Waveforms from plunges into a Schwarzschild black hole

Lorenzo Küchler

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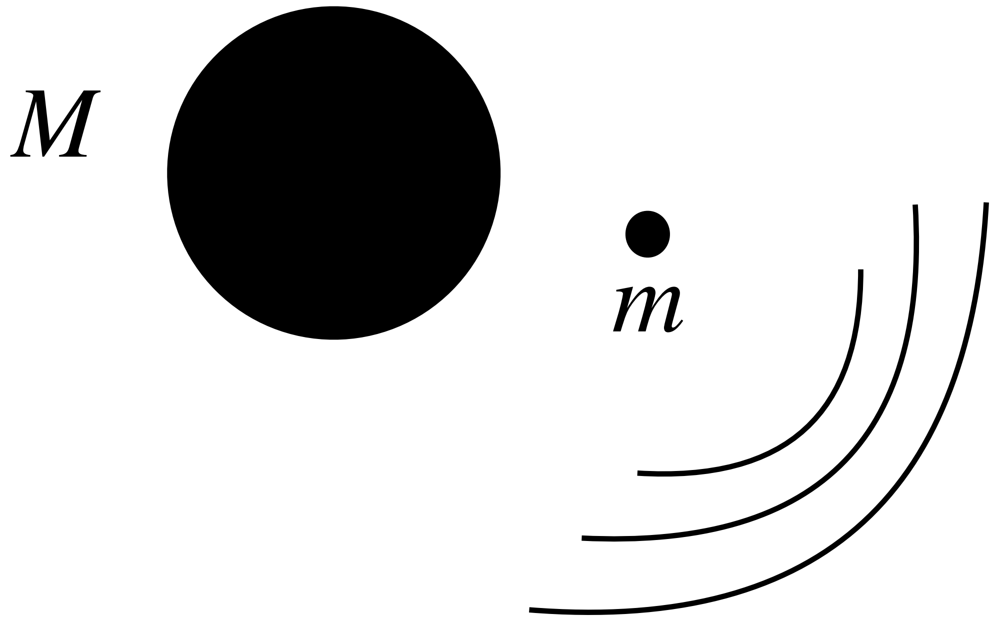
KU LEUVEN

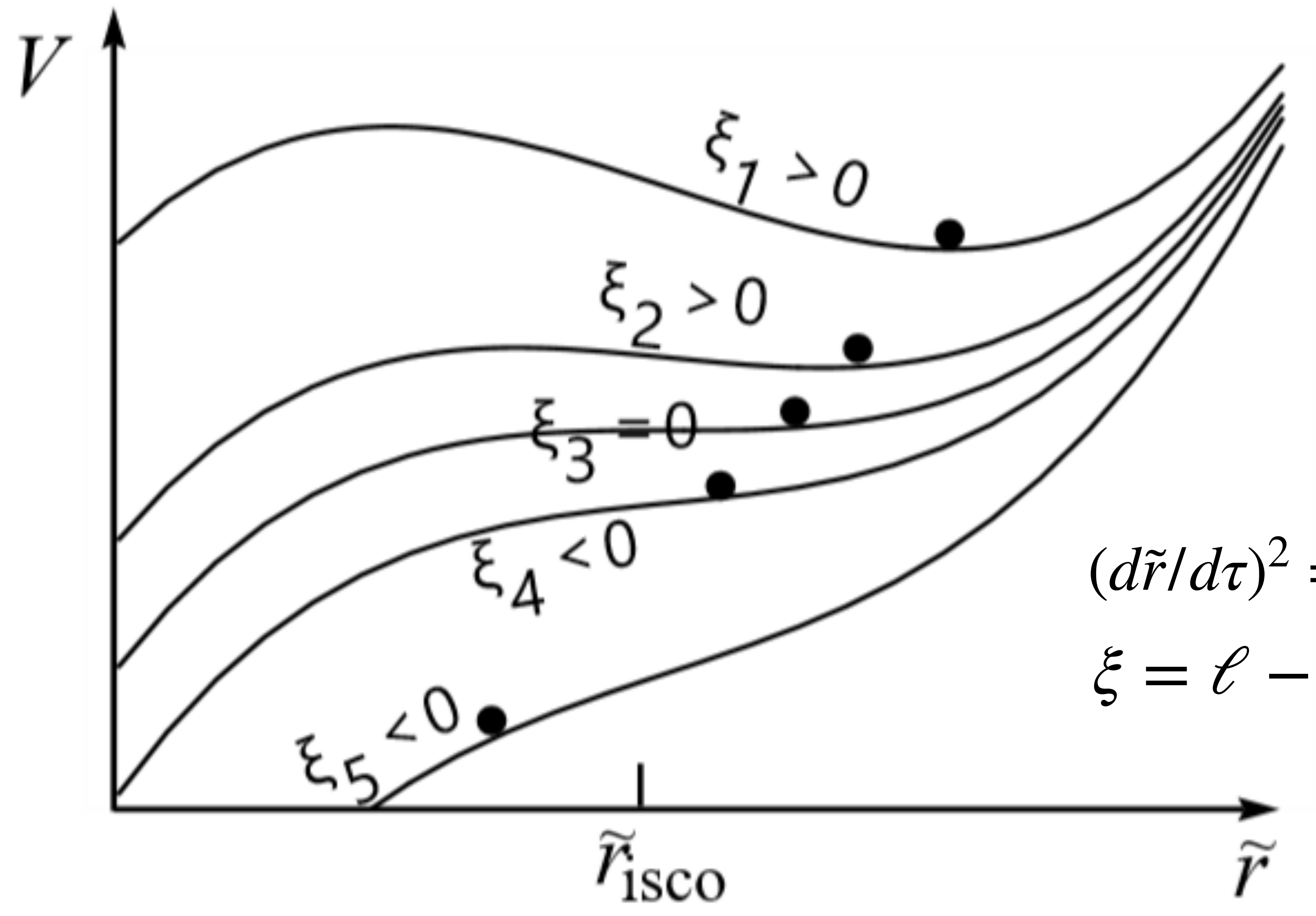
Quasi-circular and equatorial motion

- Schwarzschild background
- Neglect structure and spin of the secondary object
- Inspiral on quasi-circular and equatorial orbits



- 3 stages:

1. Inspiral (timescale ~ ε^{-1})
2. Transition-to-plunge (timescale ~ $\varepsilon^{-1/5}$)
3. Plunge (timescale ~ ε^{0})



$(d\tilde{r}/d\tau)^2 = e^2 - V(\tilde{r}, e, \ell)$

$\xi = \ell - \ell_{\text{isco}}$

[Ori & Thorne (2000)]

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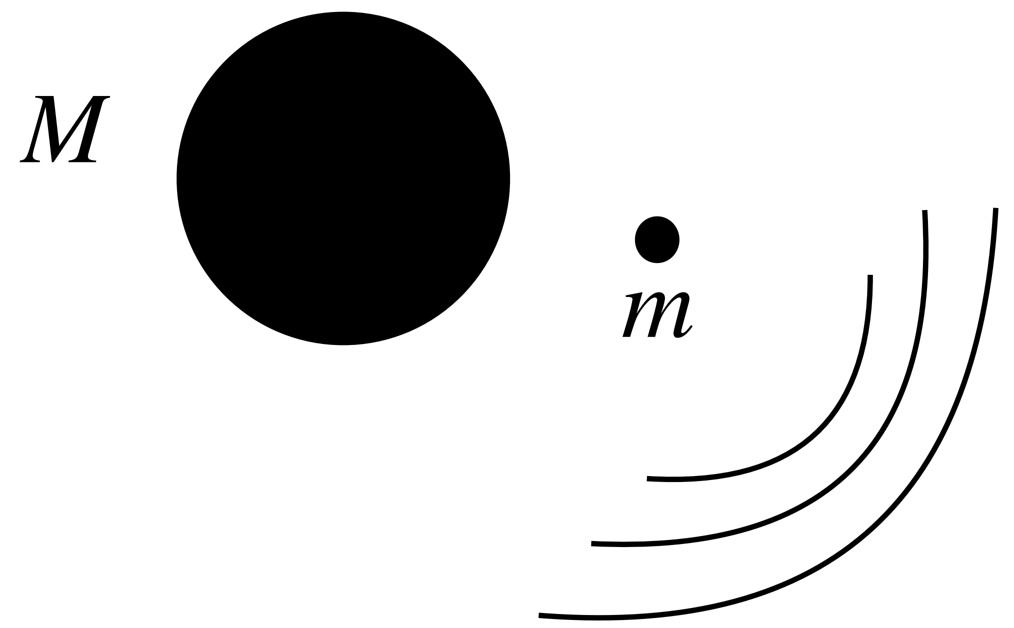

- 3 stages:

1. $\boxed{\text{Inspiral (timescale} \sim \varepsilon^{-1}\text{)}}$
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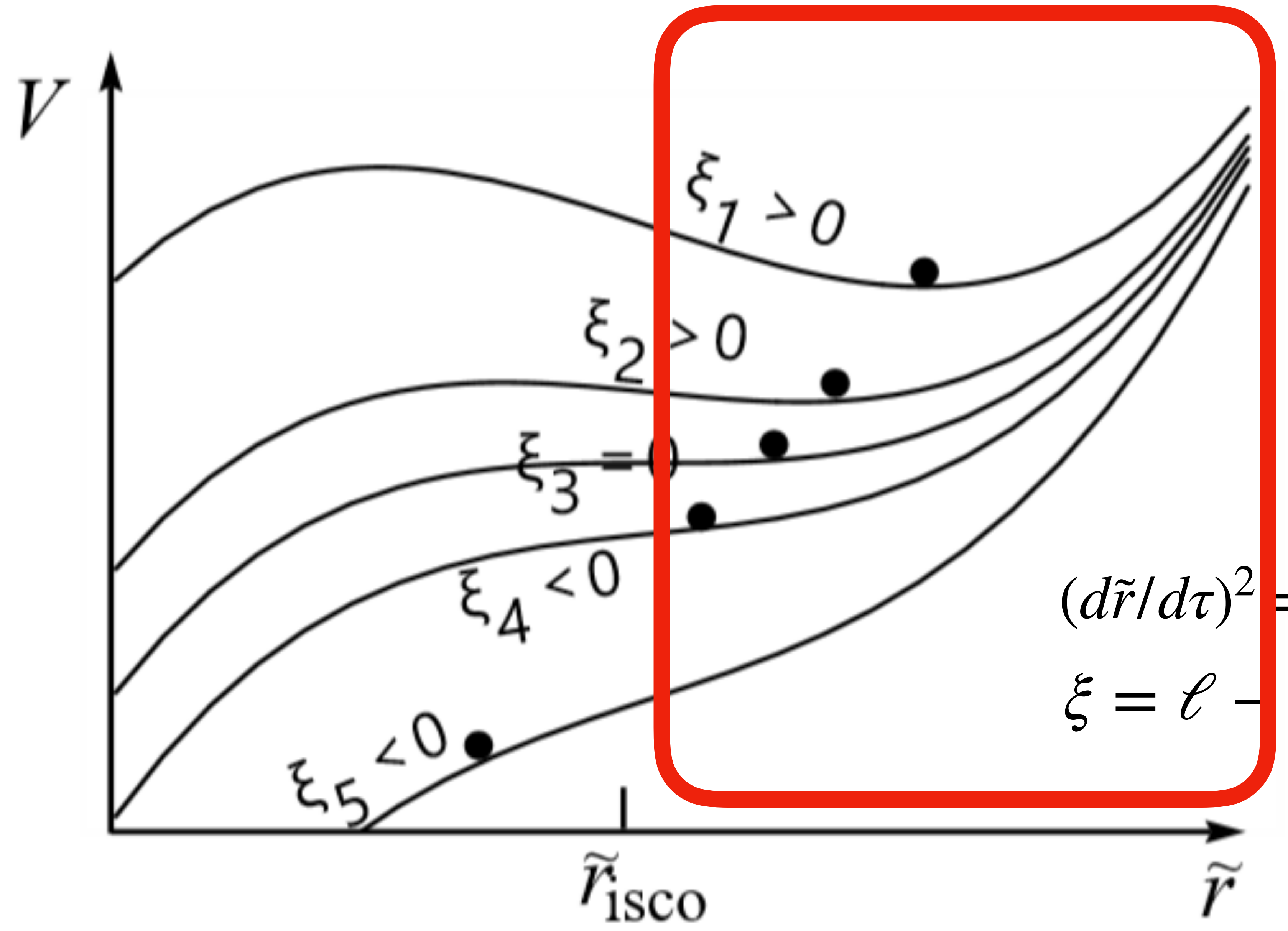

$(d\tilde{r}/d\tau)^2 = e^2 - V(\tilde{r}, e, \ell)$

$\xi = \ell - \ell_{\rm isco}$

[Ori & Thorne (2000)]

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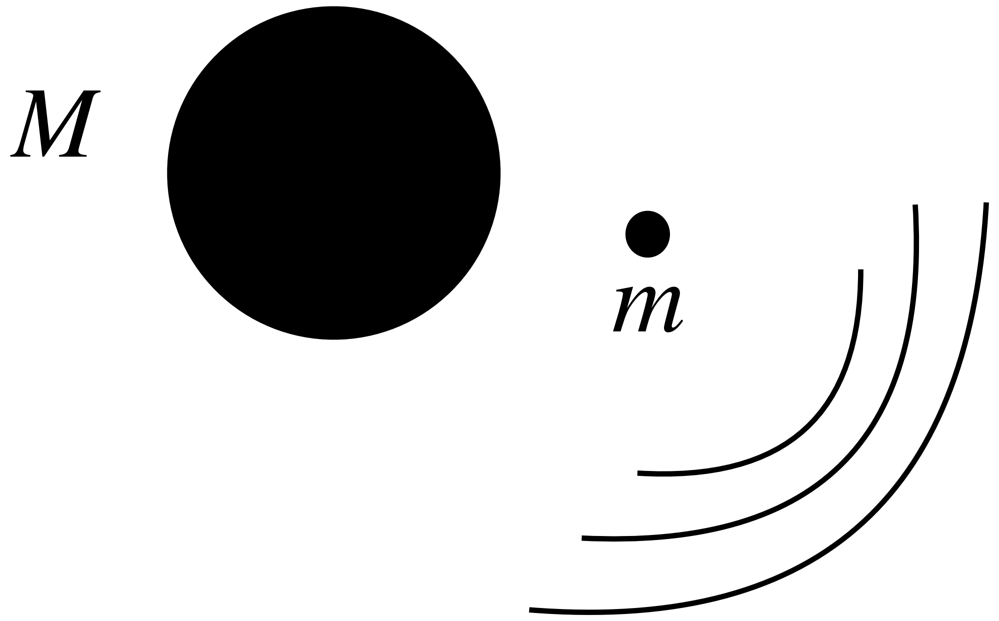

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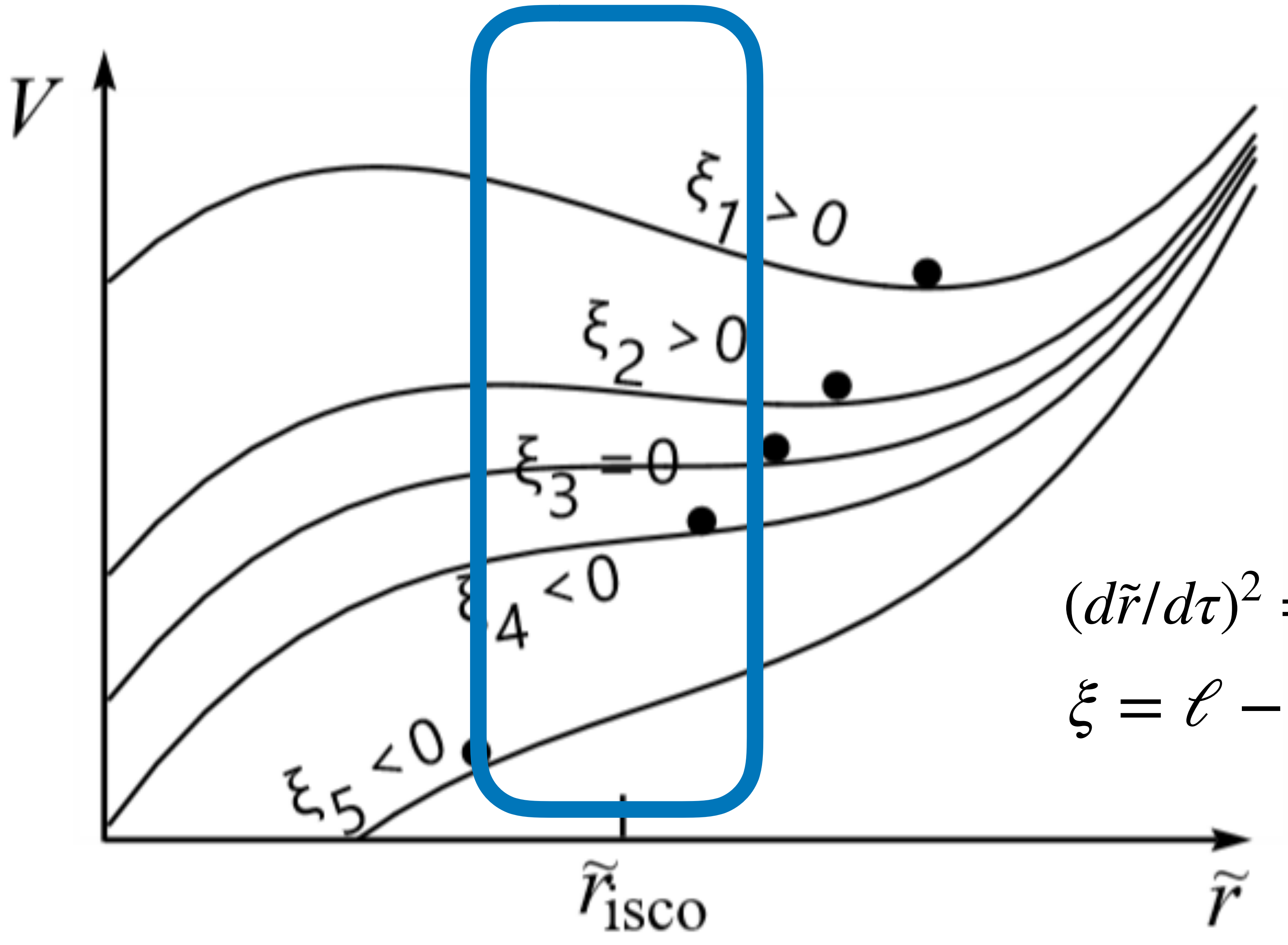

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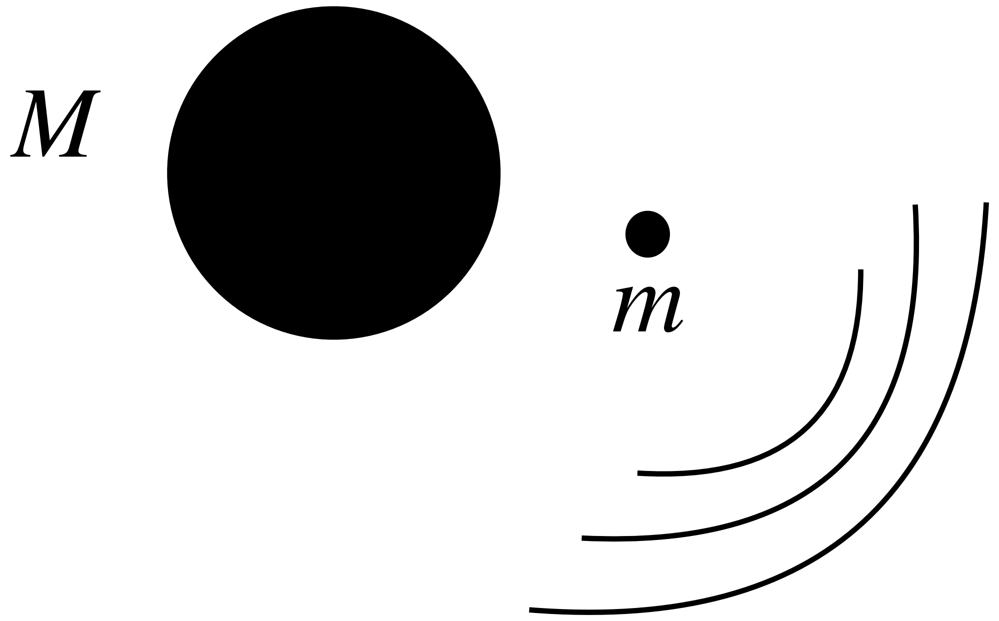

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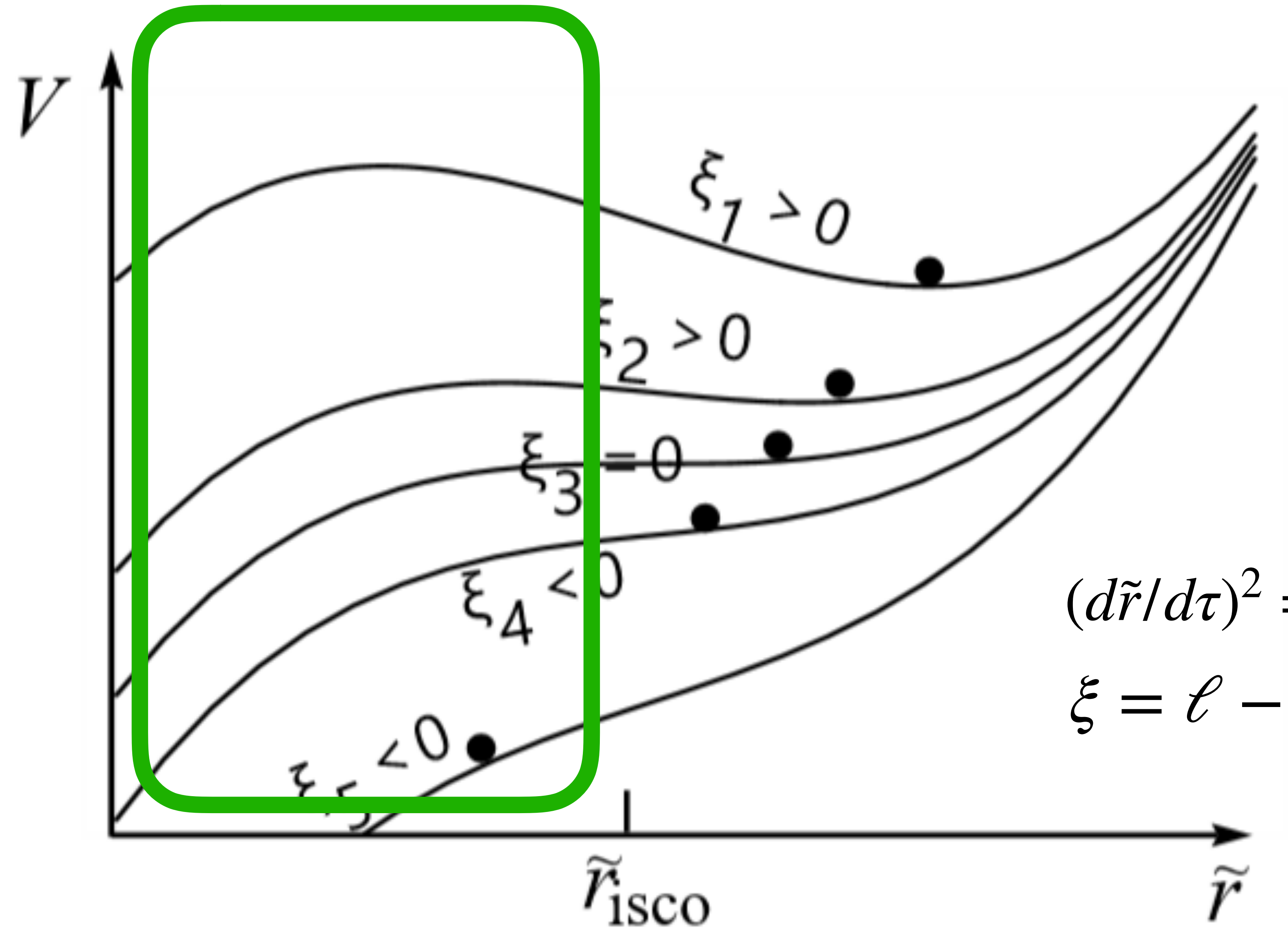


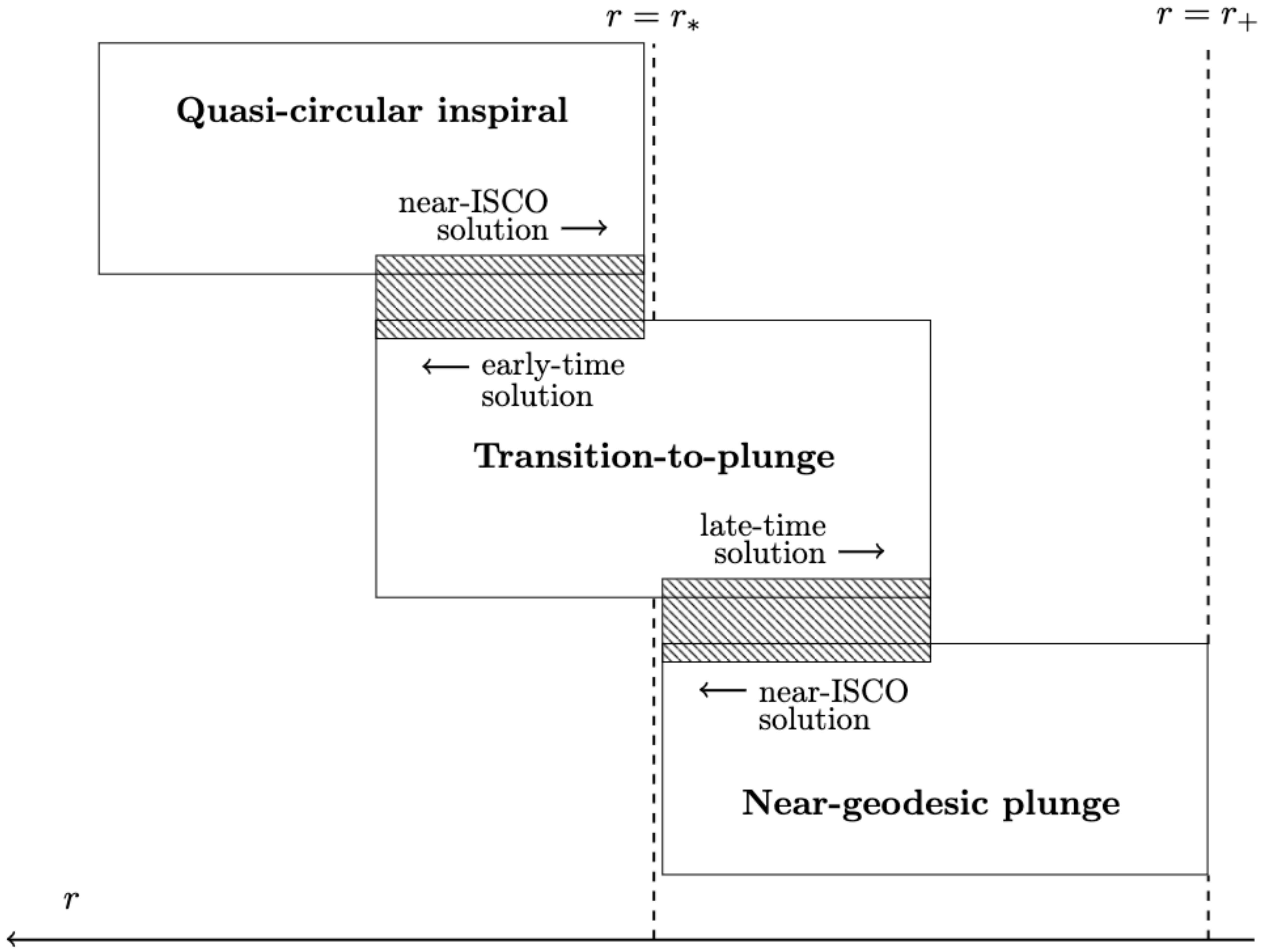
$$(d\tilde{r}/d\tau)^2 = e^2 - V(\tilde{r}, e, \ell)$$

$$\xi = \ell - \ell_{\text{isco}}$$

[Ori & Thorne (2000)]

Matched asymptotic expansions

- Modelling the motion of the secondary in each region
- Obtain composite solutions that smoothly connect one region to the other using matched asymptotic expansions
- Waveforms beyond the ISCO



Quasi-circular inspiral

- Two disparate timescales: orbital period ~ ε^0 and radiation-reaction timescale ~ ε^{-1}

- We perform a slow-timescale expansion at fixed Ω [Hinderer & Flanagan (2008), Miller & Pound (2021)]

$$X(\Omega) = X_{(0)}(\Omega) + \varepsilon\, X_{(1)}(\Omega) + O(\varepsilon^2) \qquad\qquad f^\mu = \varepsilon f^\mu_{(1)} + \varepsilon^2 f^\mu_{(2)} + O(\varepsilon^3)$$

- We introduce the slow time $\tilde{t} = \varepsilon\, t$. The main ingredients for waveform generation are:

Orbital motion

$$\frac{d\Omega}{d\tilde{t}} = F^\Omega_{(0)}(\Omega) + \varepsilon\, F^\Omega_{(1)}(\Omega) + O(\varepsilon^2)$$

$$\frac{d\phi_p}{dt} = \Omega$$

Field equations

$$h_{\mu\nu} = \sum_{n=1}^{\infty} \varepsilon^n \sum_{\ell m} R^{(n)}_{\ell m}(r,\Omega) e^{-im\phi_p} Y^{\ell m}_{\mu\nu}$$

$$(\partial_t)_r = \left(\Omega \frac{\partial}{\partial \phi_p} + \varepsilon\, \frac{d\Omega}{d\tilde{t}} \frac{\partial}{\partial \Omega} \right)$$

Transition-to-plunge

- Near-ISCO scaling [Ori & Thorne (2000)]: $\Omega - \Omega_* \sim \varepsilon^{2/5}$, $t - t_* \sim \varepsilon^{-1/5}$. We introduce the rescaled transition time

$$\lambda \equiv \varepsilon^{1/5} \qquad \hat{t} = \lambda(t - t_*) \qquad \Omega = \Omega_* + \lambda^2 \Delta\Omega$$

- Two disparate timescales: orbital period $\sim \varepsilon^0$ and ISCO-crossing timescale $\sim \varepsilon^{-1/5}$

$$X(\Delta\Omega) = X_* + \lambda^\eta \left(X_{[0]}(\Delta\Omega) + \lambda\, X_{[1]}(\Delta\Omega) + \lambda^2 X_{[2]}(\Delta\Omega) + O(\lambda^3) \right) \qquad f^\mu = \lambda^5 \sum_{i=0} \lambda^i f^\mu_{[i]}(\Delta\Omega)$$

η depends on the orbital parameter considered

- The main ingredients for waveform generation:

Orbital motion

$$\frac{d\Omega}{d\hat{t}} = \lambda^2 \sum_{i=0} \lambda^i F^{\Delta\Omega}_{[i]}(\Delta\Omega)$$

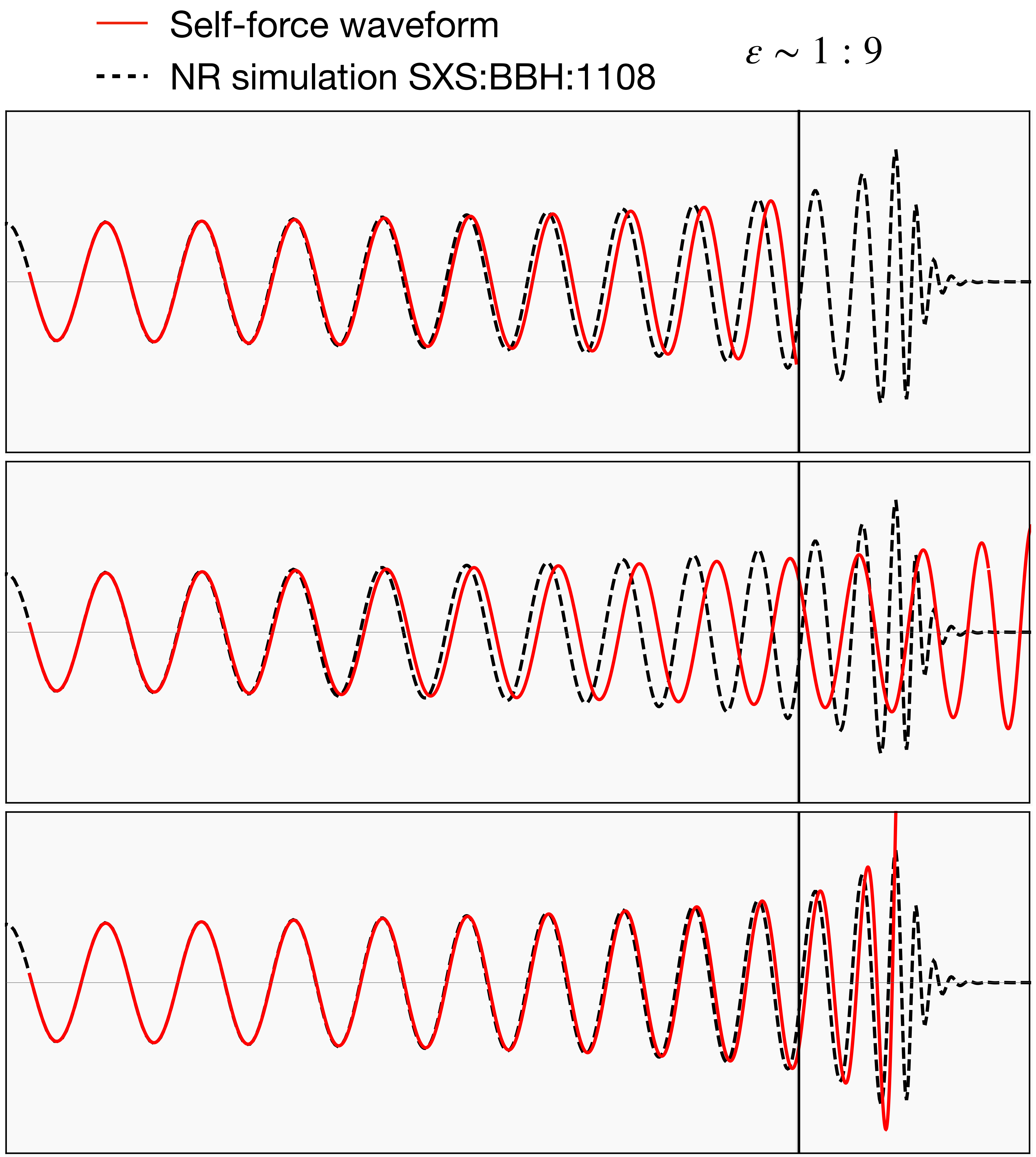
$$\frac{d\phi_p}{dt} = \Omega$$

Field equations

$$h_{\mu\nu} = \sum_{n=0}^{\infty} \lambda^{5+n} \sum_{\ell m} R^{[n]}_{\ell m}(r, \Delta\Omega) e^{-im\phi_p} Y^{\ell m}_{\mu\nu}$$

$$(\partial_t)_r = \left(\Omega \frac{\partial}{\partial \phi_p} + \lambda^3 \frac{d\Delta\Omega}{d\hat{t}} \frac{\partial}{\partial \Delta\Omega} \right)$$

Inspiral-transition waveforms



0PA

0PA
0PLT

+ 0PLT

0PA
2PLT

+ 2PLT

Plunge

Orbital motion

- The plunge extends from the ISCO up to the event horizon and occurs on the orbital timescale

- Geodesic with E_{isco} and L_{isco}
  + ε-corrections →

$$r_p = r_G(\Omega) + \varepsilon\, r_{\{1\}}(\Omega) + O(\varepsilon^2)$$

$$\frac{d\Omega}{dt} = F^{\Omega}_{G}(\Omega) + \varepsilon\, F^{\Omega}_{\{1\}}(\Omega) + O(\varepsilon^2)$$

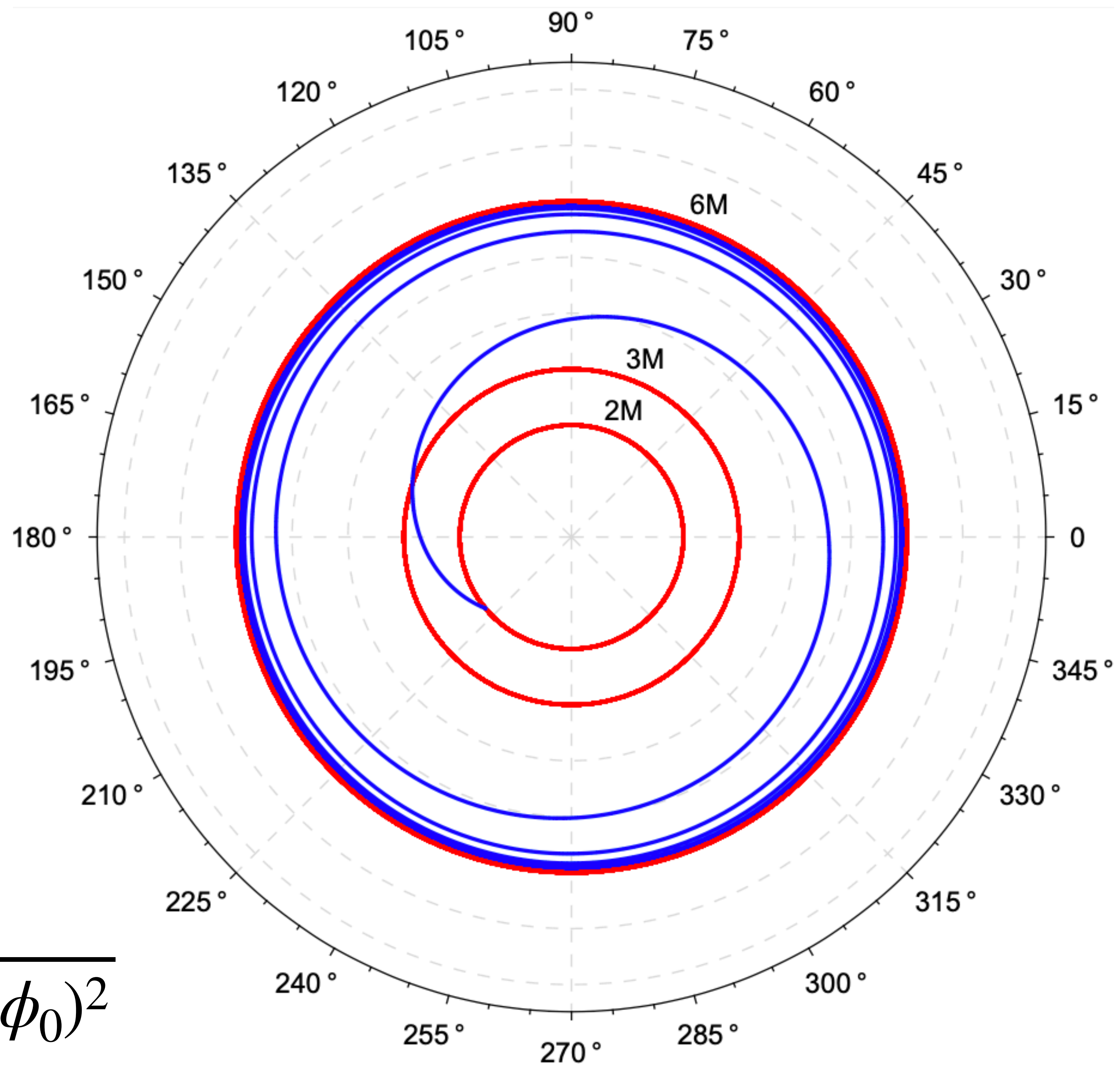
Geodesic order

- We can parametrise the orbit with the orbital radius r_G

$$\Omega(r_G) = \sqrt{\frac{3}{2}\frac{3M}{r_G^2}}f(r_G)$$

$$F^{\Omega}_{G}(r_G) = -\frac{3\sqrt{3}Mf(r_G)(3M/r_G - 1)(6M/r_G - 1)^{3/2}}{2r_G^3}$$

$$r_G(\phi_G) = \frac{6M}{1 + 12/(\phi_G - \phi_0)^2}$$



- Asymptotic match with the transition motion → composite solution to the “right” of the ISCO

Plunge

Field equations

- We consider the RWZ equation with a point-particle source $$\left(\partial_x^2 - \partial_t^2 + V\right)\Psi(t,r) = S(t,r)$$

- We use the following: $$\Psi(t,r) = R(\Omega(t),r)e^{-im\phi_p(t)}$$

$$S(t,r) = \tilde{S}(\Omega(t),r)e^{-im\phi_p(t)} \qquad \tilde{S}(\Omega(t),r) = A(\Omega)\delta(r - r_G(\Omega)) + B(\Omega)\delta'(r - r_G(\Omega))$$

- We introduce the *minimal gauge* hyperboloidal slicing: $s = t/\lambda - k(x), \quad H = d\kappa/dx$

- We work in frequency domain:

$$\hat{R}(\omega,r) = \int_{\Omega_*}^{\Omega_+} \frac{d\Omega}{F_G^\Omega\, dt/ds} R(\Omega,r)e^{i\omega s_G(\Omega) - im\phi_G(\Omega)} \qquad s_G(\Omega) = \int^{\Omega} \frac{d\Omega'}{F_G^\Omega(\Omega')}\frac{ds}{dt}(\Omega')$$

$$R(\Omega,r) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega \hat{R}(\omega,r)e^{-i\omega s_G(\Omega) + im\phi_G(\Omega)} \qquad \phi_G(\Omega) = \int^{\Omega} \frac{\Omega' d\Omega'}{F_G^\Omega(\Omega')}$$

Plunge

- The frequency-domain equation reads

$$\left(\partial_x^2 - V\right)\hat{R} + 2i\omega H \partial_x \hat{R} + i\omega \frac{dH}{dx}\hat{R} + \left(1/\lambda^2 - H^2\right)\omega^2 \hat{R} = \hat{S}(\omega, r),$$

- We want to ensure the match with the late transition: **puncture formulation**

$$R = R^{\mathscr{P}} + R^{\mathscr{R}} \qquad\qquad R^{\mathscr{P}}(\Omega, r) = R^{(1)}(\Omega_*, r)\theta(\Omega_{\mathscr{P}} - \Omega), \qquad \Omega_{\mathscr{P}} > \Omega_*$$

- We are left with the problem

$$\left(\partial_x^2 - V\right)\hat{R}^{\mathscr{R}} + 2i\omega H \partial_x \hat{R}^{\mathscr{R}} + i\omega \frac{dH}{dx}\hat{R}^{\mathscr{R}} + \left(1/\lambda^2 - H^2\right)\omega^2 \hat{R}^{\mathscr{R}} = \hat{S}_{\text{eff}}(\omega, r),$$

- The effective source contains

1) Punctured point-particle source supported on the worldline $S_{\text{eff}}^{\text{pp}} = S(\Omega, r) - S(\Omega_*, r)\theta(\Omega_{\mathscr{P}} - \Omega) \longrightarrow R_{\text{pp}}^{\mathscr{R}}$

2) The effective part containing the extended objects $R^{(1)}(\Omega_*, r) \longrightarrow R_{\text{eff}}^{\mathscr{R}}$

Ben Leather:
frequency-domain
spectral code

Plunge

Late transition amplitude:

$$R_{\rm ttp}(\Omega, r) = R^{(1)}(\Omega_*, r) + \partial_\Omega R^{(1)}(\Omega_*, r)(\Omega - \Omega_*) + \ldots$$

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Ben Leather:
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- The frequency-domain equation reads

$$\left(\partial_x^2 - V\right)\hat{R} + 2i\omega H\partial_x \hat{R} + i\omega\frac{dH}{dx}\hat{R} + \left(1/\lambda^2 - H^2\right)\omega^2\hat{R} = \hat{S}(\omega, r),$$

- We want to ensure the match with the late transition: **puncture formulation**

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- We are left with the problem

$$\left(\partial_x^2 - V\right)\hat{R}^{\mathscr{R}} + 2i\omega H\partial_x \hat{R}^{\mathscr{R}} + i\omega\frac{dH}{dx}\hat{R}^{\mathscr{R}} + \left(1/\lambda^2 - H^2\right)\omega^2\hat{R}^{\mathscr{R}} = \hat{S}_{\text{eff}}(\omega, r),$$

- The effective source contains

1) Punctured point-particle source supported on the worldline $S_{\text{eff}}^{\text{pp}} = S(\Omega, r) - S(\Omega_*, r)\theta(\Omega_{\mathscr{P}} - \Omega) \longrightarrow R_{\text{pp}}^{\mathscr{R}}$

2) The effective part containing the extended objects $R^{(1)}(\Omega_*, r) \longrightarrow R_{\text{eff}}^{\mathscr{R}}$

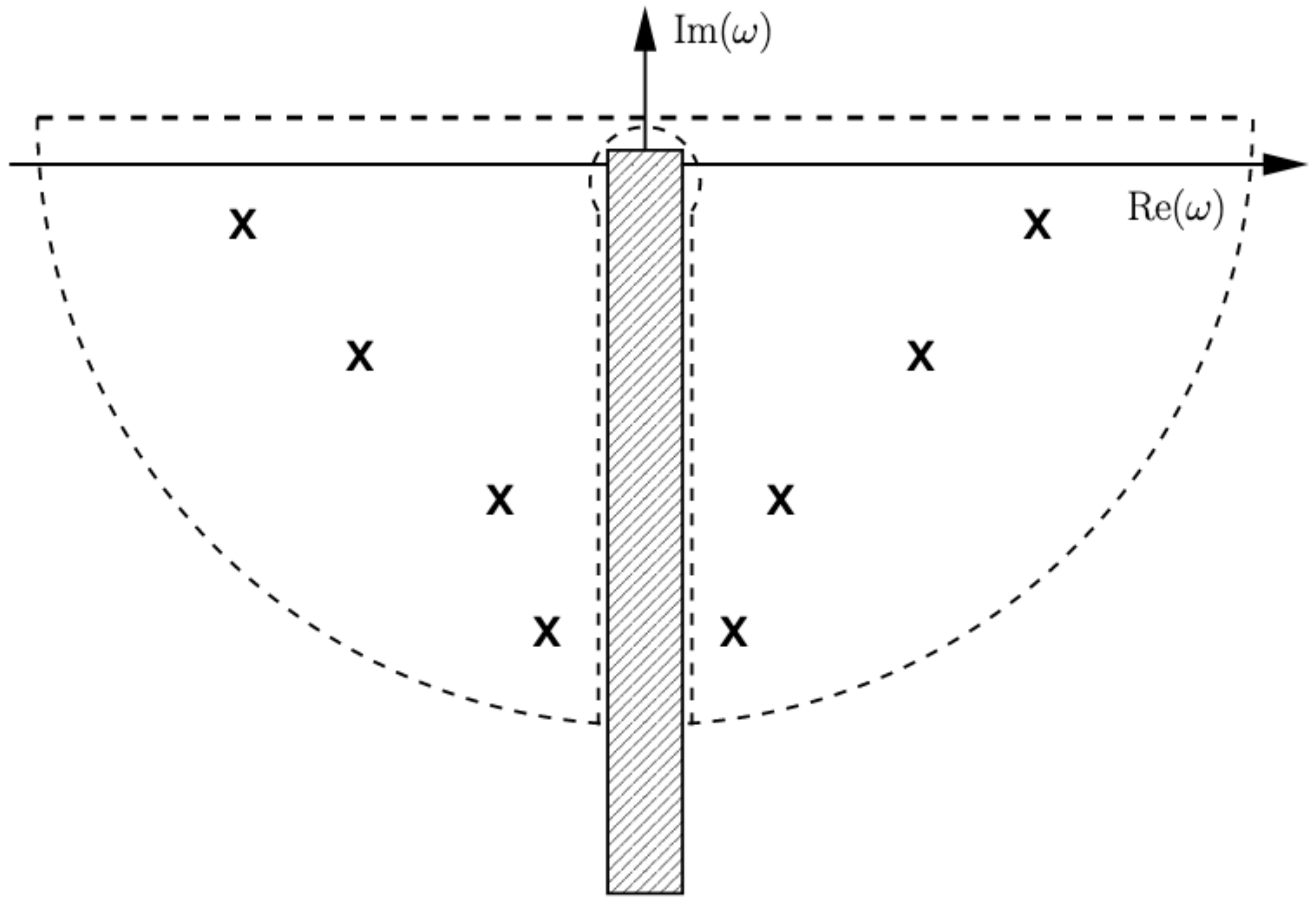
Ben Leather: frequency-domain spectral code

Plunge

- We solve for the point particle term using a Green's function

$$\hat{R}^{\mathscr{R}}_{\text{pp}}(\omega, r \to \infty) = \frac{\hat{R}_{\text{up}}(\omega, r \to \infty)}{W(\omega)} \int_{2M}^{+\infty} \frac{dr'}{f(r')} \hat{R}_{\text{in}}(\omega, r') \hat{S}^{\text{pp}}_{\text{eff}}(\omega, r') = \frac{C(\omega)}{W(\omega)}$$

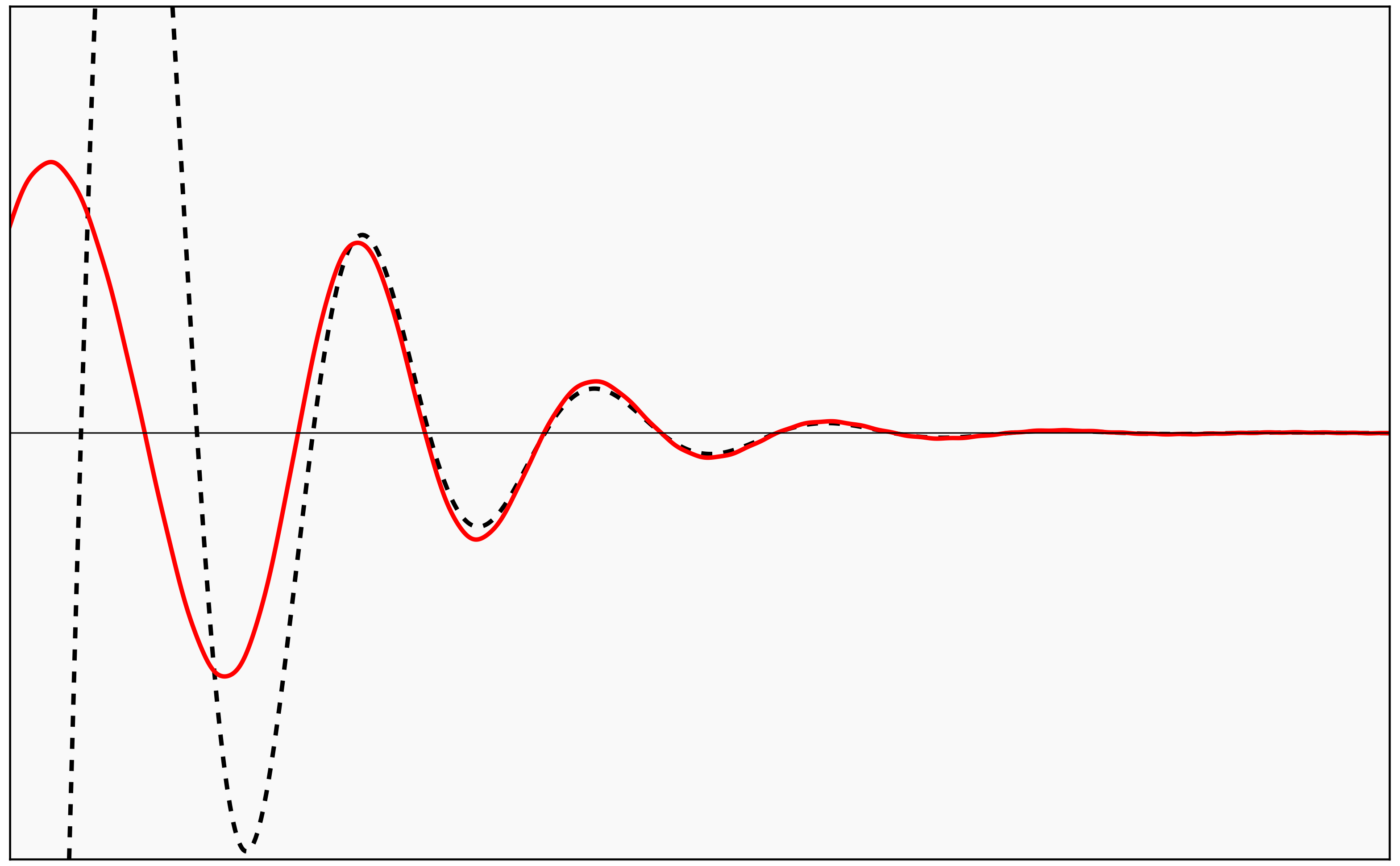
- Taking the inverse transform: $$R^{\mathscr{R}}_{\text{pp}}(\Omega, r) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega e^{-i\omega s_G(\Omega) + im\phi_G(\Omega)} \frac{C(\omega)}{W(\omega)}$$



[Berti, Cardoso and Starinets (2009)]

Contributions

- Quasi-normal modes
- Arcs
- Branch cut



Comparison of $R^{\mathscr{R}}_{\text{pp}} e^{-im\phi_G}$ with QNM waveform of [Folacci & Ould El Hadj (2018)]

- Sum up all contributions: puncture $R^{\mathscr{P}}$, point-particle term $R^{\mathscr{R}}_{\text{pp}}$, effective piece $R^{\mathscr{R}}_{\text{eff}}$

Conclusions

Summary

- Framework for complete (inspiral-transition-plunge) waveforms in Schwarzschild spacetime for quasi-circular equatorial orbits

- Adiabatic waveforms, but general enough to accept post adiabatic terms

- Promising comparison with NR simulations in the regime of nearly comparable masses

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- Framework for complete (inspiral-transition-plunge) waveforms in Schwarzschild spacetime for quasi-circular equatorial orbits
- Adiabatic waveforms, but general enough to accept post adiabatic terms
- Promising comparison with NR simulations in the regime of nearly comparable masses

To do

- Finalise the 0PA-2PLT-0PG waveform (compute missing contributions in the plunge)
- Complete 1PA waveform templates for the O4 and O5 observing runs of LIGO-Virgo-KAGRA
- Extend framework to different orbital configurations (eccentricity, inclined orbits, spin of secondary)

Thank you!

Inspiral-transition matching

Post-adiabatic expansion

Transition-timescale expansion

	0PA	1PA	2PA	...
0PLT	$F^{(3,-1)}$	-	$F^{(3,-6)}$	$\dots$
1PLT	-	-	-	$\dots$
2PLT	$F^{(5,0)}$	-	$F^{(5,-5)}$	$\dots$
3PLT	-	$F^{(6,-2)}$	-	$\dots$
4PLT	$F^{(7,1)}$	-	$F^{(7,-4)}$	$\dots$
5PLT	-	$F^{(8,-1)}$	-	$\dots$
...	$\dots$	$\dots$	$\dots$	$\dots$

Near-ISCO solution

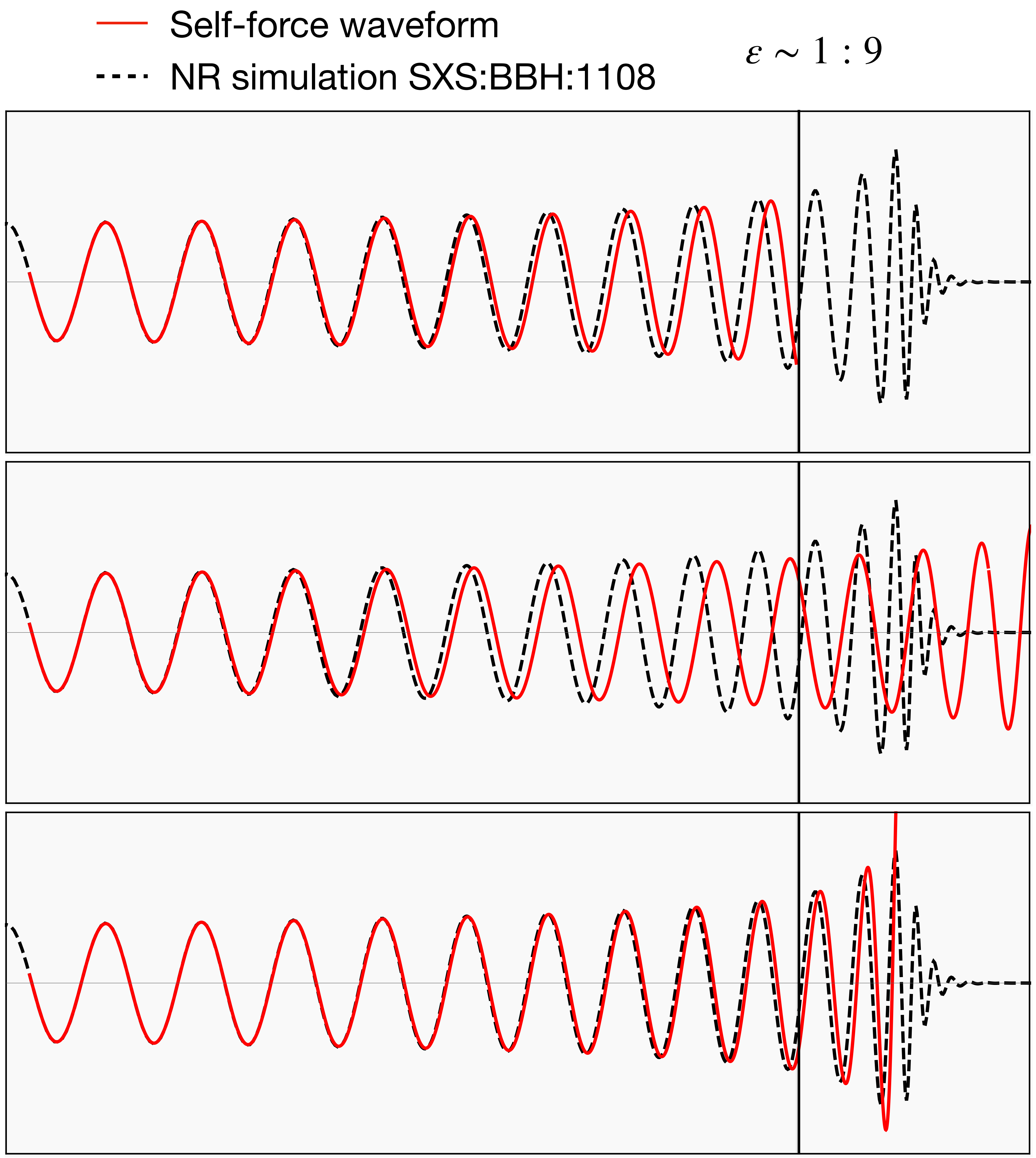
Early-time solution

Composite solution

$$
\left.\frac{d\Omega}{dt}\right|_{comp} = \lambda^5 F^{\Omega}_{(0)} + \lambda^{10} F^{\Omega}_{(1)} + \lambda^3 \sum_{i=0}^{7} \lambda^i F^{\Delta\Omega}_{[i]} + \\
-\lambda^5 \left(\frac{F^{(3,-1)}}{\Omega - \Omega_*} + F^{(5,0)} + (\Omega - \Omega_*) F^{(7,1)} + (\Omega - \Omega_*)^2 F^{(9,2)} \right) \\
-\lambda^{10} \left(\frac{F^{(6,-2)}}{(\Omega - \Omega_*)^2} + \frac{F^{(8,-1)}}{(\Omega - \Omega_*)} + F^{(10,0)} \right)
$$

- Valid for $r > 6M$ $(\Omega < \Omega_*)$
- Near the ISCO: transition approx
- At early times: inspiral approx

Inspiral-transition waveforms



$$h_{\mu\nu} = h^{(1)}_{\mu\nu}(\Omega)$$

$$\frac{d\Omega}{dt} = \lambda^5 F^{\Omega}_{(0)}$$

$$h_{\mu\nu}\Big|_{LEFT} = h^{(1)}_{\mu\nu}(\Omega)$$

$$h_{\mu\nu}\Big|_{RIGHT} = h^{[0]}_{\mu\nu}(\Delta\Omega) + \lambda^2 h^{[2]}_{\mu\nu}(\Delta\Omega)$$

$$\frac{d\Omega}{dt}\Big|_{LEFT} = \lambda^5 F^{\Omega}_{(0)} + \lambda^3 F^{\Delta\Omega}_{[0]} - \lambda^5 \frac{F^{(3,-1)}}{\Omega - \Omega_*}$$

$$\frac{d\Omega}{dt}\Big|_{RIGHT} = \lambda^3 F^{\Delta\Omega}_{[0]}$$

$$\frac{d\Omega}{dt}\Big|_{LEFT} = \lambda^5 F^{\Omega}_{(0)} + \lambda^3 F^{\Delta\Omega}_{[0]} + \lambda^5 F^{\Delta\Omega}_{[2]} - \lambda^5 \left(\frac{F^{(3,-1)}}{\Omega - \Omega_*} + F^{(5,0)} \right)$$

$$\frac{d\Omega}{dt}\Big|_{RIGHT} = \lambda^3 F^{\Delta\Omega}_{[0]} + \lambda^5 F^{\Delta\Omega}_{[2]}$$