THE EXISTENCE OF A HAMILTONIAN FORMULATION OF THE CONSERVATIVE SELF-FORCE: PROGRESS AND CHALLENGES

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CAPRA Meeting, July, 2023

State of the two-body problem

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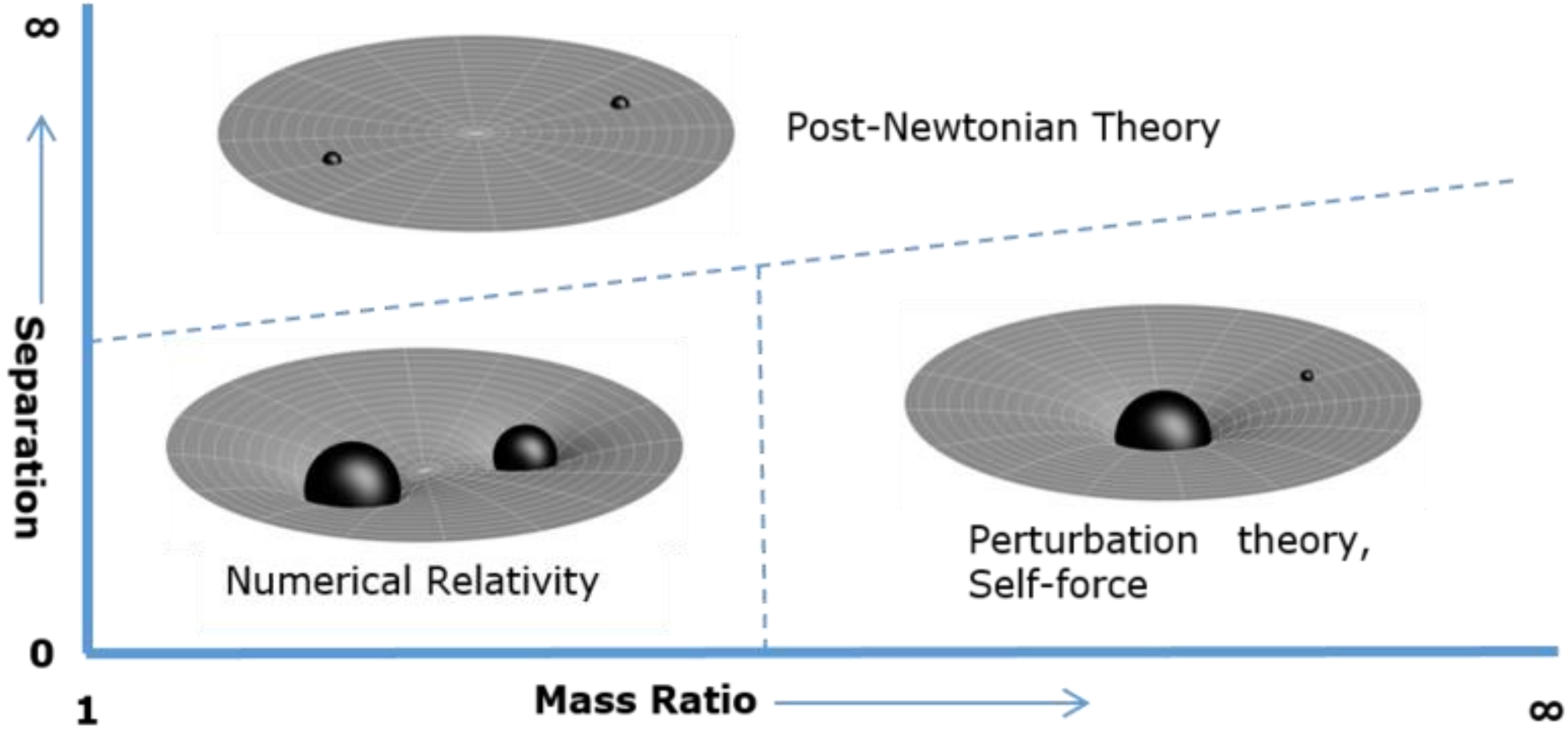

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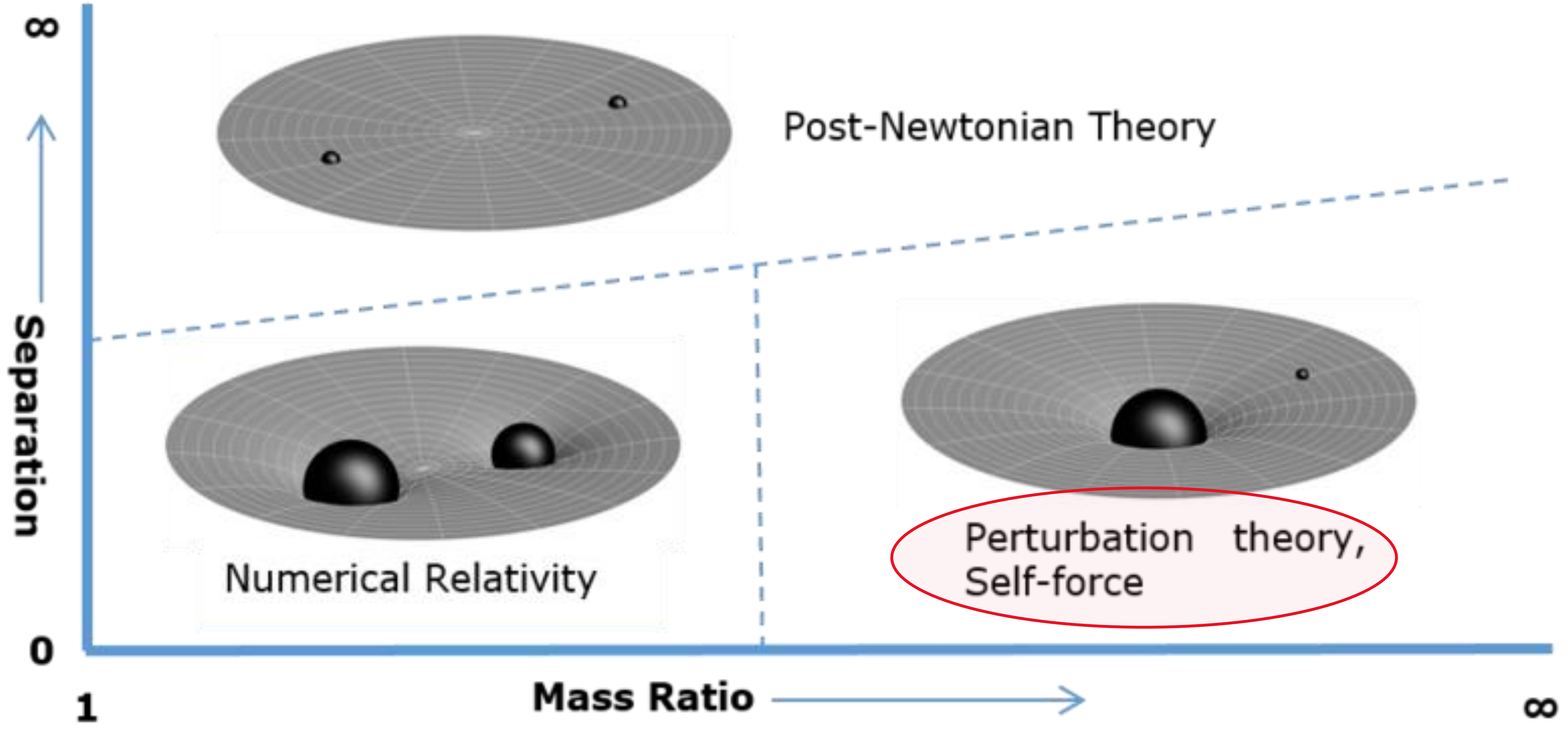

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- **Can we include spin?**
- **Can we go to 2nd order?**

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- Makes it easier to find gauge invariant quantities.
- Allows a simple derivation of the 1^{st} law of binary black hole mechanics.
- Makes comparisons with other approximations easier.

Part I: General Strategy for proving systems are Hamiltonian

Pseudo-Hamiltonian Dynamical Systems

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Hamiltonian: $$\frac{dQ^A}{d\lambda} = \Omega_0^{\mathrm{AB}} \frac{\partial}{\partial Q^B} H(Q)$$

Pseudo-Hamiltonian: $$\frac{dQ^A}{d\lambda} = \Omega_0^{\mathrm{AB}} \frac{\partial}{\partial Q^B} \boldsymbol{\mathcal{H}}(Q, Q')|_{Q=Q'}$$

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Evaluated at coincidence after taking the derivative!

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Hamiltonian System

$$\begin{cases} H(Q) = H_0(Q) + \boldsymbol{\varepsilon} H_1(Q) + \boldsymbol{\varepsilon}^2 H_2(Q) \\ \tilde{\Omega}(Q) = \tilde{\Omega}_0(Q) + \boldsymbol{\varepsilon} \tilde{\Omega}_1(Q) + \boldsymbol{\varepsilon}^2 \tilde{\Omega}_2(Q) \end{cases}$$

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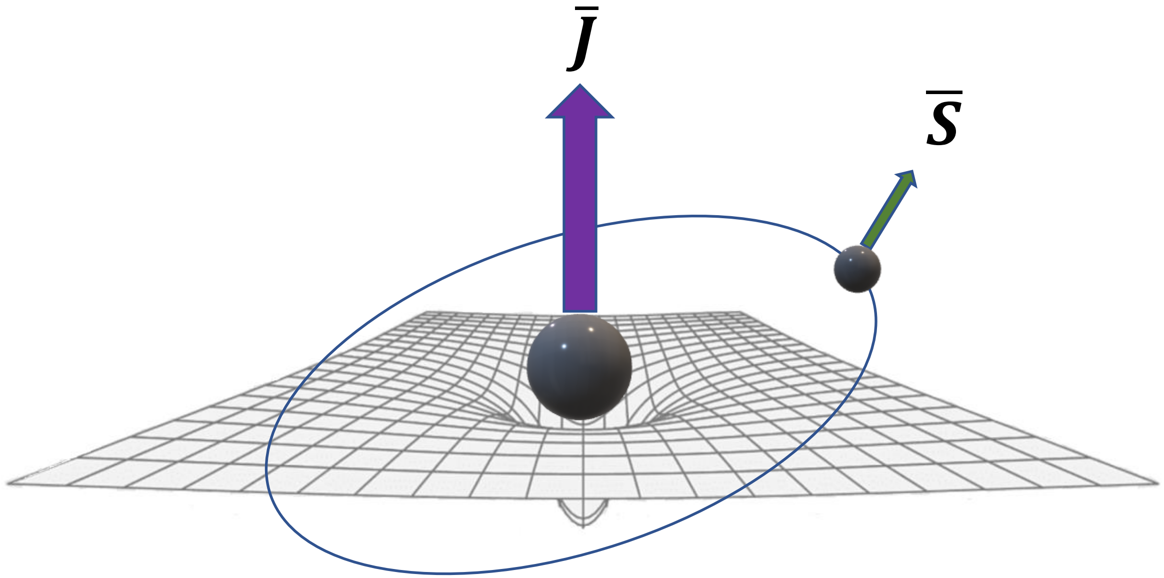
Part 2: Spin Effects

Spin in the 2-body problem



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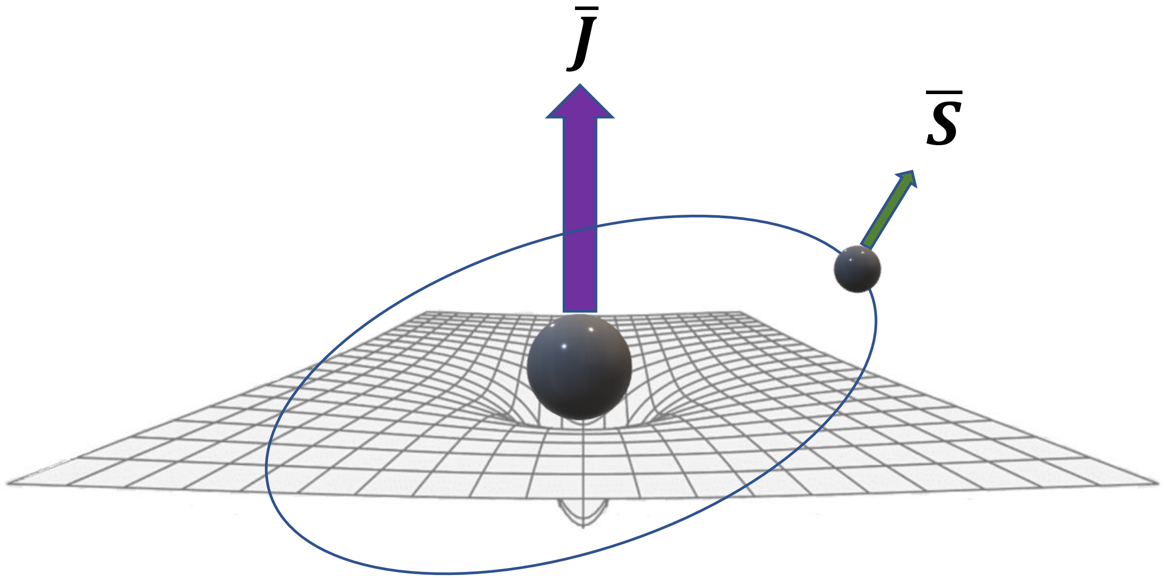

- Spin effects have been detected at LIGO.
- Spin of the secondary will be important for LISA waveforms.

Scaling of conservative effects ($\varepsilon = {}^{\mu}/_{M}$)

Conservative Effect	Interaction Energy	Accumulated Phase Shift after inspiral	Is it Hamiltonian?	In This talk!
Geodesic motion	μ	${}^{1}/_{\varepsilon}$		
1st order conservative self-force	$\mu\varepsilon$	1		
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Mathison-Papapertrou-Dixon (MPD) equations

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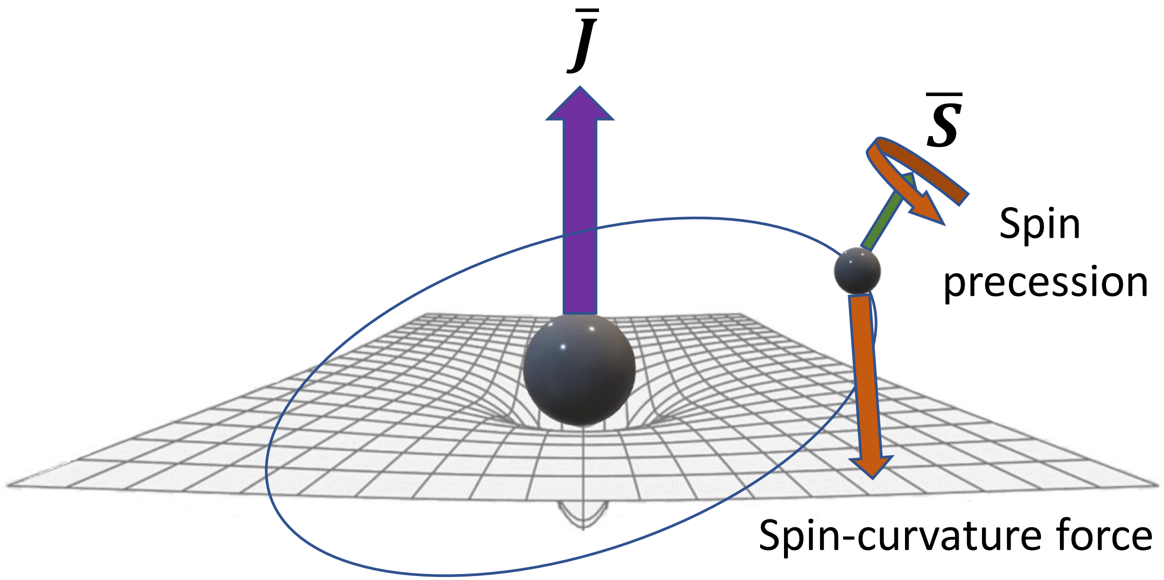

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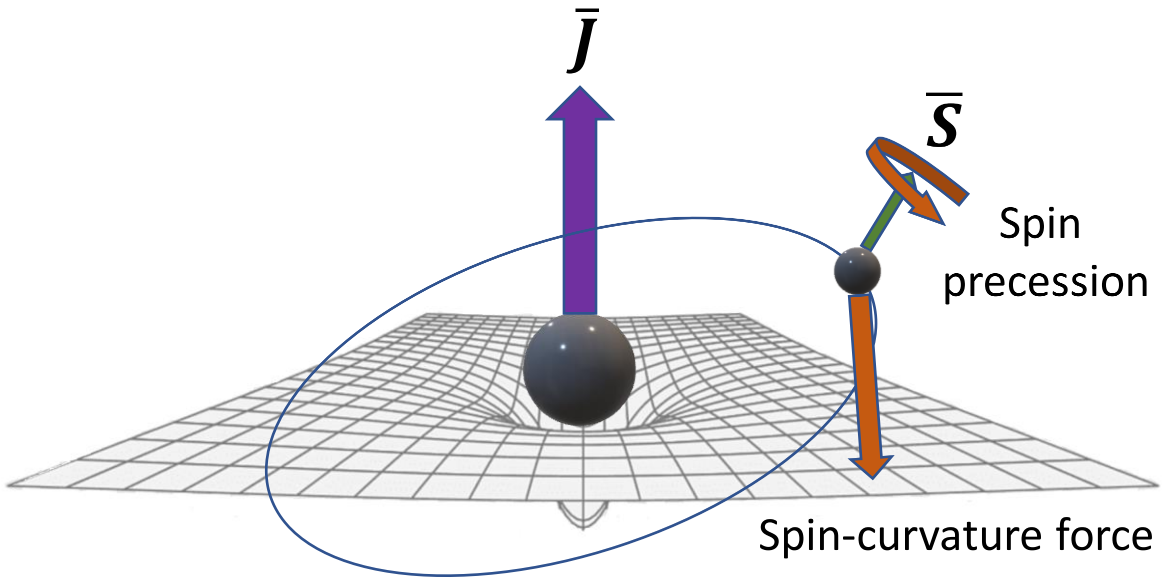

Spin supplementary condition (SSC) → $p_\alpha S^{\alpha\beta}$=0

Linear-in-Spin equations of motion

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Applying spin supplementary condition
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\begin{cases}
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$(x^\alpha, p_\alpha, S^{\alpha\beta})$ ⟶ 14D phase space

Hamiltonian formulation of spinning test particle

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(Witzany et al.)

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- There exists a Hamiltonian $H(x, p, s)$ and Poisson Brackets { , } that reproduce the Linear-in-Spin MPD equations.

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- We get a 12D phase space with nondegenerate symplectic form.

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- π_α is the momentum canonically conjugated to x.
- $S^{AB} = e^A_\mu e^B_\nu S^{\mu\nu}$ is the spin tensor in orthonormal basis e^α_A.
- $\omega_{\mu AB} = e_{A\alpha}\nabla_\mu e^\alpha_B$ is the spin connection.

What do we have so far?

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The metric perturbation $h_{\alpha\beta}$ is sourced by the particle's mass and spin

$$h_{\alpha\beta} \propto O(m) + O(S)$$

This doesn't create a Hamiltonian! There's still the extra dependence on initial condition Q'. It's a ***pseudo-Hamiltonian***.

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- O(mS)
- Responsible for Spin induced 1st order Self-Force

New Result!

This pseudo-Hamiltonian description is equivalent to a Hamiltonian system with a known Hamiltonian function and Symplectic form!

arXiv:2302.10233 (Blanco, Flanagan)

Part III: Second Order Dynamics

What about the 2^{nd} order Self-Force?

(Work in progresss with A. Pound and A. Harte)

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- And replace metric by perturbed metric

$$\boldsymbol{\mathcal{H}}(Q,Q') = -\sqrt{\left[g_{\alpha\beta}\,(x) + \boldsymbol{\varepsilon}\,h^{(1)}_{\alpha\beta}(x,Q') + \boldsymbol{\varepsilon}^2 h^{(2)}_{\alpha\beta}(x,Q')\right]^{-1} p_\alpha p_\beta}$$

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- Can this pseudo-Hamiltonian be written as integrals of symmetric 2-point and 3-point functions?

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(Blanco, Flanagan)

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$$h^{(1)}_{\alpha\beta}(x;Q') = \int d\tau' G_{\alpha\beta\rho'\sigma'}[x,x'] \frac{p^{\rho'} p^{\sigma'}}{\sqrt{-g_{\mu\nu} p^{\mu'} p^{\nu'}}}$$

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It's **symmetric**!

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- Is the 3-point function $\mathcal{G}_{\mathrm{III}}$ we get from $h^{(2)}_{\alpha\beta}$ symmetric?

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- Picking the conservative piece of $h^{(1)}_{\alpha\beta}$ means that there are standing waves at infinity. At second order, this creates an infrared divergence. How do we fix this? (A. Pound & J. Lewis are working on this)

Thanks for listening!
Any questions?

Conservative Effect	Interaction Energy	Accumulated Phase Shift after inspiral	Is it Hamiltonian?	In This talk!
Geodesic motion	μ	$1/_{\varepsilon}$	✓	✓
1st order conservative self-force	$\mu\varepsilon$	1	arXiv:2205.01667 (Blanco, Flanagan)	✓
2nd order conservative self-force	$\mu\varepsilon^2$	ε	?	?
Leading Spin-Curvature coupling	$S/_{M} \sim \mu\varepsilon$	1	arXiv:1808.06582 (Witzany et al.)	✓
Subleading Spin-Curvature coupling	$S^2/_{\mu M^2} \sim \mu\varepsilon^2$	ε	arXiv:1601.07529 (Vines et al.)	
Spin induced 1st order conservative self-force and self-torque	$\mu S/_{M^2} \sim \mu\varepsilon^2$	ε	arXiv:2302.10233 (Blanco, Flanagan)	✓