Action-Angle formalism for extreme mass ratio inspirals in Kerr spacetime

Lukáš Polcar

In collaboration with M. Kerachian, V. Skoupý and G. Loukes-Gerakopoulos

26th Capra Meeting, 6th July 2023

Motivation and outline

Why action-angle coordinates and canonical perturbation theory ?

1. The two-timescales approach to EMRI is naturally formulated in the action-angle coordinates: orbital (ψ_i) vs. inspiral timescale (J_i)

$$\frac{\mathrm{d}\psi_i}{\mathrm{d}t} = \Omega_i^{(t)}(\mathbf{J}) + qg^{(1)}(\psi, \mathbf{J}) + \mathcal{O}(q^2), \qquad \frac{\mathrm{d}J_i}{\mathrm{d}t} = qG_i^{(1)}(\psi, \mathbf{J}) + q^2 G_i^{(2)}(\psi, \mathbf{J}) + \mathcal{O}(q^3)$$

2. CPT describes the motion using precomputed analytical formulas
3. CPT can approximate nearly integrable systems by integrable ones (arXiv:2205.08516)

This presentation: Kerr metric

1. Introducing AA coordinates and CPT
2. Transformation of the Kerr Hamiltonian into AA coordinates
3. The approximate solution is analytic without any special functions (like elliptic integrals)
4. The result is illustrated on Teukolsky fluxes and a generic adiabatic Kerr inspiral
5. The limits of the approximation (eccentric orbits)

Action-angle coordinatesin integrable systems

- Assumptions
  1. Integrable system with hamiltonian $H(q_i, p_i)$
  2. Motion in the phase space is bounded.

  $\Rightarrow$ canonical transformation to action-angle coordinates

$$\mathbf{z} = (q_i, p_i) \to (\psi_i, J_i), \quad H(\mathbf{q}, \mathbf{p}) \to H(\mathbf{J})$$

  actions:

$$J_i = \frac{1}{2\pi} \oint p_i \mathrm{d} q_i$$

  angles:

$$\psi_j(t) = \Omega_j t + \psi_j(0), \qquad \Omega_i = \frac{\partial H(J_j)}{\partial J_i}$$

- Solution in original coordinates $z^i(t) = z^i(\psi(t), \mathbf{J})$
- Harmonic oscillator: $H(q,p) = \frac{p^2}{2m} + \frac{1}{2} m\Omega^2 q^2$

  Transformation: $q = \sqrt{\frac{2J}{m\Omega}} \sin(\psi)\,, \quad p = \sqrt{2J\Omega\, m} \cos(\psi)$

$$\Rightarrow \quad H(\psi, J) = \Omega J$$

Lie series

- Class of canonical transformation defined by an arbitrary generating function $\omega(q_i, p_i)$
- z_i phase space coordinates $\Rightarrow \{z_i, z_j\} = \Omega_{ij}$

  Time evolution equation $\frac{\mathrm{d}z_i}{\mathrm{d}t} = \{z_i, H\}$ solution:
  $z_i(t) = z_i + \{z_i, H\}t + \frac{1}{2}\{\{z_i, H\}, H\}t^2 + \ldots = \exp(t\pounds_H)z_i$
  where $\pounds_g f = \{f, g\}$
- Replacement $H \leftrightarrow \omega(q_i, p_i), \;\; t \leftrightarrow \varepsilon$:

$$Z_i = z_i(\varepsilon) = \exp(\varepsilon\pounds_\omega)z_i$$

- Inverse operator: $\Big(\exp(\varepsilon\pounds_\omega)\Big)^{-1} = \exp(-\varepsilon\pounds_\omega)$
- Identity $\{\exp(\pounds_\omega)f, \exp(\pounds_\omega)g\} = \exp(\pounds_\omega)\{f, g\}$

$$\Rightarrow \{z_i, z_j\} = \{Z_i, Z_j\}$$

Birkhoff normal form

- Assume we have $H^{(0)} = H_0(J_i) + \sum_{i=1} \varepsilon^i H_i^{(0)}(\psi_i, J_i)$
- First start with $\varepsilon H_1^{(0)}$: $H_1^{(0)} = Z_1(J_i) + h_1(\psi_i, J_i)$

  $\exp(\varepsilon \pounds_{\omega_1}) H^{(0)} = H_0 + \varepsilon Z_1 + \varepsilon\{H_0, \omega_1\} + \varepsilon h_1 + \mathcal{O}(\varepsilon^2)$

  $$\Rightarrow \ \{H_0, \omega_1\} + h_1 \stackrel{!}{=} 0 \text{ homological equation for } \omega_1$$
- $H^{(1)} = \exp(\varepsilon \pounds_{\omega_1}) H^{(0)}$ is independent of angles up to the first order in the perturbation parameter ε
- After r normalization steps we have $H^{(r)} = H_0(J_i) + \sum_{i=1}^{r} \varepsilon^i Z_i(J_i) + R^{(r)}(\psi_i, J_i)$,

  $R^{(r)} = \mathcal{O}(\varepsilon^{r+1})$ is a remainder.
- In total

$$H^{(n)} = \exp(\varepsilon^n \pounds_{\omega_n}) \exp(\varepsilon^{n-1} \pounds_{\omega_{n-1}}) \dots \exp(\varepsilon \pounds_{\omega_1}) H^{(0)} = U(\omega_i) H^{(0)}$$

$$\psi^{(0)} = U(\omega_i)\psi, \ \ J^{(0)} = U(\omega_i) J$$

Kerr Hamiltonian and actions

- Separation of radial and angular parts of the Hamiltonian using the Mino time $\frac{d\tau}{d\lambda} = r^2 + a^2\cos^2\theta$

$$H_{Kerr} = \frac{1}{2}\left(\Delta p_r^2 - \frac{((r^2+a^2)\,p_t + aL_z)^2}{\Delta} + \mu^2 r^2\right) + \frac{1}{2}\left(p_\theta^2 + a^2\mu^2\cos^2\theta + \frac{(L_z + a\sin^2\theta p_t)^2}{\sin^2\theta}\right).$$

- Transformation to polar-nodal coordinates to eliminate θ dependence for $a = 0$:

$$(\theta, \varphi, p_\theta, p_\phi) \to (u, \nu, p_u, p_\nu)$$

$$H_{pn}(r, u, p_t, p_r, p_u, p_\nu) = \frac{1}{2}\left(\Delta\, p_r^2 - \frac{((r^2+a^2)\,p_t + ap_\nu)^2}{\Delta} + \mu^2 r^2\right) + \frac{1}{2}\left((ap_t + p_\nu)^2 + Q\right)$$

$$Q(u, p_t, p_u, , p_\nu) = \left(1 - \frac{p_\nu^2}{p_u^2}\right)\left(p_u^2 + a^2(\mu^2 - p_t^2)\sin^2 u\right)$$

- Expressing the separated momentum components $p_r = \pm\frac{\sqrt{V_r}}{\Delta}$, $p_u = \sqrt{\frac{V_u}{2}}$, $p_\nu = L_z$.
- Kerr actions (additionally $J_t = -E$)

$$J_r = \frac{1}{2\pi}\oint \frac{\sqrt{V_r}}{\Delta}dr \quad J_u = \frac{1}{2\pi}\oint\sqrt{\frac{V_u}{2}}du \quad J_\nu = \frac{1}{2\pi}\oint p_\nu\, d\nu = L_z$$

- The actions are related to the orbital parameters $(p, e, x) \longleftrightarrow (J_r, J_u, J_\nu)$
- Separated Hamiltonian $H = H_{rad}(p_t, r, pr, L_z) + \frac{1}{2}Q$, $\qquad Q = Q(p_t, u, p_u, L_z)$

Implementing the transformations

Radial part

1. Expansion from stable fixed point (spherical orbit)
   $r = r_c + \epsilon\, d\hat{r}, \quad p_r = \epsilon\, \hat{p}_r$
   $L_z = L_{zc} + \epsilon^2 J^0_\nu, \quad p_t = p_{tc} + \epsilon^2 J^0_t, \quad Q = Q_c + \epsilon^2 \tilde{Q}$
2. harmonic oscillator in the leading order
   $r^0 = r^0(\psi^0_r, J^0_r), \quad p^0_r = p^0_r(\psi^0_r, J^0_r),$
   $H^{(0)} = \left(\Omega_{t0} J^0_t + \frac{1}{2}\tilde{Q} + \Omega_{z0} J^0_\nu + \Omega_{r0} J^0_r\right) + \mathcal{O}(\epsilon^3),$
3. Apply 10 canonical transformation to eliminate dependence on ψ_r (Q treated as constant)
   $$H^{(10)} = \prod_{i=0}^{10} \exp(\mathcal{L}_{\chi^{rad}_{10-i}}) H^{(0)} = \hat{U}_{rad} H^{(0)} =$$
   $$= Z^{(10)}_{rad}(\mathbf{J}^{(10)}) + \mathcal{O}(\varepsilon^{13})$$
4. Obtain the transformation relation for radial coordinate
   $r(\psi^{(10)}_r, \mathbf{J}^{(10)}) = \hat{U}_{rad} r(\psi^0_r, J^0_r)$

Angular part

1. Expansion from the Schwarzschild angular part
   $p_u = p_{uc} + \epsilon^2 J^0_u, \quad L_z = \tilde{L}_{zc} + \epsilon^2 \tilde{J}^0_\nu$
   $p_t = \tilde{p}_{tc} + \epsilon^2 \tilde{J}^0_t, \quad a = \epsilon^2 \tilde{a}.$
2. The leading order is already independent of angles
   $Q^{(0)} = (p^2_{uc} - \tilde{L}^2_{zc}) + 2(p_{uc} J^0_u + \tilde{L}_{zc} \tilde{J}^0_\nu)\,\epsilon^2 + \mathcal{O}(\epsilon^3)$
3. Transform to new angles: $\psi_\theta = u, \quad \psi_\phi = u + \nu.$
4. Apply 7 canonical transformation to eliminate dependence on ψ_θ
   $$Q^{(7)} = \prod_{i=0}^{7} \exp(\mathcal{L}_{\chi^{ang}_{7-i}}) Q^{(0)} = \hat{U}_{ang} Q^{(0)} =$$
   $$= Z^{(7)}_{ang}(\mathbf{J}^{(7)}) + \mathcal{O}(\varepsilon^{10})$$
5. Obtain the transformation relation for the polar coordinate $\theta(\psi^{(7)}_\theta, \mathbf{J}^{(7)}) = \hat{U}_{ang} \theta(\psi^0_\theta, \mathbf{J}^{(0)})$

$t - \phi$ coordinates

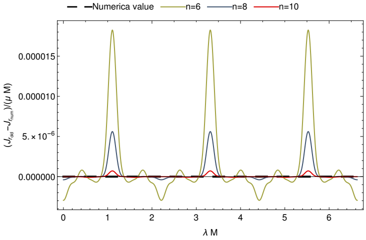
- $t = \psi^{(0)}_t \quad \phi = \psi^{(0)}_\phi + f\left(\psi^{(0)}_\theta, J^{(0)}_\theta, J^{(0)}_\phi\right)$
- $t(\psi_t, \psi_r, \psi_\theta, \mathbf{J}) = \hat{U}_{rad} \hat{U}_{ang} t(\psi^{(0)}_t) = \psi_t + \Delta t_r(\psi_r, \mathbf{J}) + \Delta t_\theta(\psi_\theta, \mathbf{J})$
- $\phi(\psi_\phi, \psi_r, \psi_\theta, \mathbf{J}) = \hat{U}_{rad} \hat{U}_{ang} \phi(\psi^{(0)}_\phi, \psi^{(0)}_\theta, \mathbf{J}^{(0)}) = \psi_\phi + \Delta\phi_r(\psi_r, \mathbf{J}) + \Delta\phi_\theta(\psi_\theta, \mathbf{J})$

Convergence of actions ?

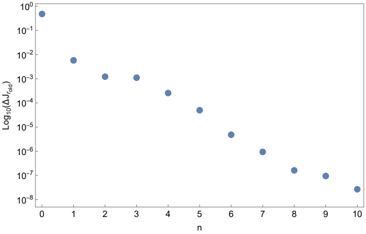
- Using the inverse transformation and the exact solution $z^i(t)$:

$$z^i(\lambda) \to J_i^{(n)}(z^i(\lambda))$$

- expected behaviour $J_i^{(n)}(\lambda) \to J_i$ for $n \to \infty$



(a) $(J_r^{(n)}(\lambda) - J_r)$



(b) $\max_\lambda (J_r^{(n)} - J_r)/J_r$

How accurate are the approximate geodesics ?

- The angular part remains accurate even for highly eccentric black hole and for $a \to 1$ ($\Delta J_\theta, \Delta\Upsilon_\theta, \Delta\Upsilon_\phi \lesssim 10^{-10}$ after 7 transformations)
- The transformation of radial Hamiltonian relies on closeness to stable fixed point, the remainder satisfies $R^{(N)}(\mathbf{z}) \leq C^{(N)} \|\mathbf{z}\|^{N+3}$, for large eccentricities ($e > 0.5$), the normalization procedure is divergent
- Table for $p = 30M, \quad \iota_0 = \frac{\pi}{3}$

a	e	$\mathcal{O}(\Delta J_{r_{\text{old}}})$	$\mathcal{O}(\delta J_{r_{\text{old}}})$	$\mathcal{O}(\delta\Upsilon_r)$
0.1	0.1	10^{-11}	10^{-11}	10^{-10}
	0.2	10^{-8}	10^{-8}	10^{-7}
	0.3	10^{-6}	10^{-6}	10^{-6}
	0.4	10^{-5}	10^{-5}	10^{-5}
	0.5	10^{-3}	10^{-3}	10^{-4}
0.3	0.1	10^{-7}	10^{-7}	10^{-10}
	0.2	10^{-8}	10^{-8}	10^{-8}
	0.3	10^{-6}	10^{-6}	10^{-6}
	0.4	10^{-5}	10^{-5}	10^{-6}
	0.5	10^{-3}	10^{-3}	10^{-4}
0.5	0.1	10^{-7}	10^{-7}	10^{-10}
	0.2	10^{-7}	10^{-8}	10^{-8}
	0.3	10^{-6}	10^{-7}	10^{-6}
	0.4	10^{-5}	10^{-5}	10^{-5}
	0.5	10^{-3}	10^{-3}	10^{-4}
0.7	0.1	10^{-7}	10^{-7}	10^{-9}
	0.2	10^{-6}	10^{-6}	10^{-7}
	0.3	10^{-7}	10^{-7}	10^{-7}
	0.4	10^{-5}	10^{-5}	10^{-6}
	0.5	10^{-3}	10^{-3}	10^{-4}
0.99	0.1	10^{-5}	10^{-5}	10^{-8}
	0.2	10^{-6}	10^{-6}	10^{-7}
	0.3	10^{-5}	10^{-5}	10^{-6}
	0.4	10^{-4}	10^{-4}	10^{-5}
	0.5	10^{-3}	10^{-3}	10^{-4}

Teukolsky amplitudes and fluxes

- Weyl scalar ψ_4 mode decomposition $\psi_4 = (r - \mathrm{i}a\cos\theta)^{-4} \int_{-\infty}^{\infty} \sum_{l,m} R_{lm\omega}(r) S_{lm}(\theta, \phi, a\omega) e^{-\mathrm{i}\omega t} \mathrm{d}\omega$
- Radial Teukolsky equation $\Delta^2 \frac{\mathrm{d}}{\mathrm{d}r}\left(\Delta^{-1}\frac{\mathrm{d}R_{lm\omega}}{\mathrm{d}r}\right) - V_{lm\omega}(r) R_{lm\omega} = -\mathcal{T}_{lm}$
- The general solution

  $$R_{lm\omega}(r) = C^{+}_{lm\omega}(r) R^{-}_{lm\omega}(r) + C^{-}_{lm\omega}(r) R^{+}_{lm\omega}(r),$$
- The Teukolsky amplitudes

  $$C^{(\pm)}_{lm\omega}(r) = \int_{r_+}^{\infty} \Theta(\pm(r - r')) \frac{R^{(\pm)}_{lm\omega}(r')}{W(r')\Delta(r')} \mathcal{T}_{lm}(r') \mathrm{d}r' \Rightarrow C^{\pm}_{lm}(r, \omega) = \sum_{nk} C^{\pm}_{nkml}(r) \delta(\omega - \omega_{nkm})$$
- The total fluxes of the three integrals of motion

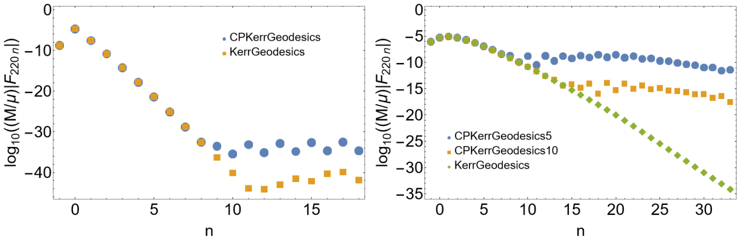
  $$F^{E} = \sum_{nklm} \frac{|C^{+}_{nkml}|^2 + \alpha_{nklm}|C^{-}_{nkml}|^2}{4\pi\omega^2_{nkm}}$$

  $$F^{L_z} = \sum_{nklm} m \frac{|C^{+}_{nkml}|^2 + \alpha_{nklm}|C^{-}_{nkml}|^2}{4\pi\omega^3_{nkm}}.$$

  $$F^{Q} = \sum_{nklm} \frac{|C^{+}_{nkml}|^2 + \alpha_{nklm}|C^{-}_{nkml}|^2}{2\pi\omega^3_{nkm}} (\mathcal{L}_{nkm} + k\Upsilon_\theta)$$

Calculating the fluxes

- (F^E, F^{L_z}, F^Q) sourced by our approximate geodesics ?
- Summing the partial fluxes $F = \sum_{l,m,k,n} F_{lmkn}$
- Checking the higher n-modes F_{222n}



(a) quasi-circular orbit

(b) $e = 0.3$

-

Relative error of fluxes

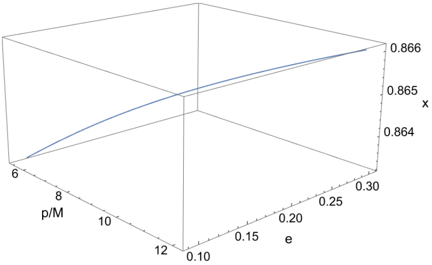
- Another scary table: relative error of fluxes caused by our approximation to the geodesic motion ($a = 0.9$, $p = 6M$).
- Breakdown of the approximation for $e > 0.5$

e	ι_0	$\delta\left\langle\frac{\mathrm{d}E^\infty}{\mathrm{d}t}\right\rangle = \delta\dot{E}^\infty$	$\delta\dot{E}^H$	$\delta\dot{L}_z^\infty$	$\delta\dot{L}_z^H$	$\delta\dot{Q}^\infty$	$\delta\dot{Q}^H$
0.1	20°	4.27×10^{-7}	1.12×10^{-8}	3.88×10^{-7}	4.45×10^{-8}	5.05×10^{-8}	2.18×10^{-7}
	40°	6.17×10^{-7}	1.10×10^{-7}	5.39×10^{-7}	1.51×10^{-7}	4.01×10^{-7}	1.88×10^{-7}
	60°	6.90×10^{-8}	2.99×10^{-8}	8.35×10^{-8}	2.61×10^{-8}	6.77×10^{-8}	1.78×10^{-7}
	80°	5.54×10^{-7}	2.88×10^{-7}	4.82×10^{-7}	2.56×10^{-7}	3.37×10^{-7}	1.02×10^{-6}
0.3	20°	1.84×10^{-4}	9.51×10^{-5}	1.85×10^{-4}	5.41×10^{-6}	1.57×10^{-4}	1.84×10^{-4}
	40°	1.45×10^{-4}	1.78×10^{-4}	1.51×10^{-4}	7.07×10^{-4}	1.22×10^{-4}	1.54×10^{-4}
	60°	1.13×10^{-4}	2.95×10^{-4}	1.28×10^{-4}	3.41×10^{-4}	8.98×10^{-5}	1.49×10^{-4}
	80°	1.60×10^{-3}	4.32×10^{-1}	1.39×10^{-3}	1.02×10^{-3}	1.15×10^{-3}	6.19×10^{-4}
0.5	20°	8.17×10^{-3}	7.09×10^{-2}	7.06×10^{-3}	1.29×10^{-2}	6.66×10^{-3}	9.77×10^{-3}
	40°	1.01×10^{-2}	9.61×10^{-2}	8.30×10^{-3}	1.25×10^{-2}	8.25×10^{-3}	9.50×10^{-3}
	60°	1.83×10^{-2}	1.22×10^{-1}	1.56×10^{-2}	1.94×10^{-2}	1.43×10^{-2}	1.14×10^{-2}
	80°	1.77×10^{0}	1.40×10^{-2}	1.56×10^{0}	2.19×10^{-1}	1.28×10^{0}	3.84×10^{0}

Adiabatic inspiral

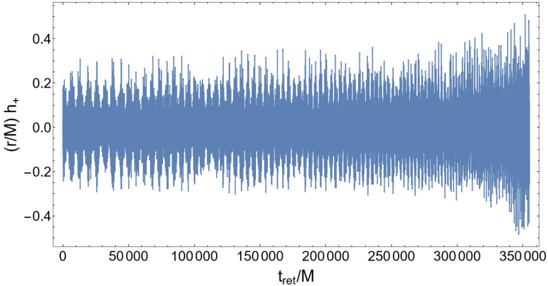
- Evolution of orbital parameters $(E, L_z, Q) \Rightarrow (p, e, x) = \mathbf{I}$ $\qquad \frac{\mathrm{d}I_i(t)}{\mathrm{d}t} = q\langle F_i^{(I)}\rangle(\mathbf{I}(\mathbf{t}))$
- A generic adiabatic inspiral for $a = 0.5$:

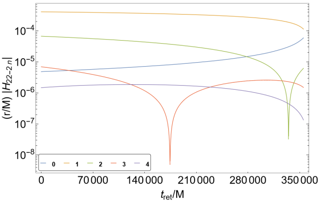
(p, e, x): $(12M, 0.3, 0.8660) \rightarrow (6.084M, 0.106, 0.86357)$

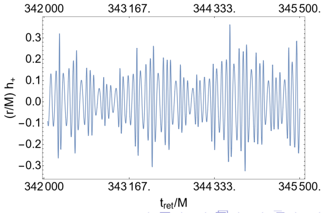


Waveforms for illustrations

- $$h_+ - \mathrm{i}h_\times = -\frac{2}{r}\sum_{lmkn}\frac{C^+_{lmkn}(t)}{\omega^2_{mkn}(t)}\,{}_{-2}S_{lm}(\theta, a\omega_{mkn}(t))e^{-i\Phi_{mkn}(t)+im\phi} \qquad \Phi_{mkn}(t) = \int_{t_0}^{t}\omega_{mkn}(t')\mathrm{d}t'$$







Summary and future work

1. We have managed to transform the Kerr Hamiltonian to action-angle coordinates in sufficient accuracy. The geodesic approximation is reliable only for eccentricities sufficiently small.
2. We calculated the Teukolsky fluxes on a grid in (p, e, x) space $(a = 0.5)$, the fluxes are accurate enough for $e < 0.3$.
3. We adiabatically evolved a generic Kerr inspiral and extracted the waveform as an illustration.

Possible future work

To repeat this procedure with Kerr or Schwwarzschild solution perturbed by additional matter.

M. Kerachian, L. Polcar, V. Skoupý, C. Efthymiopoulos, and G. Lukes-Gerakopoulos, "Action-Angle formalism for extreme mass ratio inspirals in Kerr spacetime," 2023.