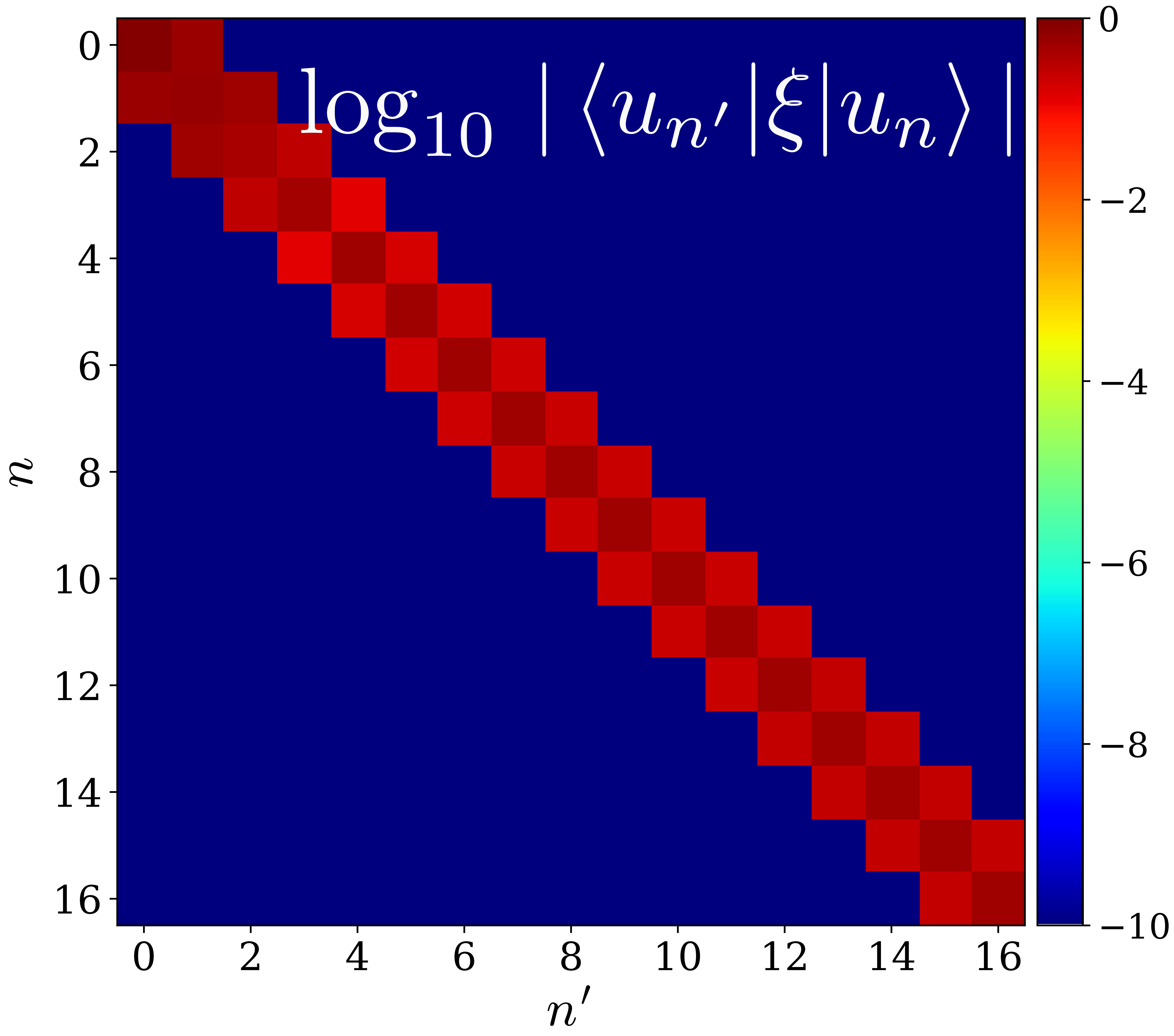


- Dominant eigenvalues are typically distinct
- Parabolic dependence can be qualitatively understood

$$\lambda_{nk} = n\,(\,n + C_4 - 1\,)\,\frac{\langle y_{00}|\xi\ |y_{nk}\rangle}{\langle y_{00}\,|\,y_{nk}\rangle}$$

Since the polynomials are "non-classical", they are not sufficient to simplify the physical problem. **It happens that this is not a dead end ….**

y at fixed polynomial order	Behavior of eigenvalues	Develop orthonormal seq

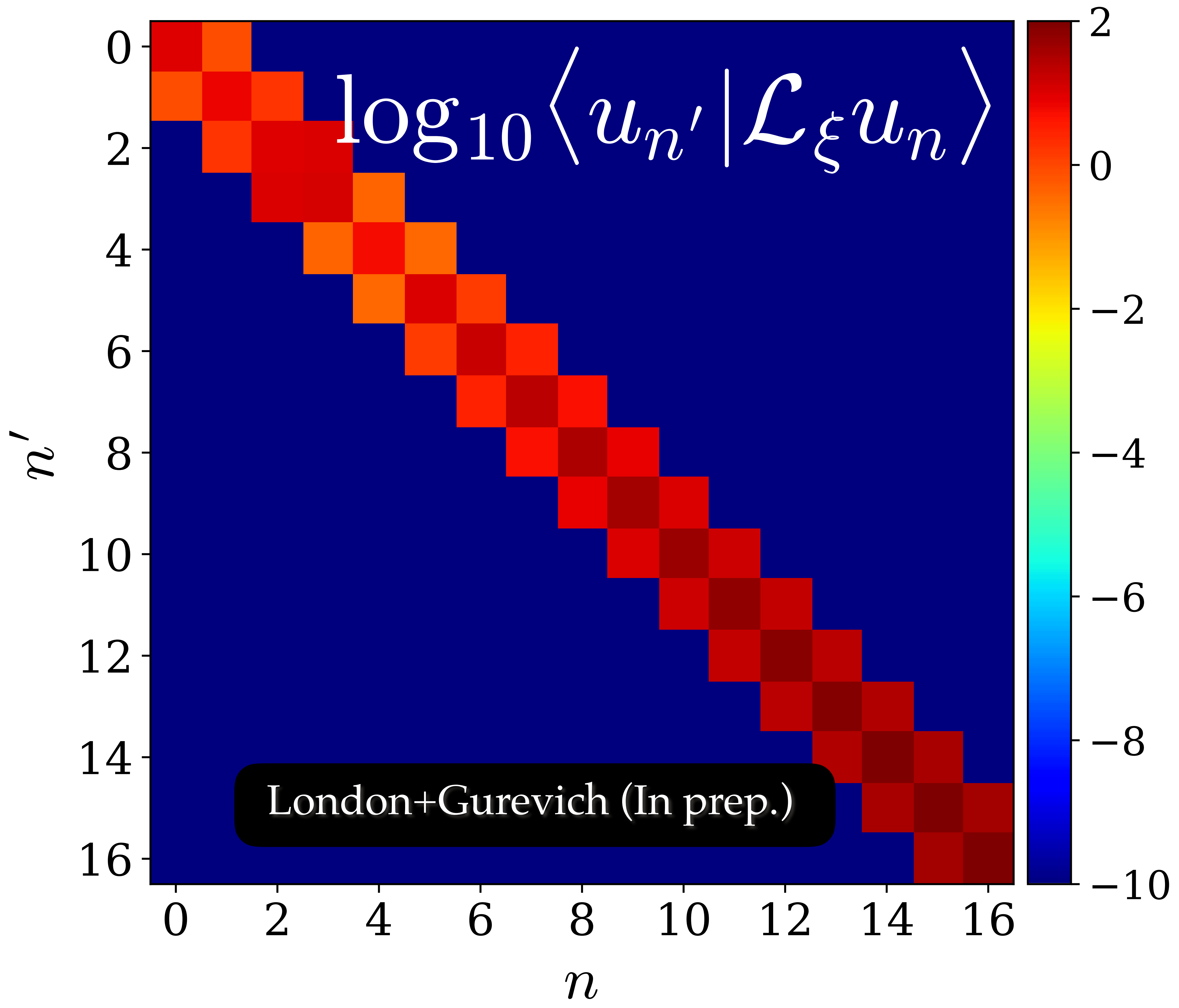


$$u_n(\xi) = \sum_{k=0}^{n} b_{nk}\, y_{nk}(\xi)$$

We have a multitude of options …

- "Canonical construction" (novel)
- Gram Schmidt …
- **All have three term recursion**

ehavior of eigenvalues | Develop orthonormal sequence | Apply to the radial equa



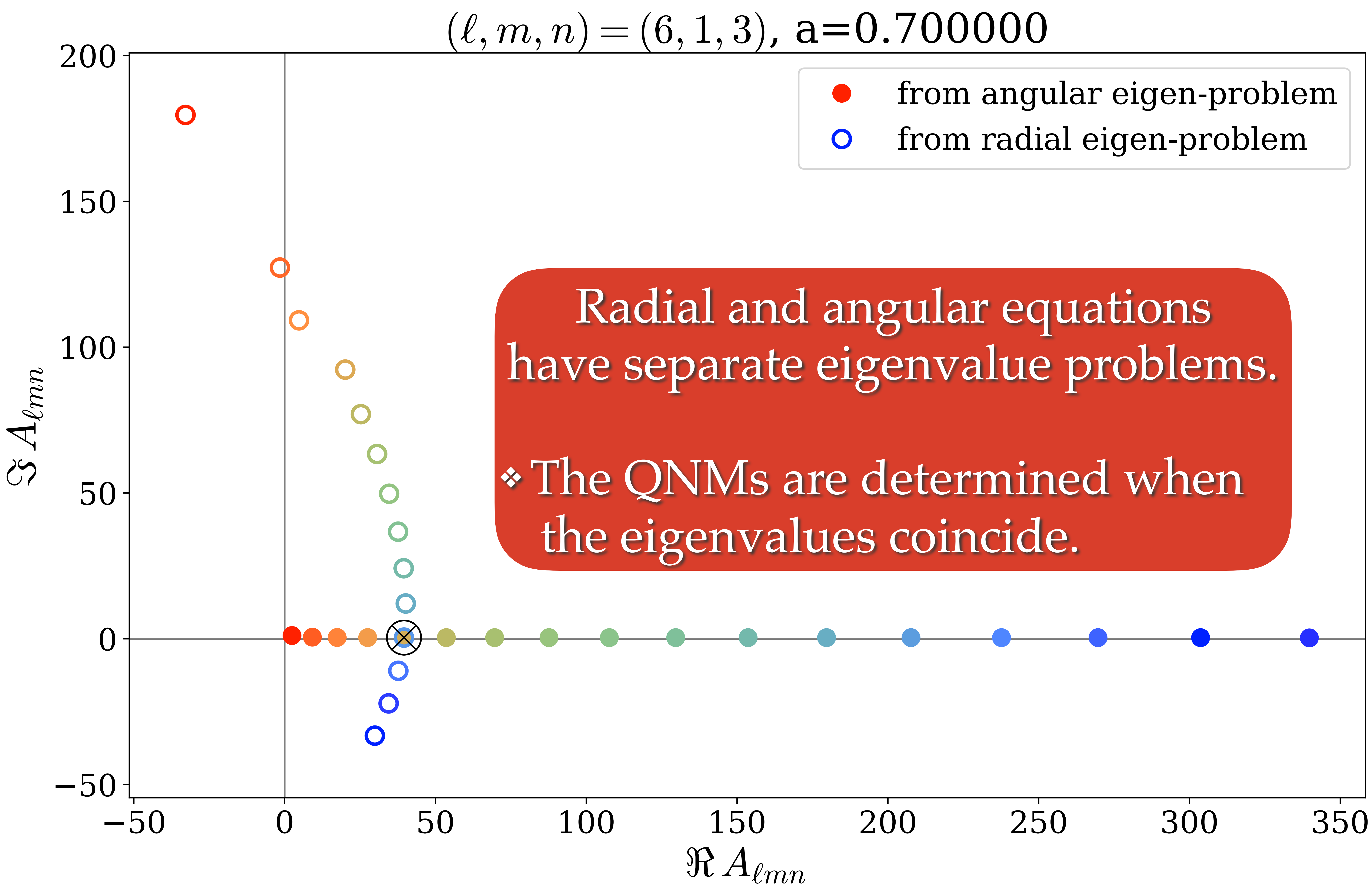
$$\mathcal{L}_\xi\, f(\xi) \;=\; A\, f(\xi)$$

$$f(\xi) \;=\; \sum_{n=0}^{\infty} c_n\, u_n(\xi)$$

- **Hermitian nature of operator**
- **three-term recursion of polynomials**
- **orthogonality polynomials**

elop orthogonal sequence

Apply to the radial equation



Radial and angular equations have separate eigenvalue problems.

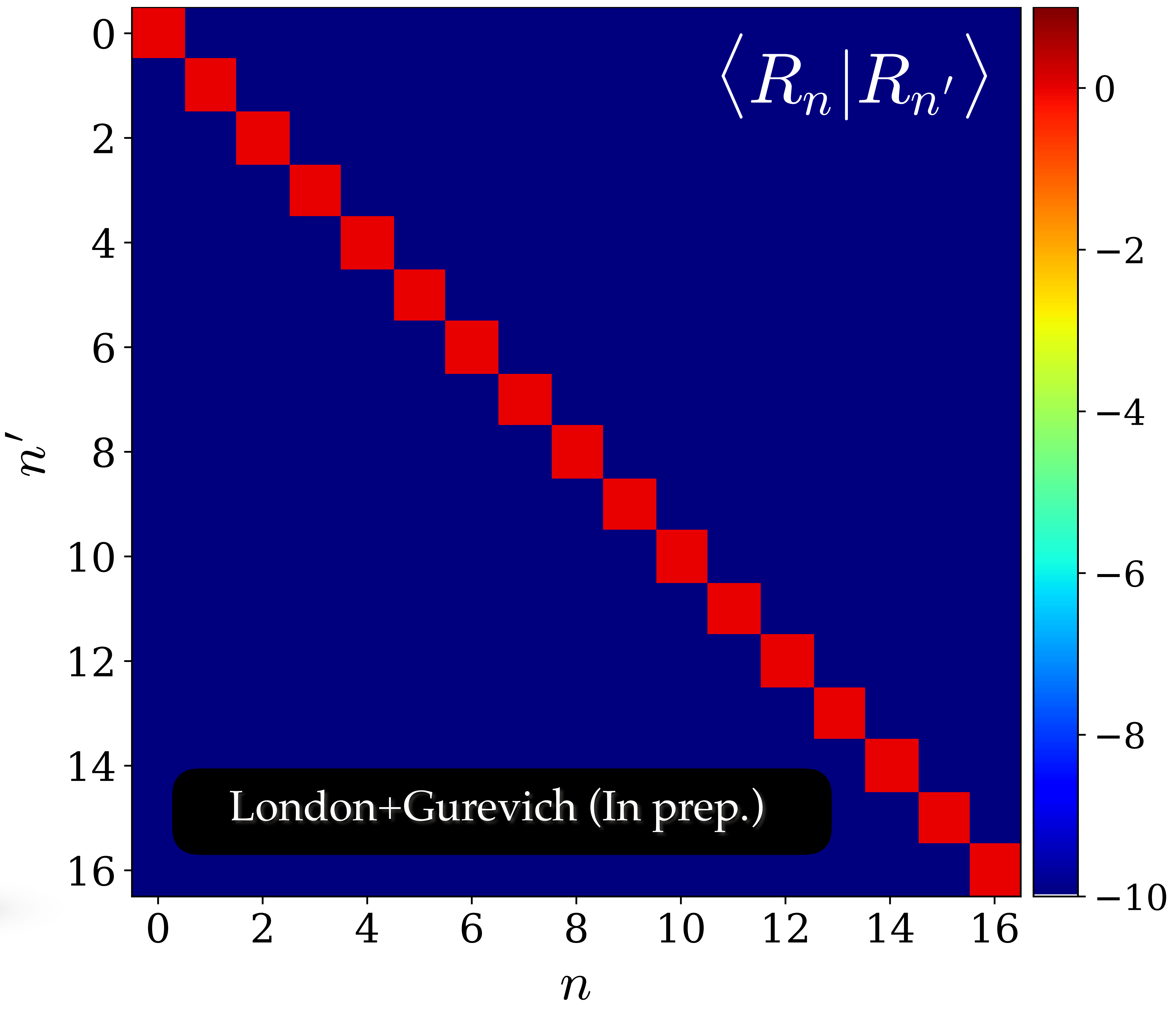
- The QNMs are determined when the eigenvalues coincide.

elop orthogonal sequence

Apply to the radial equation

Maybe in Singapore …

- A few papers published (all in prep now)
- Various comparisons with other methods …
- Better analytic understanding of confluent Heun polynomials
- Better understanding of our orthogonal polynomials
- Applications to non-QNM scenarios? …



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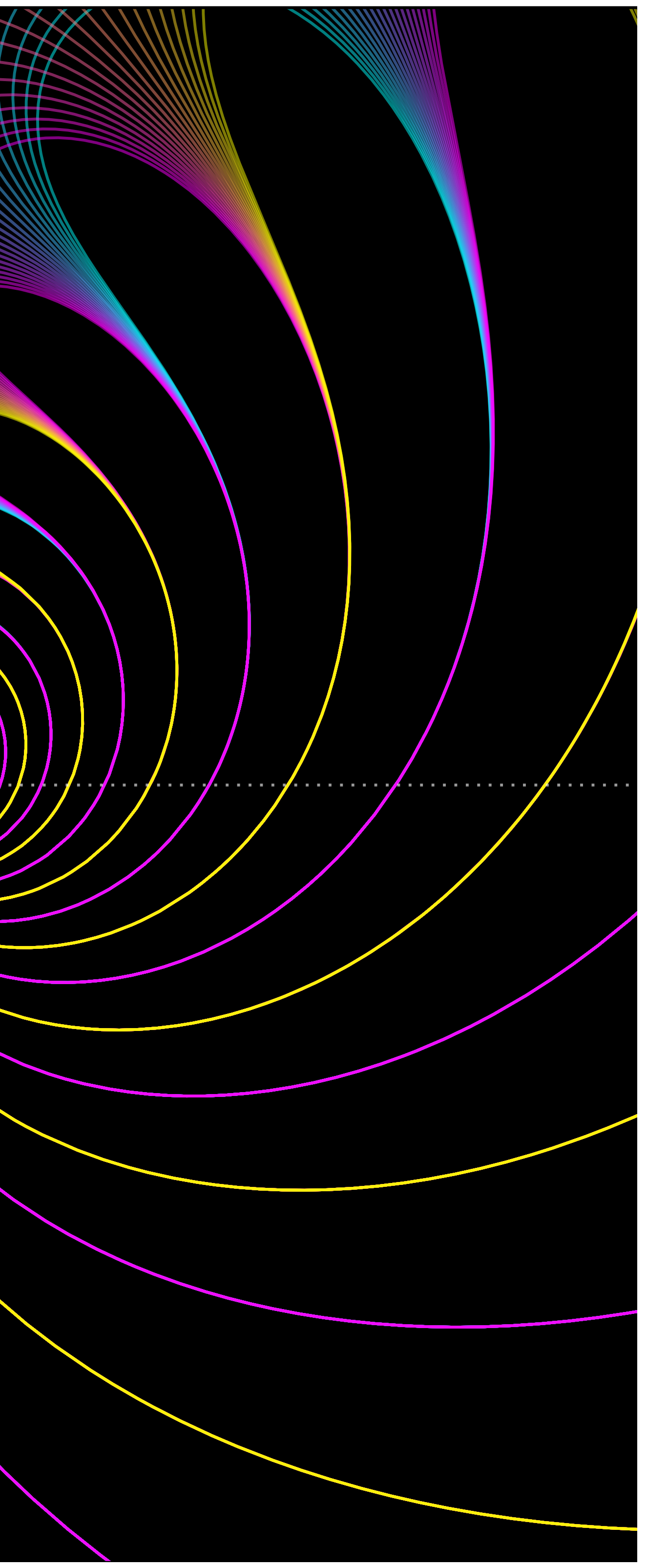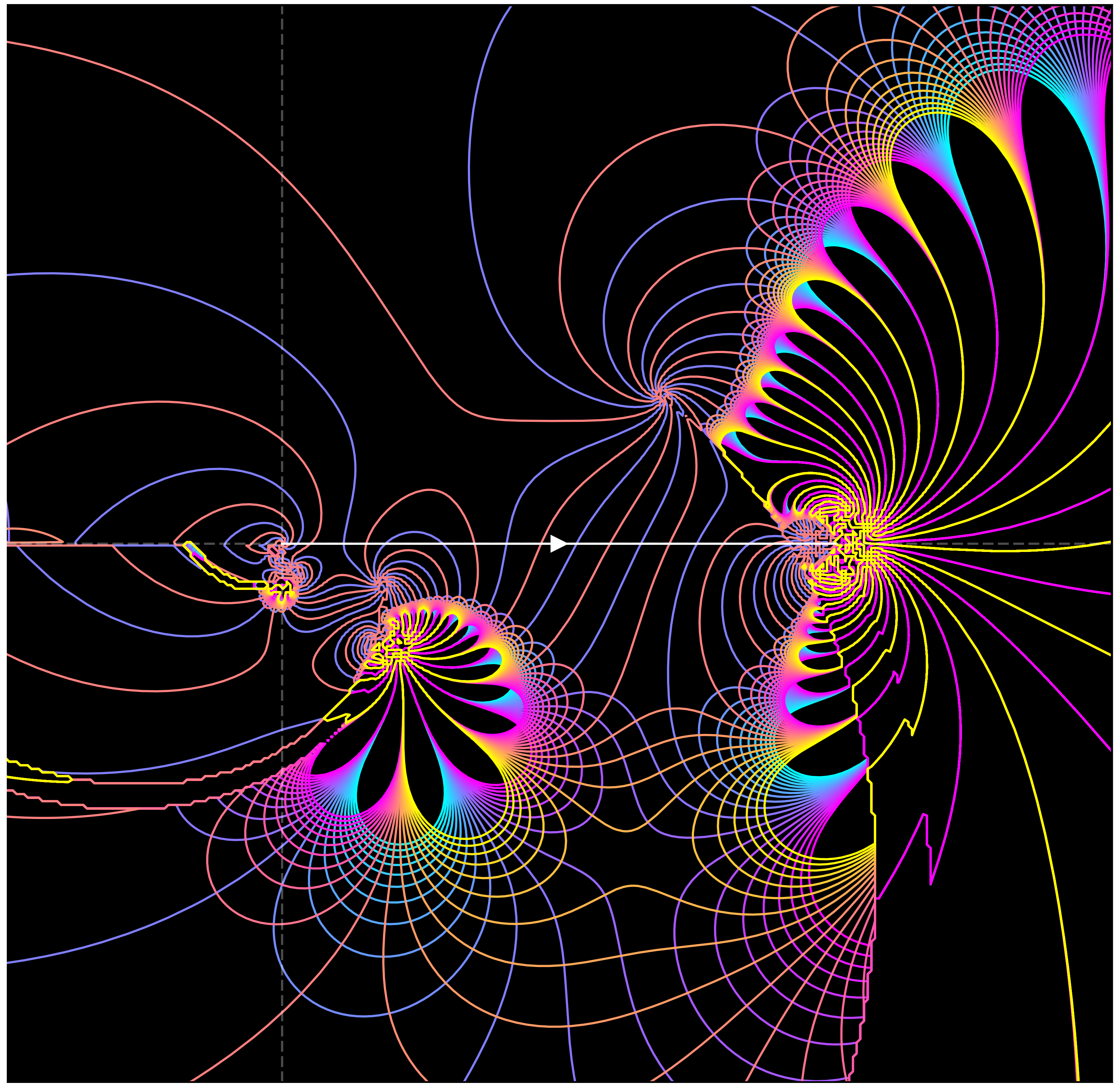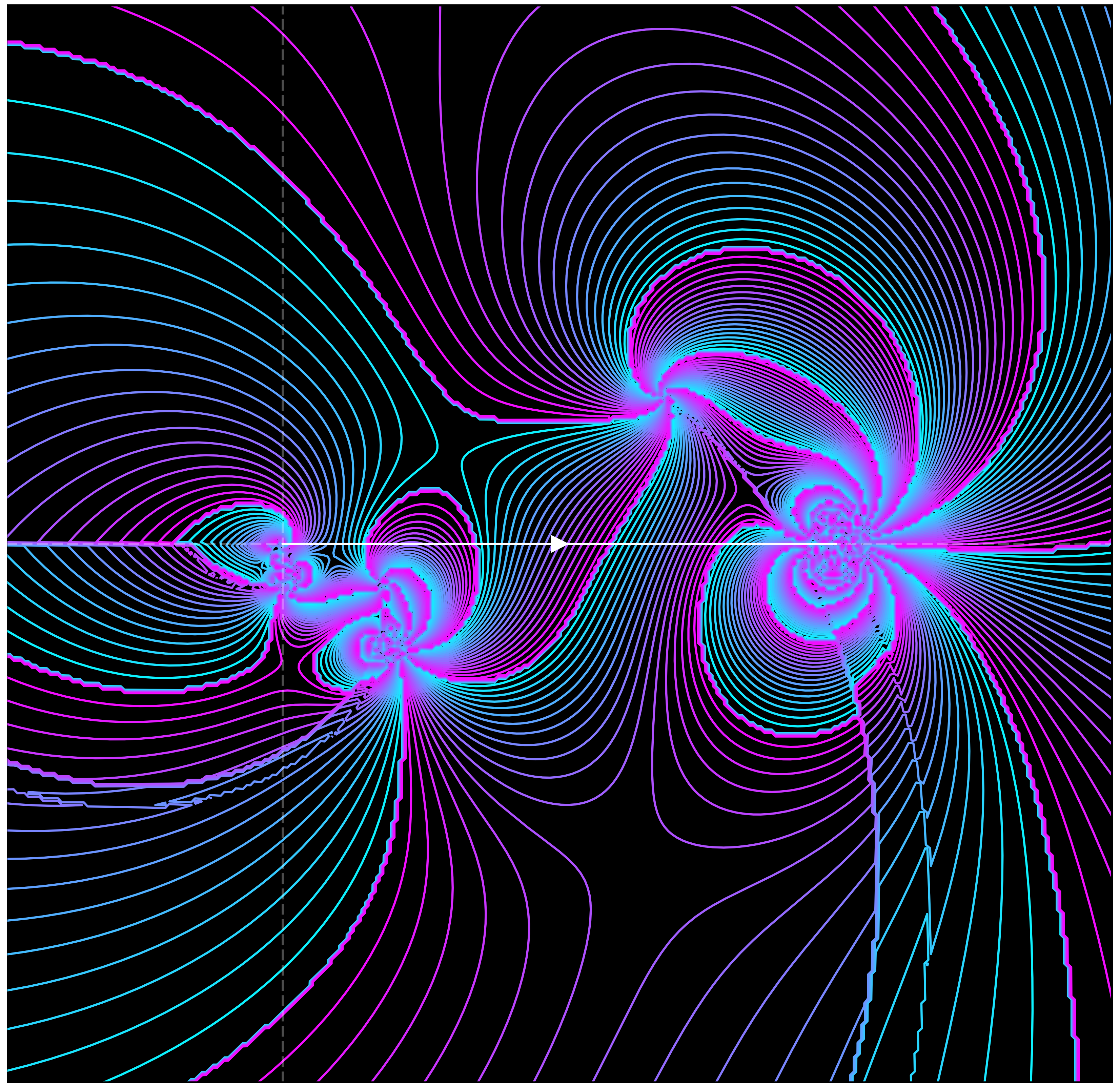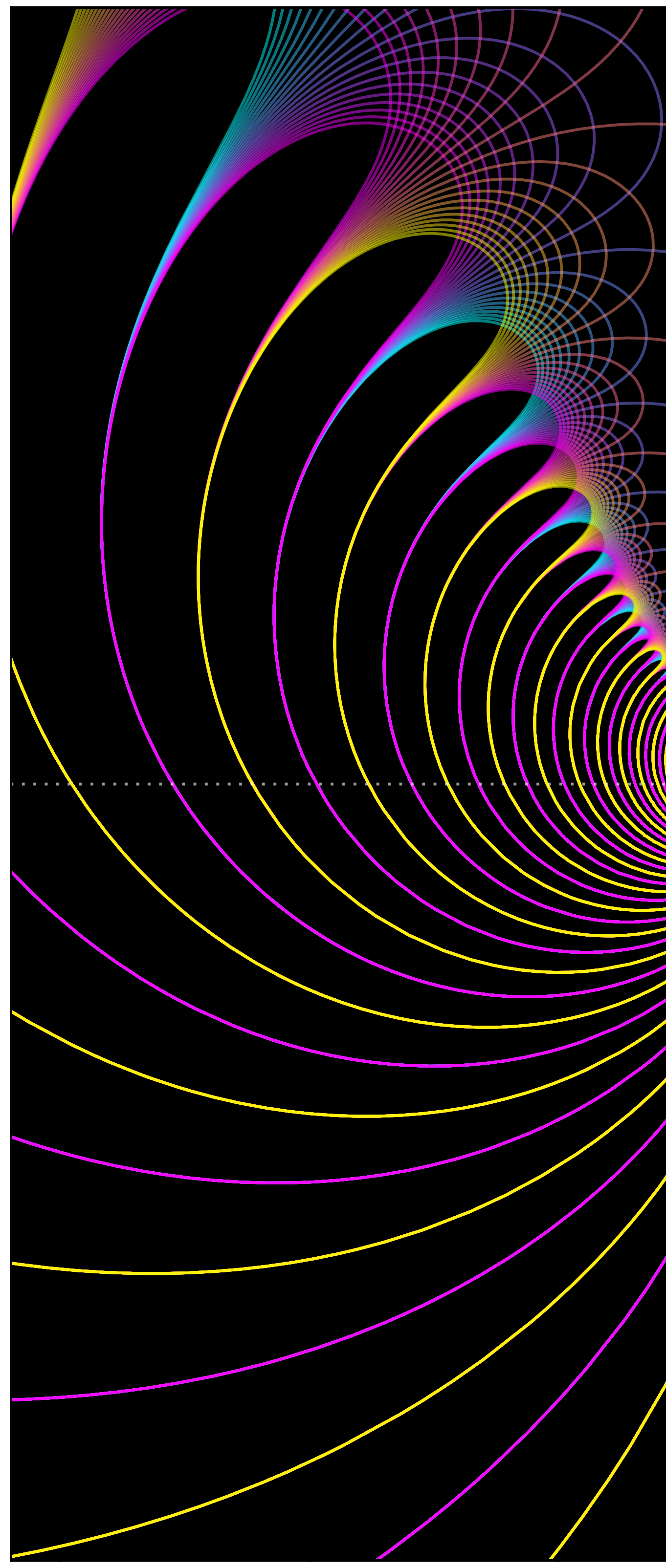

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