Metric perturbations of Kerr spacetime in Lorenz gauge with separation of variables

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Work in collaboration with **Leanne Durkan**, **Chris Kavanagh** and **Barry Wardell**:

- DKW, "*Gravitational Perturbations of Rotating Black Holes in Lorenz Gauge*", Phys. Rev. Lett. 128, 151101 (2022).
- DDKW, "*Metric perturbations of Kerr spacetime in Lorenz gauge: Circular equatorial orbits*", arXiv:2306.16459 (2023).

Overview

1. **Motivation**: Lorenz gauge and metric reconstruction

2. Lorenz-gauge modes in vacuum: $s = 0$, 1 and 2 modes

3. Metric perturbations for circular orbits on Kerr.
   "**Jigsaw**": build a regular MP from a set of vacuum modes

4. Static modes ($\omega = 0$) and completion pieces.

5. Prospects and future work.

Motivation: Why Lorenz gauge?

- A metric perturbation **without** discontinuities & distributions.
- A sufficiently regular gauge for calculating $h_{\mu\nu}^{(1)R}, \partial_\sigma h_{\mu\nu}^{(1)R}, \ldots$ as inputs for **second-order self-force** calculations**.**
- Metric perturbation data already exists from **2+1D time-domain code** (cf. SD, Barack & Wardell, circa 2014).
- To seek a deeper understanding of the structure of the metric perturbation (e.g. relationship between **radiation gauge** & Lorenz).
- Hyperbolic PDEs; 1/r divergence near particle worldline; asymptotics at horizon and infinity are well understood.

Motivation: Why separation of variables?

- The metric perturbation is reconstructed from (differential operators on) functions of **one** variable, e.g. $P_{+2}(r)$ and $S_{+2}(\theta)$, that satisfy **ordinary** differential equations in the frequency domain.

- This brings clear advantages in **accuracy** and **efficiency**.

- In the static $\omega = 0$ sector, the functions are known in **closed form**. For $\omega \neq 0$ the functions are available via BHPToolkit (apart from a M2af-like scalar).

- Working in the frequency domain tames the $\ell = 0$ and $\ell = 1$ linear-in-t **instabilities** in the time domain (cf J. Thornburg's work).

Formulation

The linearized Einstein equations

Einstein equations:

$$G_{\mu\nu}[g] \;=\frac{8\pi G}{c^4}T_{\mu\nu}.$$

Perturbation theory:

$$g^{\text{exact}}_{\mu\nu} = g_{\mu\nu} + \epsilon h^{(1)}_{\mu\nu} + \epsilon^2 h^{(2)}_{\mu\nu} + O(\epsilon^3).$$

Linearised Einstein equations:

$$\Box \bar{h}^{(i)}_{\mu\nu} + 2R^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu}\bar{h}^{(i)}_{\alpha\beta} + g_{\mu\nu}\nabla_\sigma Z^{\sigma}_{(i)} - 2\nabla_{(\mu}Z^{(i)}_{\nu)} = S^{(i)}_{\mu\nu}[h^{(i-1)}, \cdots, h^{(1)}, T_{\mu\nu}].$$

Gauge choice:

$$Z^{(i)}_{\mu} \equiv \nabla^{\nu}\bar{h}_{\mu\nu} \qquad \text{Lorenz gauge: } Z^{(i)}_{\mu} = 0.$$

Weyl scalars and Teukolsky equations

The Weyl tensor $C_{\mu\nu\sigma\lambda}$ is decomposed into five complex Weyl scalars Ψ_i $(i = 0 \ldots 4)$.
On the Kerr spacetime with *principal null tetrad*,

$$\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0, \quad \Psi_2 = -M/\overline{\rho}^3, \qquad \overline{\rho} \equiv r - ia\cos\theta.$$

Of particular importance in the following are the Weyl scalars of maximal spin weight,

$$\Psi_0 = C_{\mu\nu\sigma\lambda} l^\mu m^\nu l^\sigma m^\lambda$$
$$\Psi_4 = C_{\mu\nu\sigma\lambda} n^\mu \overline{m}^\nu n^\sigma \overline{m}^\lambda$$

They satisfy decoupled, separable Teukolsky equations

$$\left[\Delta\mathcal{D}_1\mathcal{D}_2^\dagger + \mathcal{L}_{-1}^\dagger\mathcal{L}_2 + 6i\omega\rho\right]\Psi_0 = -8\pi\Sigma T_0,$$
$$\left[\Delta\mathcal{D}_1^\dagger\mathcal{D}_2 + \mathcal{L}_{-1}\mathcal{L}_2^\dagger - 6i\omega\rho\right]\widetilde{\Psi}_4 = -8\pi\Sigma\widetilde{T}_4.$$

Setup

Metric:

$$g^{\mu\nu}_{(\mathrm{Kerr})} = \frac{1}{\Sigma}\left\{\Delta l_+^{(\mu} l_-^{\nu)} + m_+^{(\mu} m_-^{\nu)}\right\}$$

with $\rho = r + ia\cos\theta, \quad \Sigma = \rho\bar{\rho} = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 - 2Mr + a^2.$

Principal null tetrad and directional derivatives:

$$l_\pm^\mu = \left[\tfrac{r^2+a^2}{\Delta}, 1, 0, \tfrac{a}{\Delta}\right] \qquad \mathcal{D}\psi \equiv l_+^\mu \partial_\mu \psi$$

$$m_\pm^\mu = [\pm ia\sin\theta, 0, 1, \pm i\csc\theta] \qquad \mathcal{L}^\dagger\psi \equiv m_+^\mu \partial_\mu \psi.$$

Separation of variables:

$$\Psi_0 = \Delta^{-2} P_{+2}(r) S_{+2}(\theta) e^{-i\omega t + im\phi}.$$

The mode functions satisfy **ODEs**. In vacuum,

$$\left(\Delta_r \mathcal{D}_{-1}\mathcal{D}^\dagger + 6i\omega r - \Lambda\right) P_{+2}(r) = 0, \qquad \left(\mathcal{L}_{-1}^\dagger \mathcal{L}_2 - 6a\omega\cos\theta + \Lambda\right) S_{+2}(\theta) = 0.$$

Reconstruction: Wald's adjoint method (1979)

- Operator identity:

$$\boxed{\hat{\mathcal{S}}\hat{\mathcal{E}} = \hat{\mathcal{O}}\hat{\mathcal{T}}.}$$

  - $\hat{\mathcal{T}}$: $h_{ab} \to \Psi_{\pm 2}$ (or $A_a \to \phi_{\pm 1}$ in E.M.)
  - $\hat{\mathcal{O}}$: Teukolsky wave operator.
  - $\hat{\mathcal{E}}$: field equation (e.g. linearized Einstein equation).
  - $\hat{\mathcal{S}}$: Teukolsky source operator.

- Taking the *adjoint*:

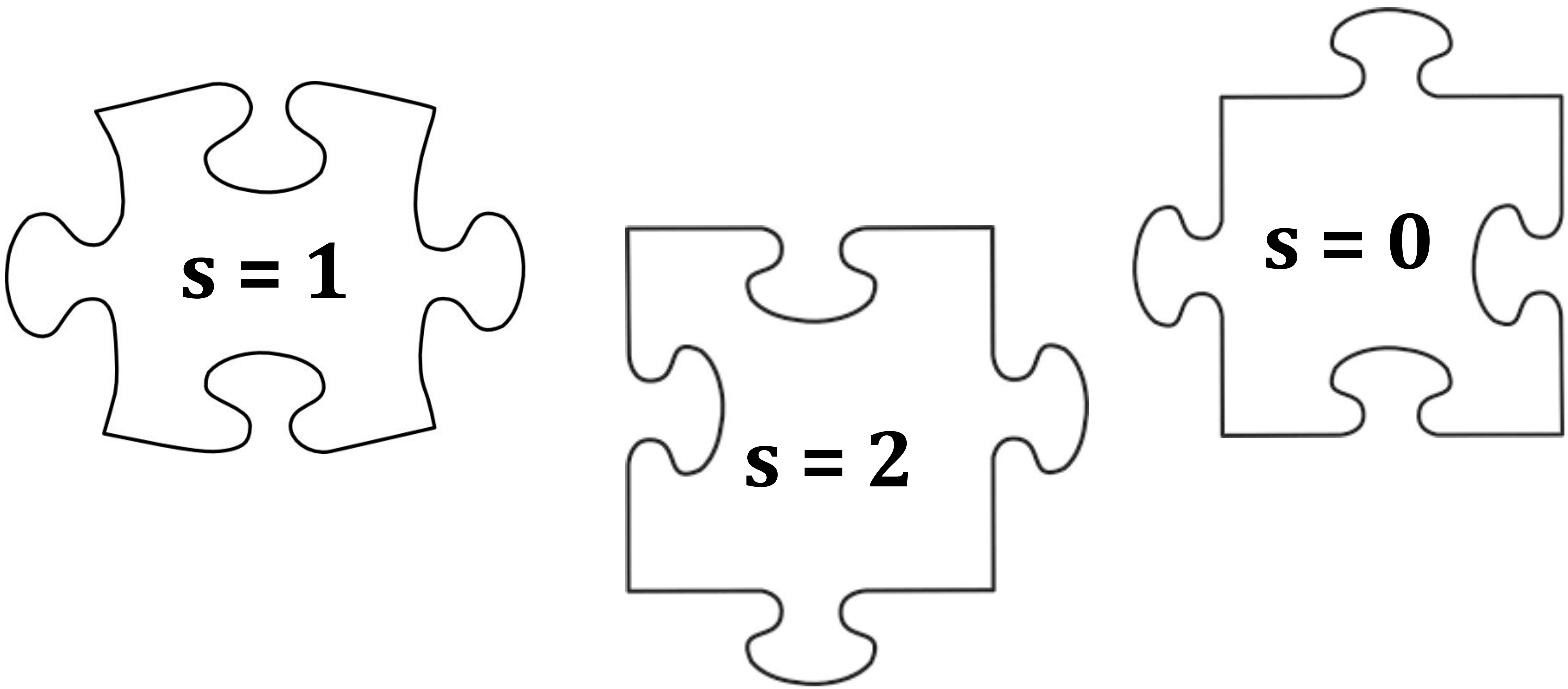
$$\boxed{\hat{\mathcal{E}}^\dagger\hat{\mathcal{S}}^\dagger = \hat{\mathcal{T}}^\dagger\hat{\mathcal{O}}^\dagger}$$

- The field-equation operator is self-adjoint: $\hat{\mathcal{E}}^\dagger = \hat{\mathcal{E}}$

- If $\hat{\mathcal{O}}^\dagger\psi = 0$ then $h = \hat{\mathcal{S}}^\dagger\psi$ satisfies the vacuum field equation $\hat{\mathcal{E}}h = 0$.

- ψ is called the *Hertz potential.*

Lorenz-gauge radiative modes ($\omega \neq 0$) in vacuum



Radiation to Lorenz gauge: spin 1

- Vector potential A^μ from Hertz potential φ satisfying $s=-1$ vacuum Teuk eq.

$$\boxed{A^\mu = \bar{\rho}^2 \nabla_\nu \left(\frac{2\varphi}{\bar{\rho}\,\Sigma} \, l_+^{[\mu} m_+^{\nu]} \right)}$$

- The vector potential is in (ingoing) **radiation gauge**: $A \cdot l_+ = A \cdot m_+ = 0.$
- Transformation to **Lorenz gauge**: $\boxed{A^\mu_{(L)} = A^\mu - \nabla^\mu \chi}$ such that $\nabla_\mu A^\mu_{(L)} = 0.$
- Applying the **source-free** Teukolsky equation, $\Box\chi = \nabla_\mu A^\mu$ has an elementary solution:

$$\boxed{\chi = \frac{1}{i\omega} \mathcal{D} \mathcal{L}_1^\dagger \varphi.}$$

- $A^\mu_{(L)}$ serves as a **gauge vector** to generate a metric perturbation in Lorenz gauge,

$$h^{(s=1)}_{\mu\nu} = 2 \nabla_{(\mu} A^{(L)}_{\nu)}.$$

Radiation to Lorenz gauge: spin 2

Ingoing radiation-gauge $h^{\mu\nu}$ from Hertz potential ψ_- :

$$h^{\mu\nu} = -\nabla_{(\sigma}\,\bar{\rho}^4\nabla_{\lambda)}\left(\frac{2\psi_-}{\bar{\rho}^2\Sigma^2}\,l_+^{[\mu}m_+^{\sigma]}l_+^{[\nu}m_+^{\lambda]}\right).$$

Key properties:

$$h_{\mu\nu}l_+^\nu = h_{\mu\nu}m_+^\nu = 0, \qquad h^\mu{}_\mu = 0, \qquad \nabla_\mu\nabla_\nu h^{\mu\nu} = 0.$$

Transformation to **traceless Lorenz** gauge:

$$h_L^{\mu\nu} = h^{\mu\nu} + 2\xi^{(\mu;\nu)}.$$

where the gauge vector is

$$\xi^\mu = \bar{\rho}^2\nabla_\nu\left(\frac{\mathcal{D}\mathcal{L}_2^\dagger\psi_-}{3i\omega\bar{\rho}\Sigma}l_+^{[\mu}m_+^{\nu]}\right) + g^{\mu\nu}\partial_\nu\left(\frac{1}{24\omega^2}\mathcal{D}\mathcal{D}\mathcal{L}_1^\dagger\mathcal{L}_2^\dagger\psi_-\right).$$

Trace and scalar modes: s = 0

- How to construct a metric perturbation with a trace h?
- In the source-free case $\Box h = 0$, and the trace mode is generated from a gauge vector.
- We find that:

$$\boxed{\xi^a = \frac{1}{2i\omega} \mathfrak{f}^{ab} \nabla_b h + 2\nabla^a \kappa}$$

  with

$$\Box \kappa = \tfrac{1}{2} h.$$

- This satisfies the **Lorenz condition** $\Box \xi_a = 0$ and the **trace condition** $h = -2\nabla_a \xi^a$.
- Here $\mathfrak{f}_{ab}$ is the **principal tensor** (a.k.a **conformal Killing-Yano tensor**) satisfying

$$\mathfrak{f}_{ab;c} = g_{bc} T_a - g_{ac} T_b.$$

Inversion

	IRG	ORG
$\Psi_0 =$	0	$+6iM\omega\Delta^{-2}\psi_+$
$\widetilde{\Psi}_4 =$	$-6iM\omega\Delta^{-2}\psi_-$	0
$\Psi_0' =$	$-\frac{1}{2}\mathcal{DDDD}\psi_-$	$-\frac{1}{2}\Delta^{-2}\mathcal{L}_{-1}\mathcal{L}\mathcal{L}_1\mathcal{L}_2\psi_+$
$\widetilde{\Psi}_4' =$	$-\frac{1}{2}\Delta^{-2}\mathcal{L}_{-1}^\dagger\mathcal{L}^\dagger\mathcal{L}_1^\dagger\mathcal{L}_2^\dagger\psi_-$	$-\frac{1}{2}\mathcal{D}^\dagger\mathcal{D}^\dagger\mathcal{D}^\dagger\mathcal{D}^\dagger\psi_+$

TABLE I. The Weyl scalars for the ingoing and outgoing radiation-gauge MPs.

- Form the **difference** between IRG and ORG MPs after transforming to Lorenz gauge,

$$h_{\mu\nu}^{L(-)} \equiv h_{\mu\nu}^{IRG,L} - h_{\mu\nu}^{ORG,L},$$

and now set the two Hertz potentials in proportion to the Weyl scalars,

$$\Delta^{-2}\psi_+ = (-6iM\omega)^{-1}\Psi_0,$$
$$\Delta^{-2}\psi_- = (-6iM\omega)^{-1}\tilde{\Psi}_4.$$

- Inversion breaks down for $M = 0$ or $\omega = 0$.

Metric perturbations: spin 2

$$h^{L(-)}_{l_+l_+} = \frac{-1}{6\omega^2\Delta^2}P_{+2}\left((-1)^{\ell+m}\mathcal{L}_1^\dagger\mathcal{L}_2^\dagger S_{-2} + \mathcal{L}_1\mathcal{L}_2 S_{+2}\right)$$

$$h^{L(-)}_{l_-l_-} = \frac{-1}{6\omega^2\Delta^2}P_{-2}\left(\mathcal{L}_1^\dagger\mathcal{L}_2^\dagger S_{-2} + (-1)^{\ell+m}\mathcal{L}_1\mathcal{L}_2 S_{+2}\right)$$

$$h^{L(-)}_{m_+m_+} = \frac{-1}{6\omega^2}\left((-1)^{\ell+m}\mathcal{D}\mathcal{D}P_{-2} + \mathcal{D}^\dagger\mathcal{D}^\dagger P_{+2}\right)S_{+2}$$

$$h^{L(-)}_{m_-m_-} = \frac{-1}{6\omega^2}\left(\mathcal{D}\mathcal{D}P_{-2} + (-1)^{\ell+m}\mathcal{D}^\dagger\mathcal{D}^\dagger P_{+2}\right)S_{-2}$$

$$\alpha_- = \frac{1}{3i\omega\bar{\rho}}\mathcal{D}\mathcal{L}_2^\dagger\psi_-,$$

$$\alpha_+ = \frac{1}{-3i\omega\bar{\rho}}\mathcal{D}^\dagger\mathcal{L}_2\psi_+,$$

$$\chi_- = \frac{1}{24\omega^2}\mathcal{D}\mathcal{D}\mathcal{L}_1^\dagger\mathcal{L}_2^\dagger\psi_-,$$

$$\chi_+ = \frac{1}{24\omega^2}\mathcal{D}^\dagger\mathcal{D}^\dagger\mathcal{L}_1\mathcal{L}_2\psi_+.$$

$$h^{L(-)}_{l_+m_+} = (-1)^{\ell+m}\left\{2\left(\mathcal{D}\mathcal{L}^\dagger - \frac{1}{\rho}\mathcal{L}^\dagger - \frac{\rho_{,\theta}}{\rho}\mathcal{D}\right)(\chi'_- - \chi'_+) - \frac{\rho^2}{\Delta}\left(\Delta\mathcal{D}\mathcal{D}\alpha'_- + \mathcal{L}^\dagger\mathcal{L}_1^\dagger\alpha'_+\right)\right\}, \quad (89e)$$

$$h^{L(-)}_{l_-m_-} = (-1)^{\ell+m}\left\{2\left(\mathcal{D}^\dagger\mathcal{L} - \frac{1}{\rho}\mathcal{L} - \frac{\rho_{,\theta}}{\rho}\mathcal{D}^\dagger\right)(\chi'_- - \chi'_+) + \frac{\rho^2}{\Delta}\left(\Delta\mathcal{D}^\dagger\mathcal{D}^\dagger\alpha'_+ + \mathcal{L}\mathcal{L}_1\alpha'_-\right)\right\} \quad (89f)$$

$$h^{L(-)}_{l_+m_-} = \left(\mathcal{D} - \frac{2}{\bar{\rho}}\right)\left(\mathcal{L}(\chi_- - \chi_+) - \bar{\rho}^2\mathcal{D}\alpha_-\right) + \left(\mathcal{L} - \frac{2\bar{\rho}_{,\theta}}{\bar{\rho}}\right)\left(\mathcal{D}(\chi_- - \chi_+) - \bar{\rho}^2\Delta^{-1}\mathcal{L}_1\alpha_+\right), \quad (89g)$$

$$h^{L(-)}_{l_-m_+} = \left(\mathcal{D}^\dagger - \frac{2}{\bar{\rho}}\right)\left(\mathcal{L}^\dagger(\chi_- - \chi_+) + \bar{\rho}^2\mathcal{D}^\dagger\alpha_+\right) + \left(\mathcal{L}^\dagger - \frac{2\bar{\rho}_{,\theta}}{\bar{\rho}}\right)\left(\mathcal{D}^\dagger(\chi_- - \chi_+) + \bar{\rho}^2\Delta^{-1}\mathcal{L}_1^\dagger\alpha_-\right), \quad (89h)$$

$$h^{L(-)}_{l_+l_-} = \frac{1}{\Delta\Sigma}\left(\Sigma(\mathcal{D}\Delta\mathcal{D}^\dagger + \mathcal{D}^\dagger\Delta\mathcal{D}) - 2\Delta\Sigma_{,r}\partial_r + 2\Sigma_{,\theta}\partial_\theta\right)(\chi_- - \chi_+) \quad (89i)$$

$$+ \frac{1}{\Delta\Sigma}\left(\Sigma(\mathcal{D}\bar{\rho}^2\mathcal{L}_1^\dagger\alpha_- - \mathcal{D}^\dagger\bar{\rho}^2\mathcal{L}_1\alpha_+) - \bar{\rho}^2\Sigma_{,r}(\mathcal{L}_1^\dagger\alpha_- - \mathcal{L}_1\alpha_+) + \bar{\rho}^2\Sigma_{,\theta}(\mathcal{D}^\dagger\alpha_+ - \mathcal{D}\alpha_-)\right), \quad (89j)$$

$$h^{L(-)}_{m_+m_-} = -\Delta h^{L(-)}_{l_+l_-}. \quad (89k)$$

Metric perturbations: spin 1

$$h^{(s=1)}_{l_+l_+} = \pm 2\Delta^{-1}\mathcal{D}_{-1}P_{+1}\mathcal{S},$$
$$h^{(s=1)}_{l_-l_-} = \quad 2\Delta^{-1}\mathcal{D}^\dagger_{-1}P_{-1}\mathcal{S},$$
$$h^{(s=1)}_{m_+m_+} = \pm 2\mathcal{P}\mathcal{L}^\dagger_{-1}S_{+1},$$
$$h^{(s=1)}_{m_-m_-} = -2\mathcal{P}\mathcal{L}_{-1}S_{-1},$$
$$h^{(s=1)}_{l_+m_+} = \pm\left(\mathcal{D} - \frac{2}{\rho}\right)\mathcal{P}S_{+1} \pm \Delta^{-1}P_{+1}\left(\mathcal{L}^\dagger - \frac{2\rho_{,\theta}}{\rho}\right)\mathcal{S},$$
$$h^{(s=1)}_{l_+m_-} = -\left(\mathcal{D} - \frac{2}{\overline{\rho}}\right)\mathcal{P}S_{-1} \pm \Delta^{-1}P_{+1}\left(\mathcal{L} - \frac{2\overline{\rho}_{,\theta}}{\overline{\rho}}\right)\mathcal{S},$$
$$h^{(s=1)}_{l_-m_+} = \pm\left(\mathcal{D}^\dagger - \frac{2}{\overline{\rho}}\right)\mathcal{P}S_{+1} + \Delta^{-1}P_{-1}\left(\mathcal{L}^\dagger - \frac{2\overline{\rho}_{,\theta}}{\overline{\rho}}\right)\mathcal{S},$$
$$h^{(s=1)}_{l_-m_-} = -\left(\mathcal{D}^\dagger - \frac{2}{\rho}\right)\mathcal{P}S_{-1} + \Delta^{-1}P_{-1}\left(\mathcal{L} - \frac{2\rho_{,\theta}}{\rho}\right)\mathcal{S},$$
$$\Delta h^{(s=1)}_{l_+l_-} = \mathcal{P}\mathcal{S} - \frac{\Sigma_{,r}}{\Sigma}\left(P_{-1} \pm P_{+1}\right)\mathcal{S} \pm \mathcal{P}\frac{\Sigma_{,\theta}}{\Sigma}\left(S_{+1} \mp S_{-1}\right),$$
$$h^{(s=1)}_{m_+m_-} = -\Delta h^{(s=1)}_{l_+l_-}.$$

$$\mathcal{P} \equiv \mathcal{D}P_{-1} \pm \mathcal{D}^\dagger P_{+1}$$
$$\mathcal{S} \equiv \mathcal{L}^\dagger_1 S_{-1} \mp \mathcal{L}_1 S_{+1}.$$

Metric perturbations: spin 2

$$\Box h = 0,$$
$$\Box \kappa = \tfrac{1}{2} h,$$

$$h^{(s=0)}_{l_+l_+} = -\left(\frac{1}{i\omega}\mathcal{D}r\mathcal{D}h + 4\mathcal{D}\mathcal{D}\kappa\right),$$

$$h^{(s=0)}_{l_-l_-} = -\left(-\frac{1}{i\omega}\mathcal{D}^\dagger r\mathcal{D}^\dagger h + 4\mathcal{D}^\dagger\mathcal{D}^\dagger\kappa\right),$$

$$h^{(s=0)}_{m_+m_+} = -\mathcal{L}^\dagger_{-1}\left(-\frac{a\cos\theta}{\omega}\mathcal{L}^\dagger h + 4\mathcal{L}^\dagger\kappa\right),$$

$$h^{(s=0)}_{m_-m_-} = -\mathcal{L}_{-1}\left(+\frac{a\cos\theta}{\omega}\mathcal{L}h + 4\mathcal{L}\kappa\right),$$

$$\rho\, h^{(s=0)}_{l_+m_+} = -\left[\frac{\Sigma}{2i\omega}\mathcal{D}\mathcal{L}^\dagger h + \frac{a}{\omega}\left(r\sin\theta\mathcal{D} + \cos\theta\mathcal{L}^\dagger\right)h + 4\rho\mathcal{D}\mathcal{L}^\dagger\kappa - 4\left(\mathcal{L}^\dagger - ia\sin\theta\mathcal{D}\right)\kappa\right],$$

$$\bar{\rho}\, h^{(s=0)}_{l_+m_-} = -\left[\frac{\Sigma}{2i\omega}\mathcal{D}\mathcal{L}h - \frac{a}{\omega}\left(r\sin\theta\mathcal{D} + \cos\theta\mathcal{L}\right)h + 4\bar{\rho}\mathcal{D}\mathcal{L}\kappa - 4\left(\mathcal{L} + ia\sin\theta\mathcal{D}\right)\kappa\right],$$

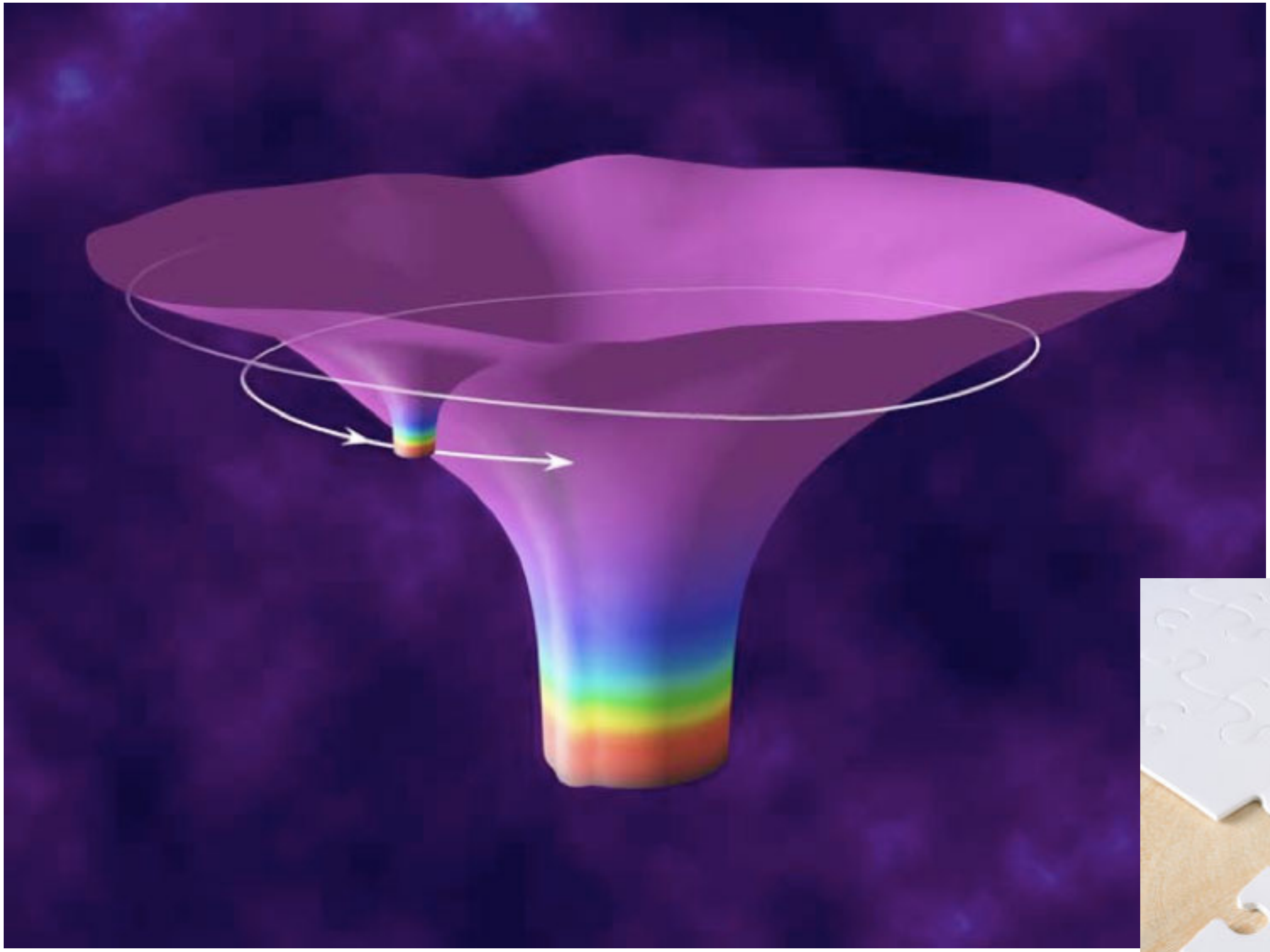
$$\bar{\rho}\, h^{(s=0)}_{l_-m_+} = -\left[-\frac{\Sigma}{2i\omega}\mathcal{D}^\dagger\mathcal{L}^\dagger h + \frac{a}{\omega}\left(r\sin\theta\mathcal{D}^\dagger + \cos\theta\mathcal{L}^\dagger\right)h + 4\bar{\rho}\mathcal{D}^\dagger\mathcal{L}^\dagger\kappa - 4\left(\mathcal{L}^\dagger + ia\sin\theta\mathcal{D}^\dagger\right)\kappa\right]$$

$$\rho\, h^{(s=0)}_{l_-m_-} = -\left[-\frac{\Sigma}{2i\omega}\mathcal{D}^\dagger\mathcal{L}h - \frac{a}{\omega}\left(r\sin\theta\mathcal{D}^\dagger + \cos\theta\mathcal{L}\right)h + 4\rho\mathcal{D}^\dagger\mathcal{L}\kappa - 4\left(\mathcal{L} - ia\sin\theta\mathcal{D}^\dagger\right)\kappa\right],$$

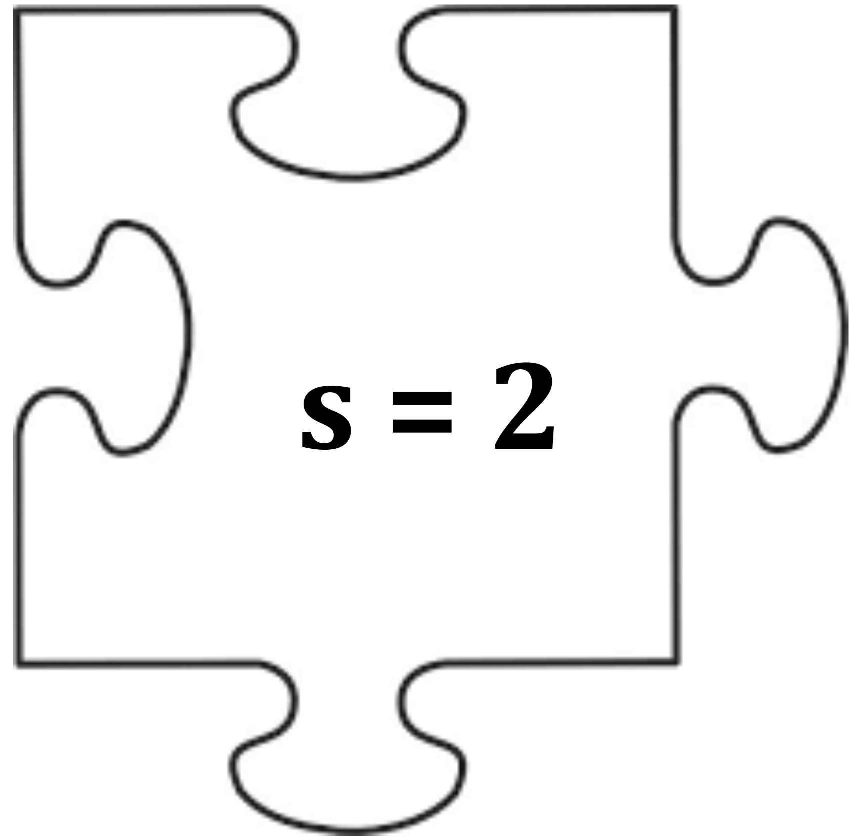
$$h^{(s=0)}_{m_+m_-} = -\left(\frac{a\sin\theta Q}{\omega} - 2\Sigma\right)h - 2\left(\mathcal{L}_1\mathcal{L}^\dagger + \mathcal{L}^\dagger_1\mathcal{L}\right)\kappa + \frac{4}{\Sigma}\left(-\Sigma_{,r}\Delta\kappa_{,r} + \Sigma_{,\theta}\kappa_{,\theta}\right),$$

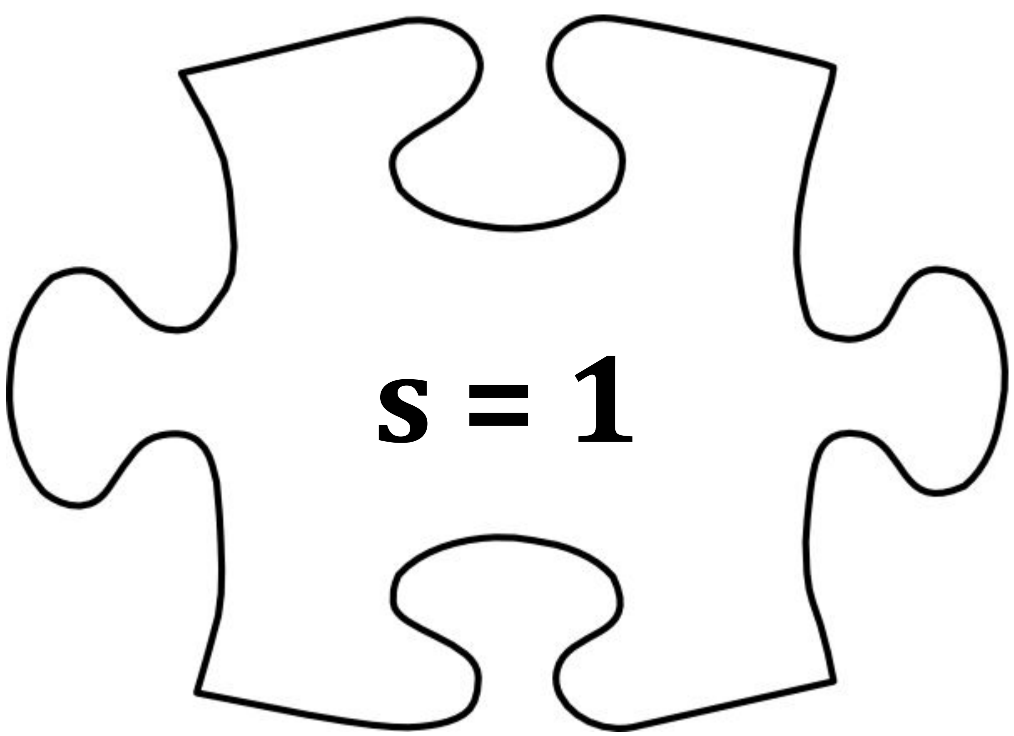
$$\Delta h^{(s=0)}_{l_+l_-} = -\left(\frac{K_r}{\omega} - 2\Sigma\right)h - 2\left(\mathcal{D}\Delta\mathcal{D}^\dagger + \mathcal{D}^\dagger\Delta\mathcal{D}\right)\kappa - \frac{4}{\Sigma}\left(-\Sigma_{,r}\Delta\kappa_{,r} + \Sigma_{,\theta}\kappa_{,\theta}\right).$$

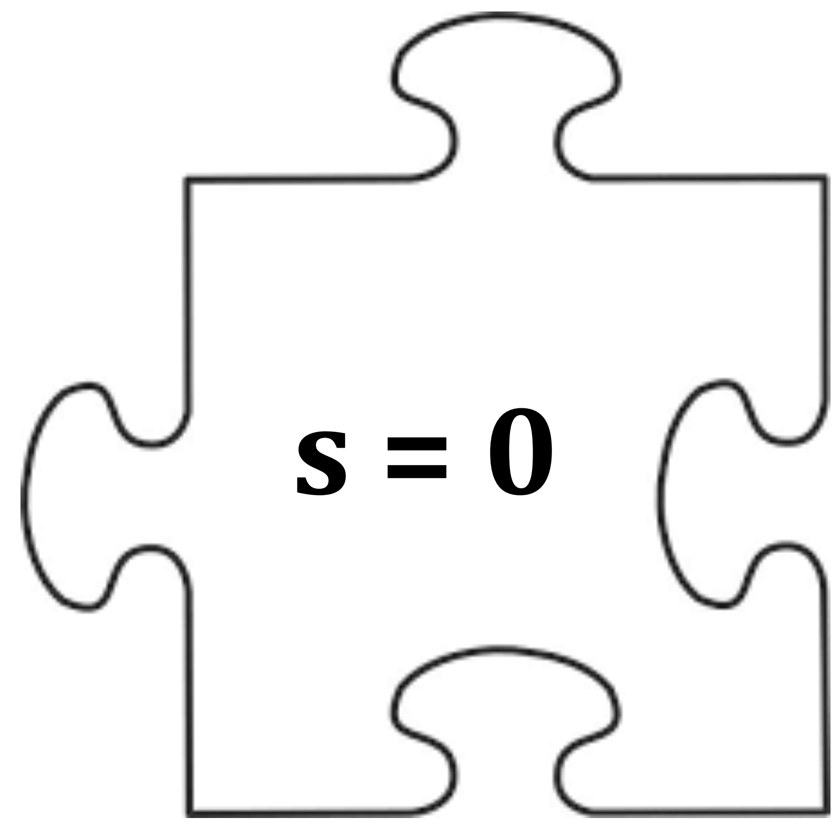
Metric perturbation for a particle on a circular orbit



Fit pieces together at $r = r_0$







Project metric components onto a common basis of spin-weighted **spherical** harmonics

Metric perturbation for a particle on a circular orbit

Conjecture: The (radiative part of) the Lorenz-gauge metric perturbation for a particle on a circular equatorial orbit on Kerr spacetime is, in vacuum regions, the sum of:

1. The IRG-minus-ORG perturbation in Lorenz gauge (s = 2).

2. A perturbation that generates the correct trace (s = 0).

3. A vector perturbation (s=1) [traceless, pure-gauge].

4. A traceless scalar perturbation (s=0): $h_{ab} = \nabla_{(a}\nabla_{b)}\kappa$, Box kappa = 0.

- Part 1 is **fixed** by the Weyl scalars (determined from Teukolsky equations)
- Part 2 is **fixed** by the trace equation.
- Parts 3 and 4 are fixed by demanding that the metric perturbation is sufficiently regular at $r = r_0$ & satisfies the Lorenz-gauge field equations.

Assembling the jigsaw

- The metric perturbation for a source on an **equatorial circular orbit** at $r = r_0$ is constructed by **glueing** the UP and IN vacuum solutions h^+_{ab} and h^-_{ab} at $r = r_0$.

$$h_{ab} = \sum_{lms} h^+_{ab}\Theta(r - r_0) + h^-_{ab}\Theta(r_0 - r)$$

- The vacuum solutions are made from sums of modes:

$$h^\pm_{ab} = h^{(s=2)\pm}_{ab} + h^{(s=0,trace)\pm}_{ab} + h^{(s=1)\pm}_{ab} + h^{(s=0,traceless)\pm}_{ab}$$

- The $s = 1$ and traceless $s = 0$ parts have undetermined coefficients:

$$h^{(s=1)\pm}_{ab} = \sum_{lm} \alpha^{(s=1)\pm}_{lm} h^{(s=1)lm}_{ab}, \qquad h^{(s=0,traceless)\pm}_{ab} = \sum_{lm} \alpha^{(s=0)\pm}_{lm} h^{(s=0,traceless)lm}_{ab}$$

- The Lorenz-gauge MP is continuous on the sphere except at $\theta = \pi/2$; moreover, from the source term $-16\pi T_{ab}$, we infer 'jumps' in the radial derivatives $\partial_r h^{(s)}_{ab}$ at $r = r_0$.
- To set up a (overdetermined) linear system of equations for $\alpha^{(s=1)\pm}_{lm} h^{(s=1)lm}_{ab}$ and $\alpha^{(s=0)\pm}_{lm} h^{(s=1)lm}_{ab}$, we project the metric components on a (spin-weighted) **spherical basis**.

Assembling the jigsaw

- To determine the 'jumps' in the $s=1$ and $s=0$ parts **uniquely**, we need consider only **two components** of the metric perturbation, and their radial derivatives:

$$h^{(s=2)}_{l_+l_+} = \frac{-1}{6\omega^2\Delta^2}P_{+2}\left(\pm\mathcal{L}_1^\dagger\mathcal{L}_2^\dagger S_{-2} + \mathcal{L}_1\mathcal{L}_2 S_{+2}\right), \quad h^{(s=2)}_{m_+m_+} = \frac{-1}{6\omega^2}\left(\pm\mathcal{D}\mathcal{D}P_{-2} + \mathcal{D}^\dagger\mathcal{D}^\dagger P_{+2}\right)S_{+2}.$$

$$h^{(s=1)}_{l_+l_+} = \pm 2\Delta^{-1}\mathcal{D}_{-1}P_{+1}(\mathcal{L}_1^\dagger S_{-1} \mp \mathcal{L}_1 S_{+1}), \quad h^{(s=1)}_{m_+m_+} = \pm 2\left(\mathcal{D}P_{-1} \pm \mathcal{D}^\dagger P_{+1}\right)\mathcal{L}^\dagger_{-1}S_{+1},$$

$$h^{(s=0)}_{l_+l_+} = -\left(\frac{1}{i\omega}\mathcal{D}r\mathcal{D}h + 4\mathcal{D}\mathcal{D}\kappa\right), \quad h^{(s=0)}_{m_+m_+} = -\mathcal{L}^\dagger_{-1}\left(-\frac{a\cos\theta}{\omega}\mathcal{L}^\dagger h + 4\mathcal{L}^\dagger\kappa\right).$$

- The remaining 8 components are used as **consistency checks**.

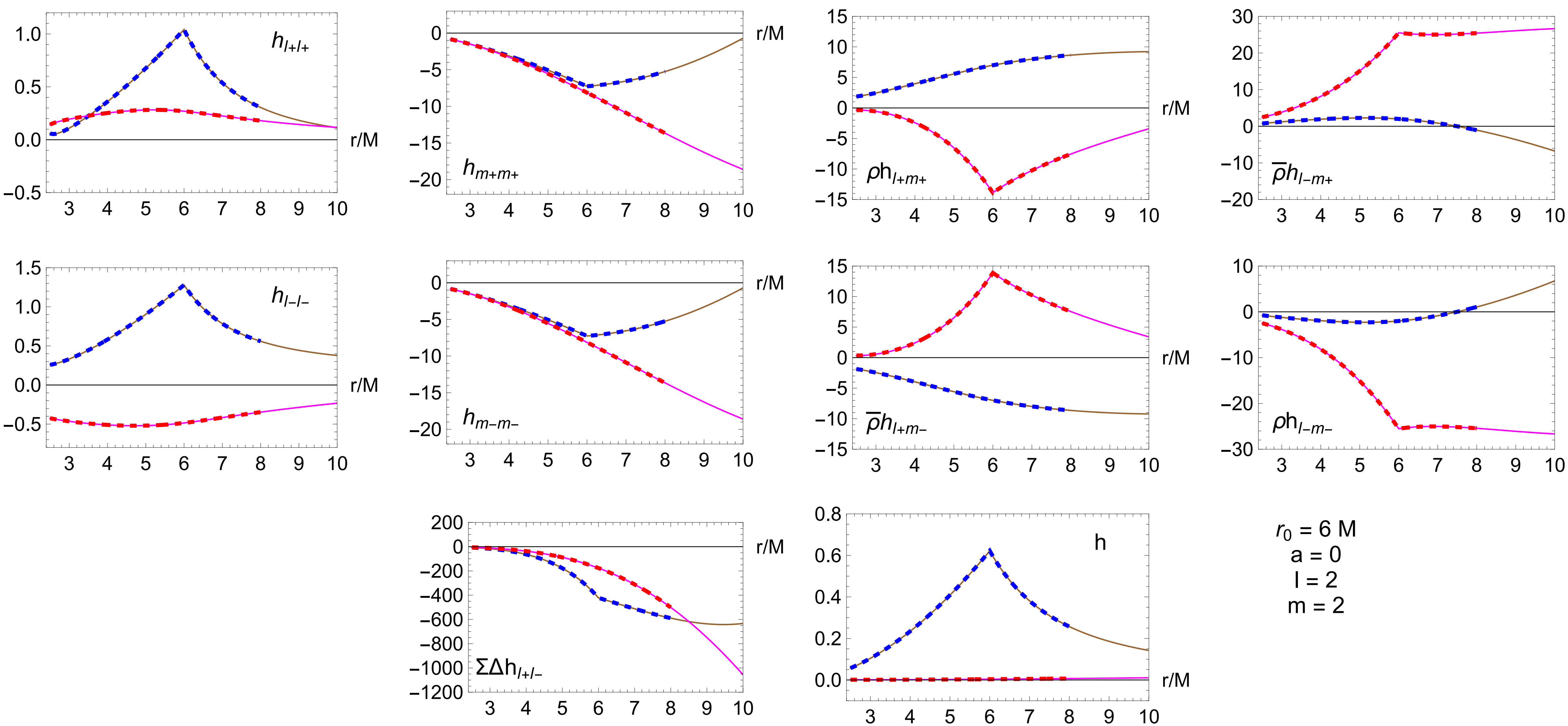
- Projections of components onto the **spherical basis** involve matrix multiplication.

Example projection: $$h^{(s=2)}_{l_+l_+} = -\frac{1}{6\omega^2\Delta^2}\mathbf{P_{+2}}^T\mathbf{S}^{(s=2)}_{l_+l_+}\mathbf{Y}_0,$$

$$\mathbf{S}^{(s=2)}_{l_+l_+} = \mathbf{b}_{+2}\left(\hat{\mathbf{\Lambda}} - 2a\omega\hat{\boldsymbol{\lambda}}_2\mathfrak{s}_{10} + a^2\omega^2(\mathfrak{s}^2)_{20}\right) \pm \mathbf{b}_{-2}\left(\hat{\mathbf{\Lambda}} - 2a\omega\hat{\boldsymbol{\lambda}}_2\mathfrak{s}_{-10} + a^2\omega^2(\mathfrak{s}^2)_{-20}\right)$$

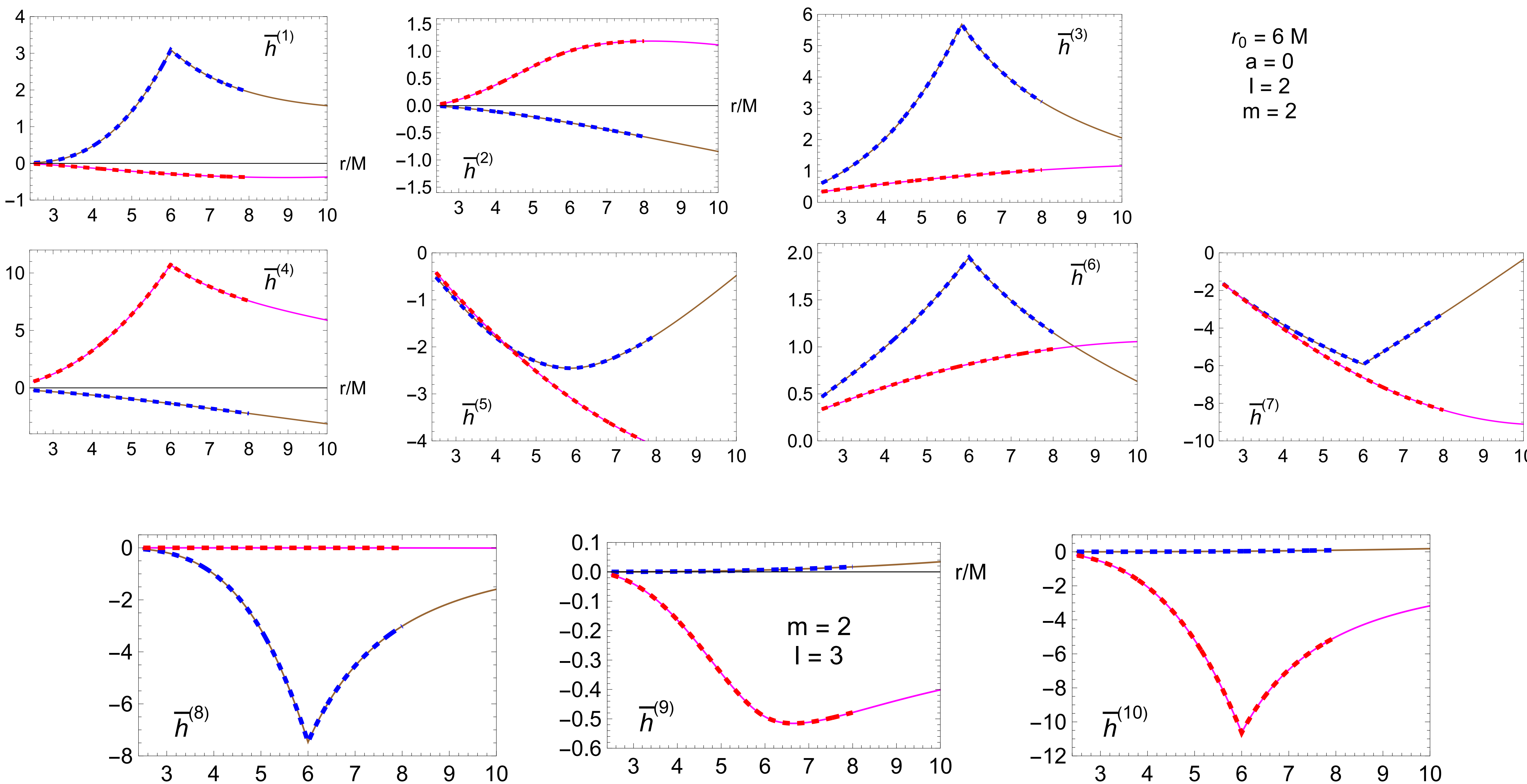
Numerical results: radiative modes ($\omega \neq 0$)

Comparison with Schwarzschild



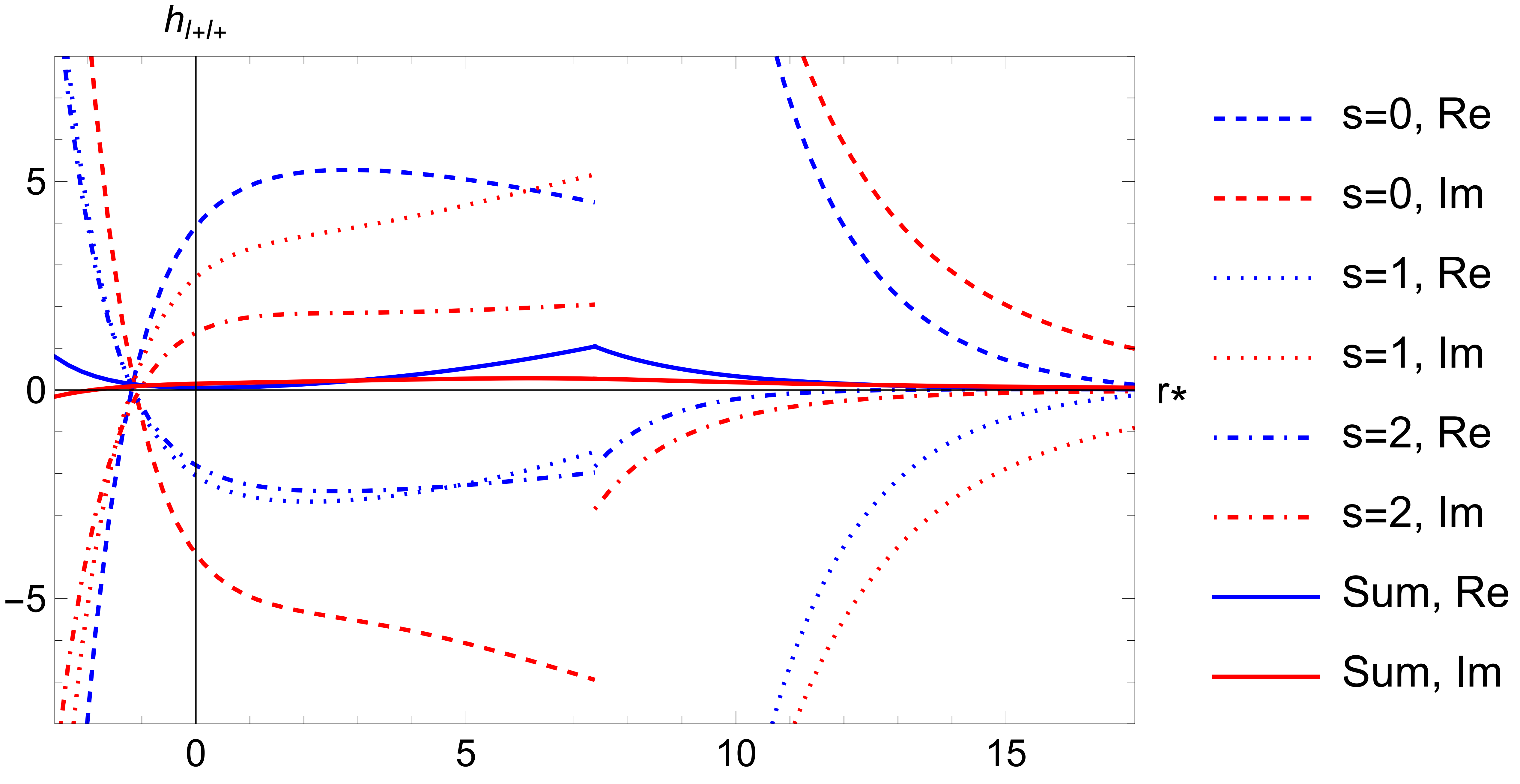
10 projections of the metric onto unnormalised null tetrad.
New results [dashed] vs comparison data [solid] from N Warburton.

Comparison with Schwarzschild

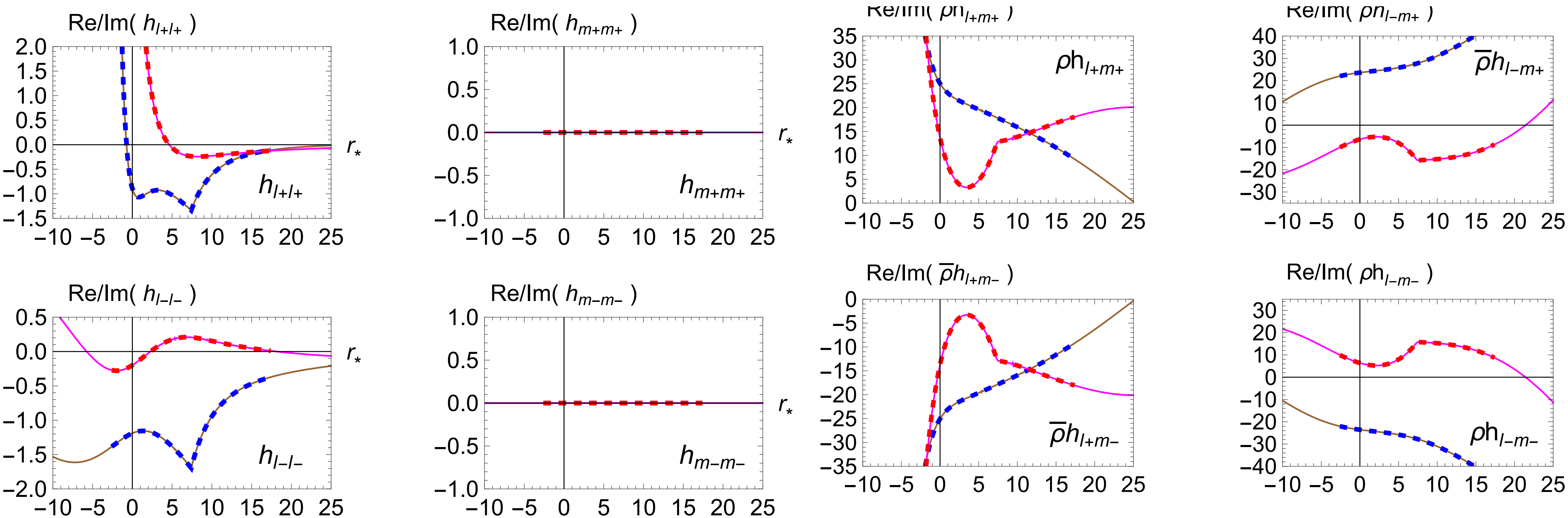


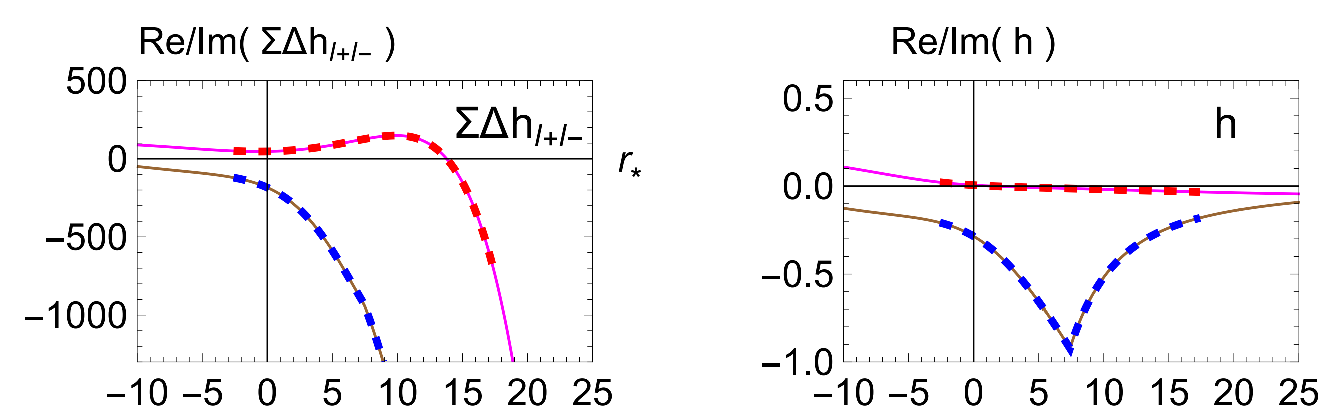
Barack and Lousto variables

Spins 0, 1 and 2



Comparison with Kerr data



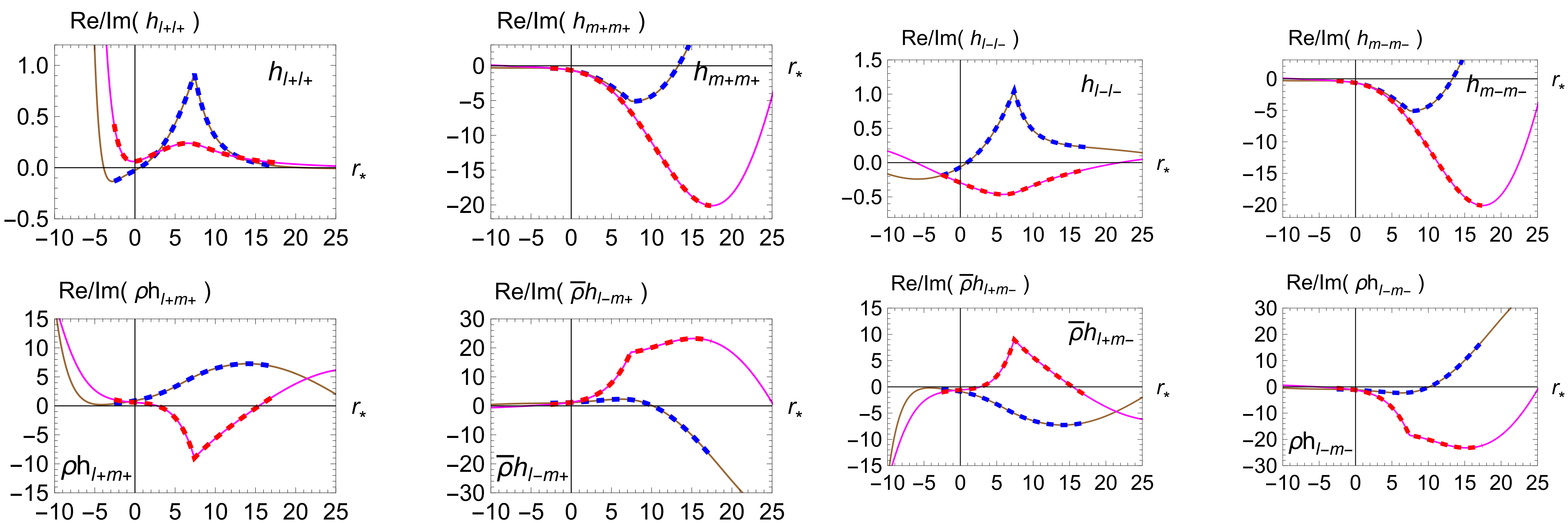


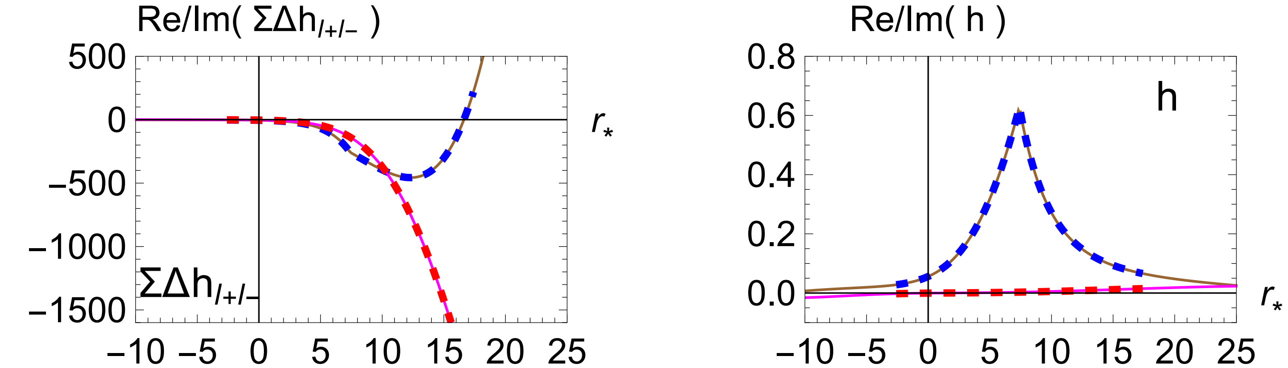
l = m = 1 radial modes.

a = **0.6 M**, r0 = 6M

New results [dashed] vs
2+1D time domain data [solid]

Comparison with Kerr data



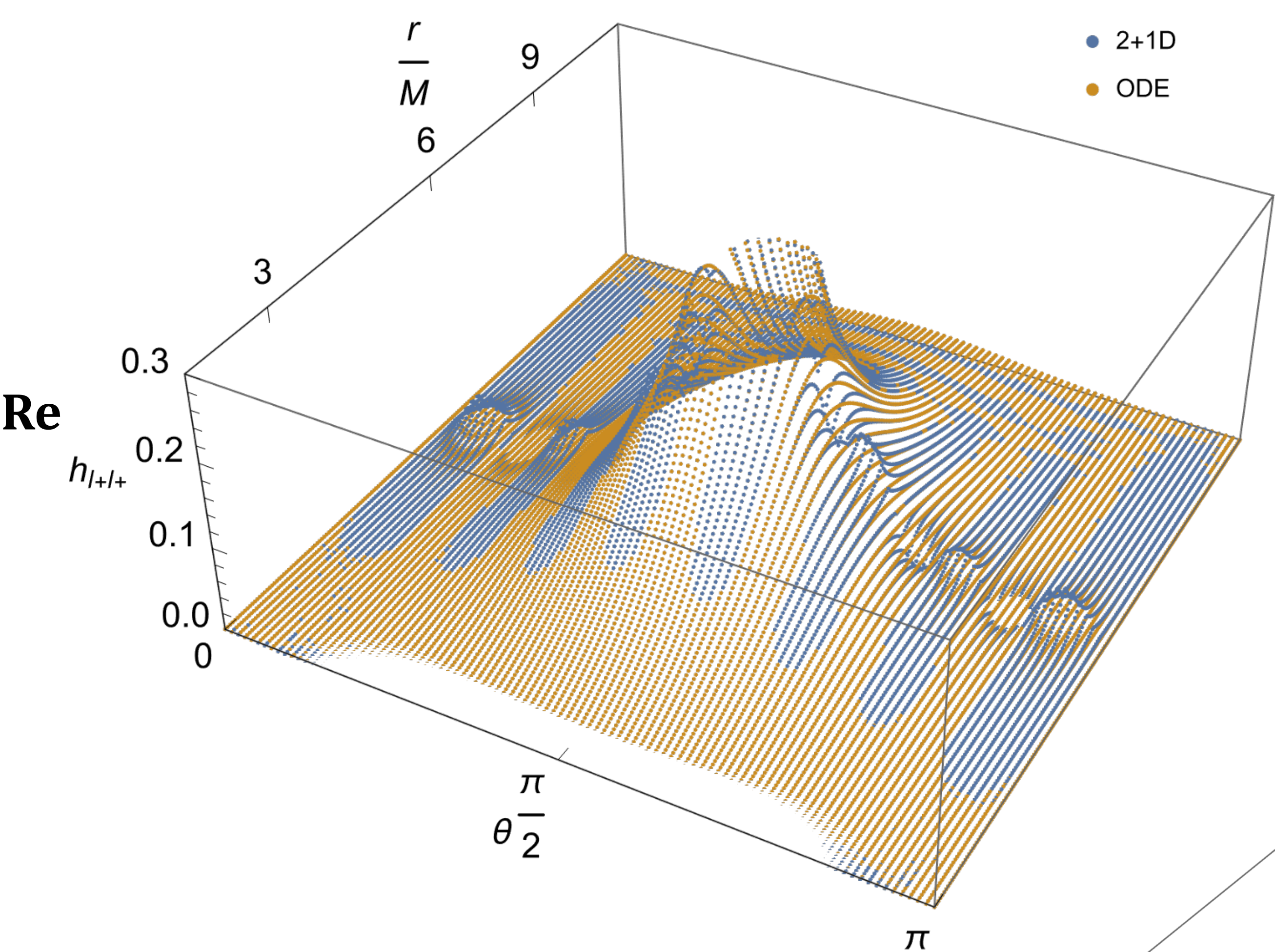


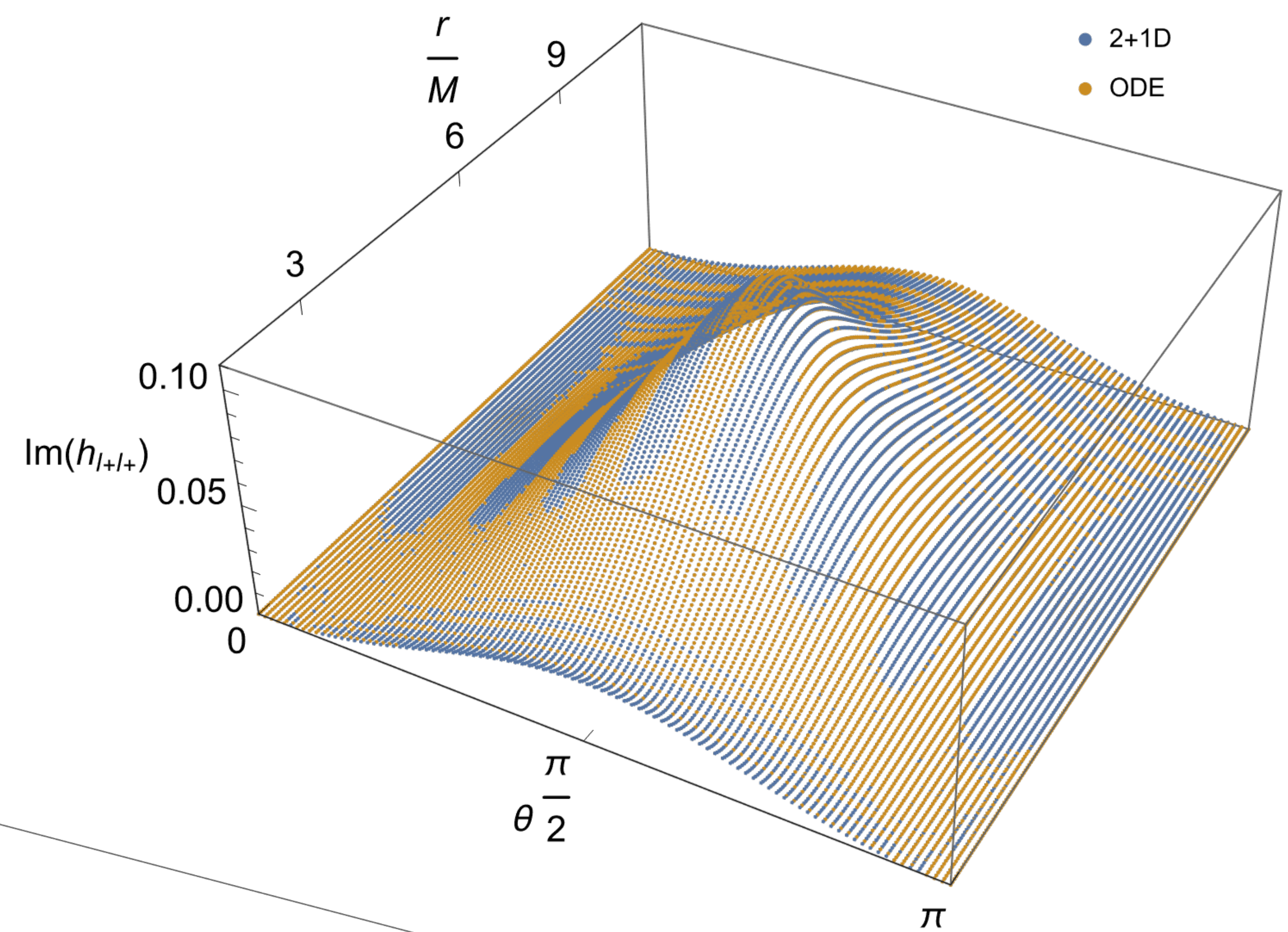
l = m = 2 radial modes.

a = **0.6 M**, r0 = 6M

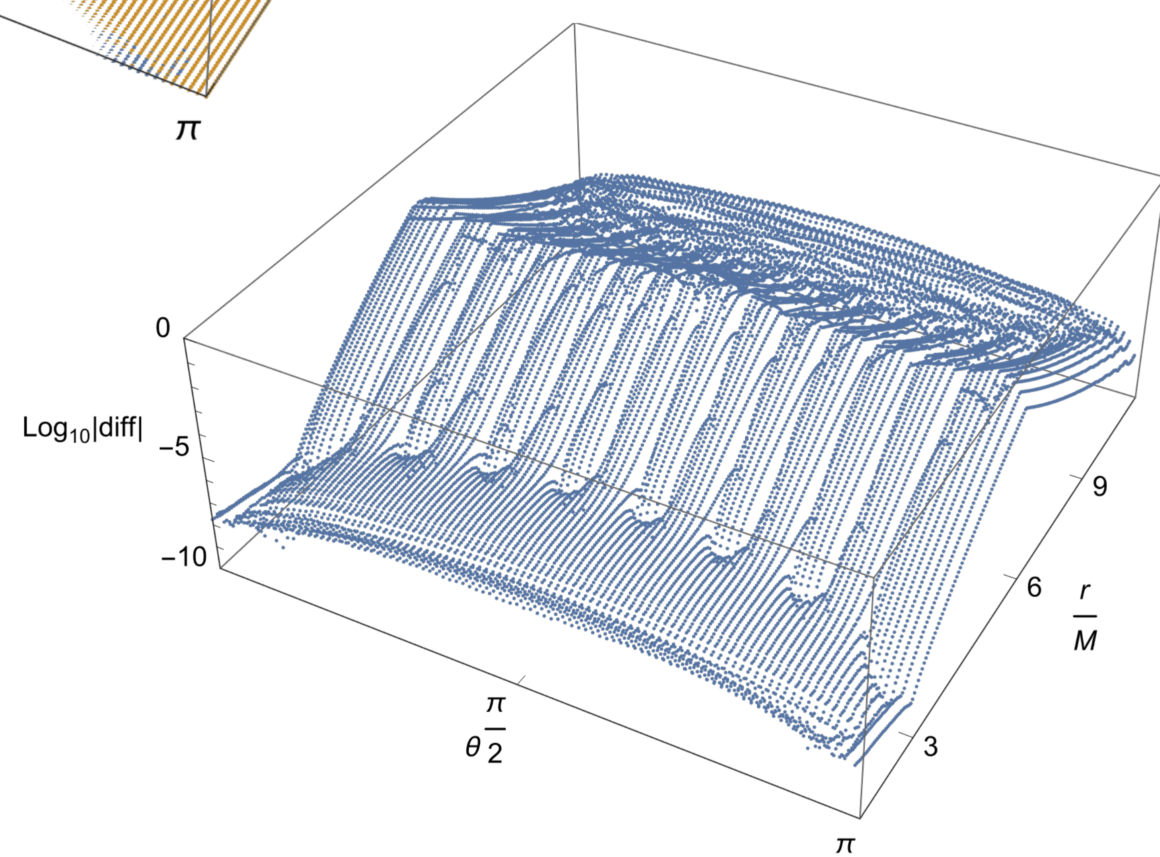
New results [dashed] vs
2+1D time domain data [solid]

Results: m=2 in (r, θ) domain



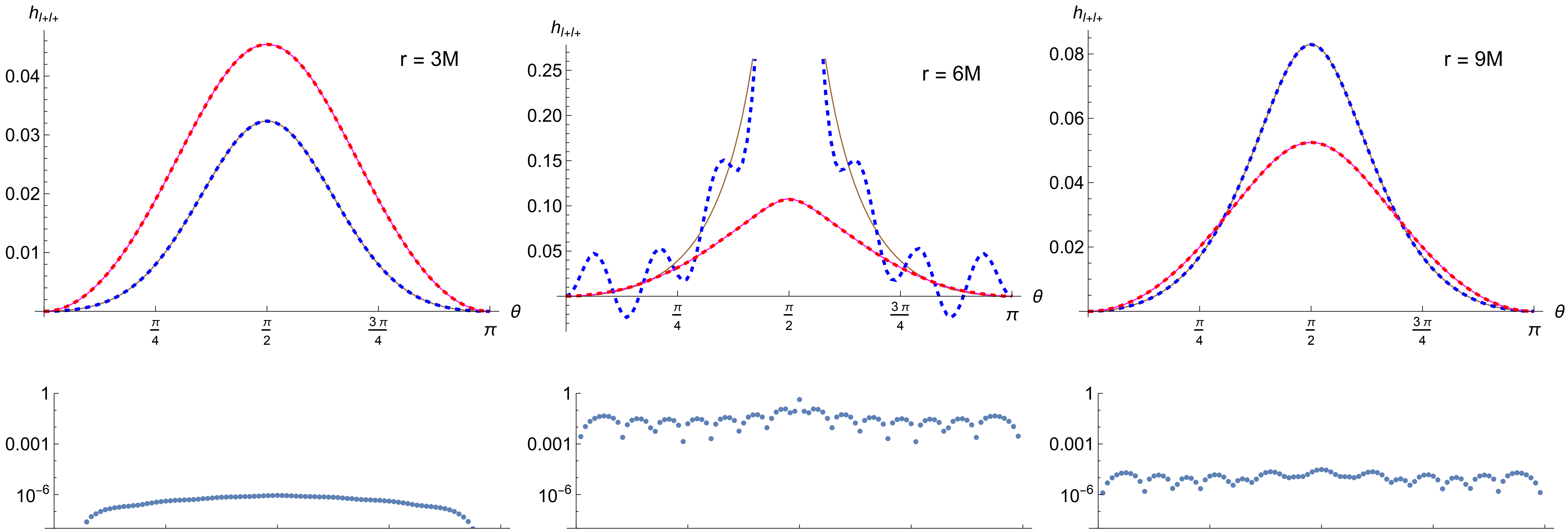


difference between
2+1D data & new data
(log scale)



$\ell_{\max} = 14$

Results: m=2



angular profiles at

r = 3M $r = r_0 = 6M$ r = 9M.

$\ell_{max} = 14$

Static axisymmetric modes:

$$\omega = m = 0$$

Static modes

- The radiation-gauge MPs are well-defined for $\omega = 0$, but the transformation to Lorenz gauge **breaks down** for $\omega = 0$.

- For $\omega = 0$ the difference between IRG and ORG MPs is trivial.

- We found an **alternative** gauge transformation from radiation to Lorenz gauge in this case, **and** an alternative construction for the s=0 trace modes (see paper / last year's Capra talk).

- The m = 0 (axisymmetric) metric perturbation is made from:

  - s = 2 Radiation-gauge -> Lorenz-gauge modes.

  - s = 0 trace modes.

  - **Completion pieces** in the $\ell = 0$, $\ell = 1$ sector.

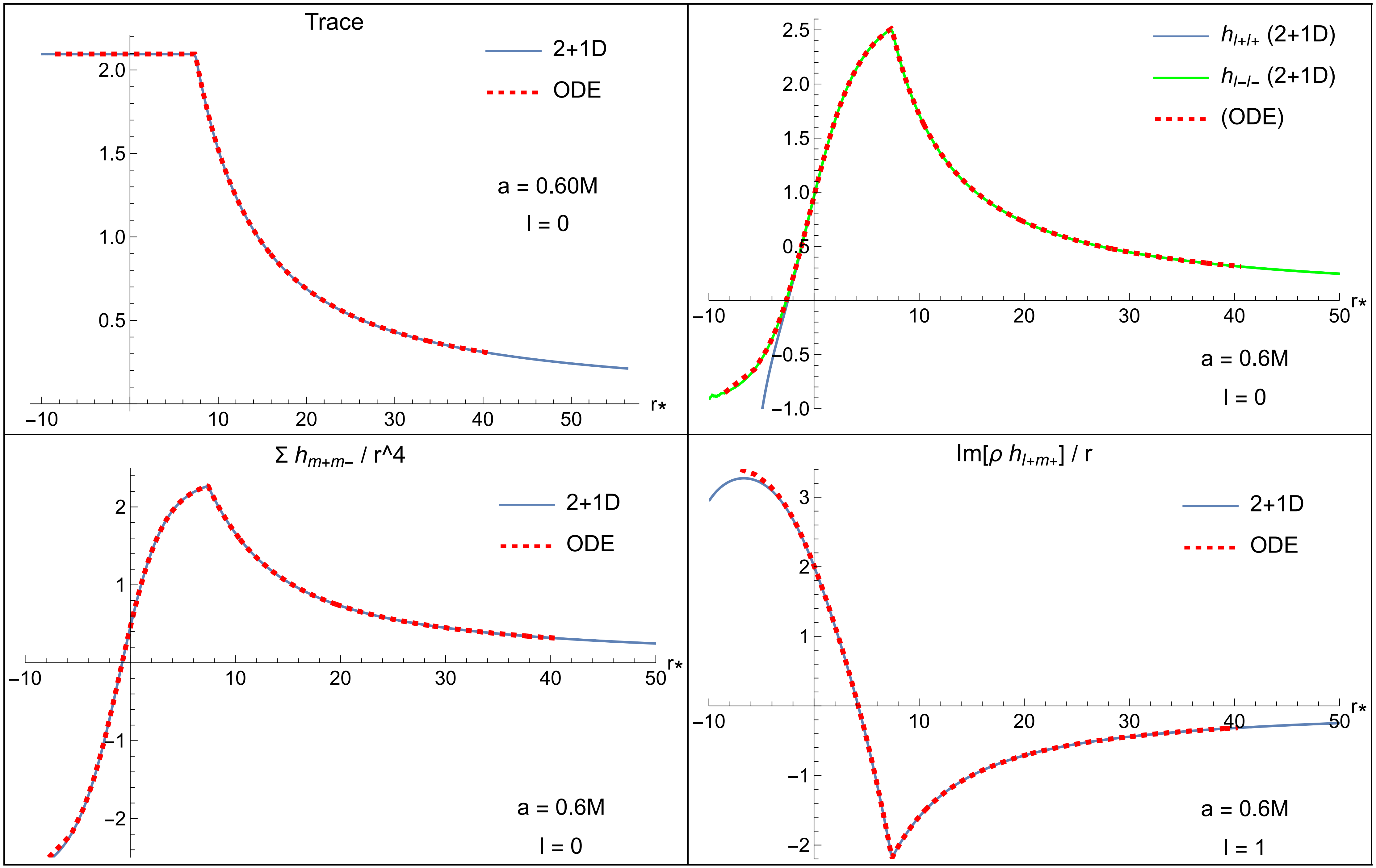
Completion pieces

Mode	Metric pert. or gauge vector	Trace h	$Q_{(t)}$	$Q_{(\phi)}$
(A)	$g_{\mu\nu}$	$h=4$	$M/2$	$-aM$
(B)	$\xi_\mu = \left[0, \frac{r(r^2+a^2)}{\Delta}, a^2 \sin\theta\cos\theta, 0\right] - 2Mr_+^2 \xi_\mu^{(C)}$	$h=6$	0	0
(C)	$\xi_\mu = [0, 1/\Delta, 0, 0]$	$h=0$	0	0
(D)	$\xi^\mu = [t, \frac{Ma^2\cos^2\theta}{\Sigma}, 0, 0] + M\nabla^\mu y$	$h = 2 + \frac{4M}{(r_+-r_-)} \ln\left(\frac{r-r_+}{r-r_-}\right)$	0	0
(E)	$\frac{\partial}{\partial M} g_{\mu\nu} - 2\nabla_\mu \nabla_\nu y$	$h = -\frac{4}{(r_+-r_-)} \ln\left(\frac{r-r_+}{r-r_-}\right)$	1	$-a$
(F)	$\xi^\mu = \left[2at, -\frac{aM(r^2-a^2)\cos^2\theta}{\Sigma}, 0, t\right] + \nabla^\mu z$	$h = 4\,a$	0	0
(G)	$\frac{\partial}{\partial a} g_{\mu\nu} - 2\xi_{(\mu;\nu)}, \quad \xi_\mu = [0, \frac{a(r\sin^2\theta + M\cos^2\theta)}{\Delta}, a\sin\theta\cos\theta, 0]$	$h=0$	0	$-M$

- The "jumps" in pieces (E) and (G) across $r = r_0$ are
  $\Delta c_E = E, \qquad \Delta c_G = (L - aE)/M$

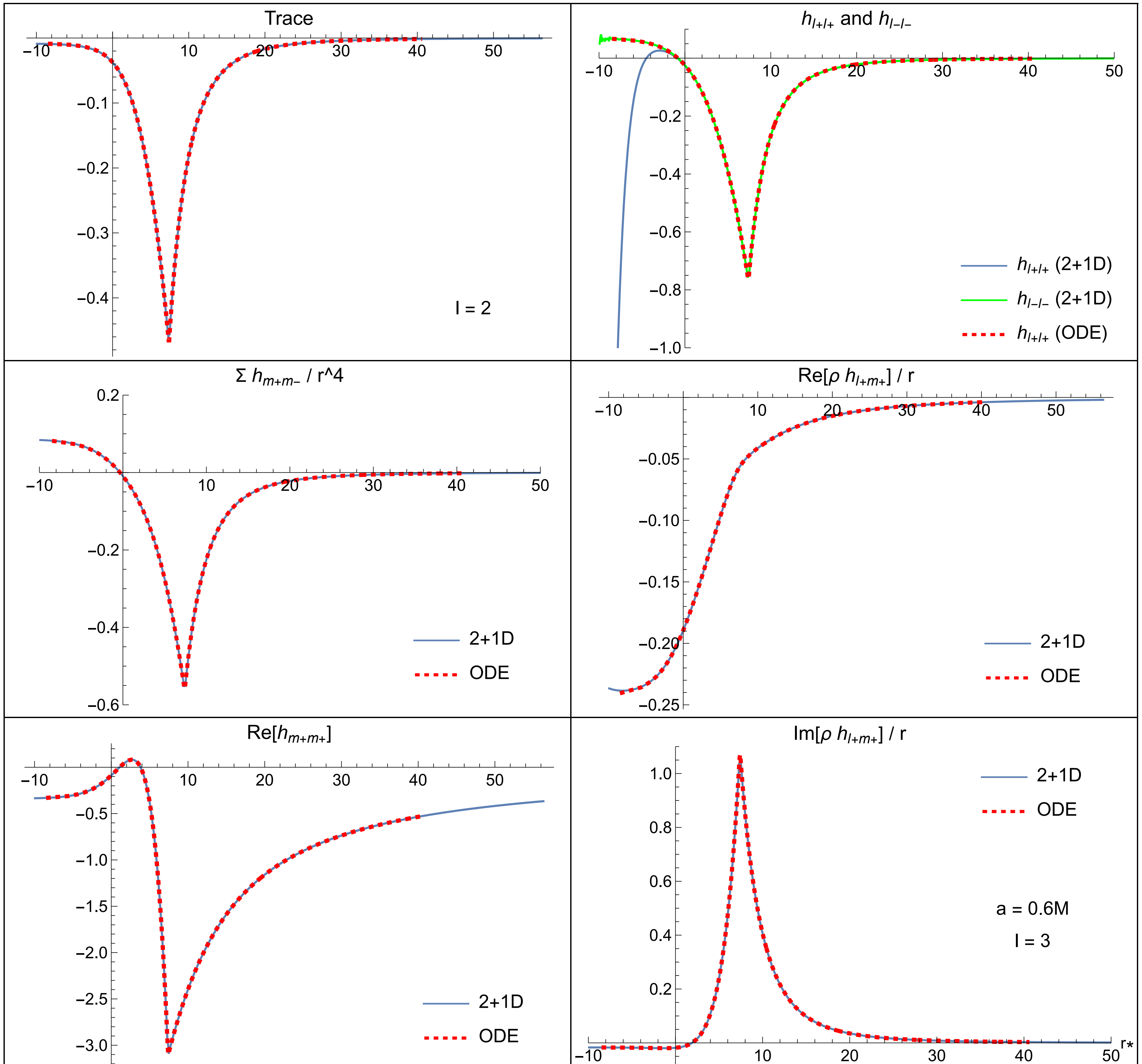
- The remaining jumps are determined from the regularity of the metric perturbation.

Results: m = 0 static sector



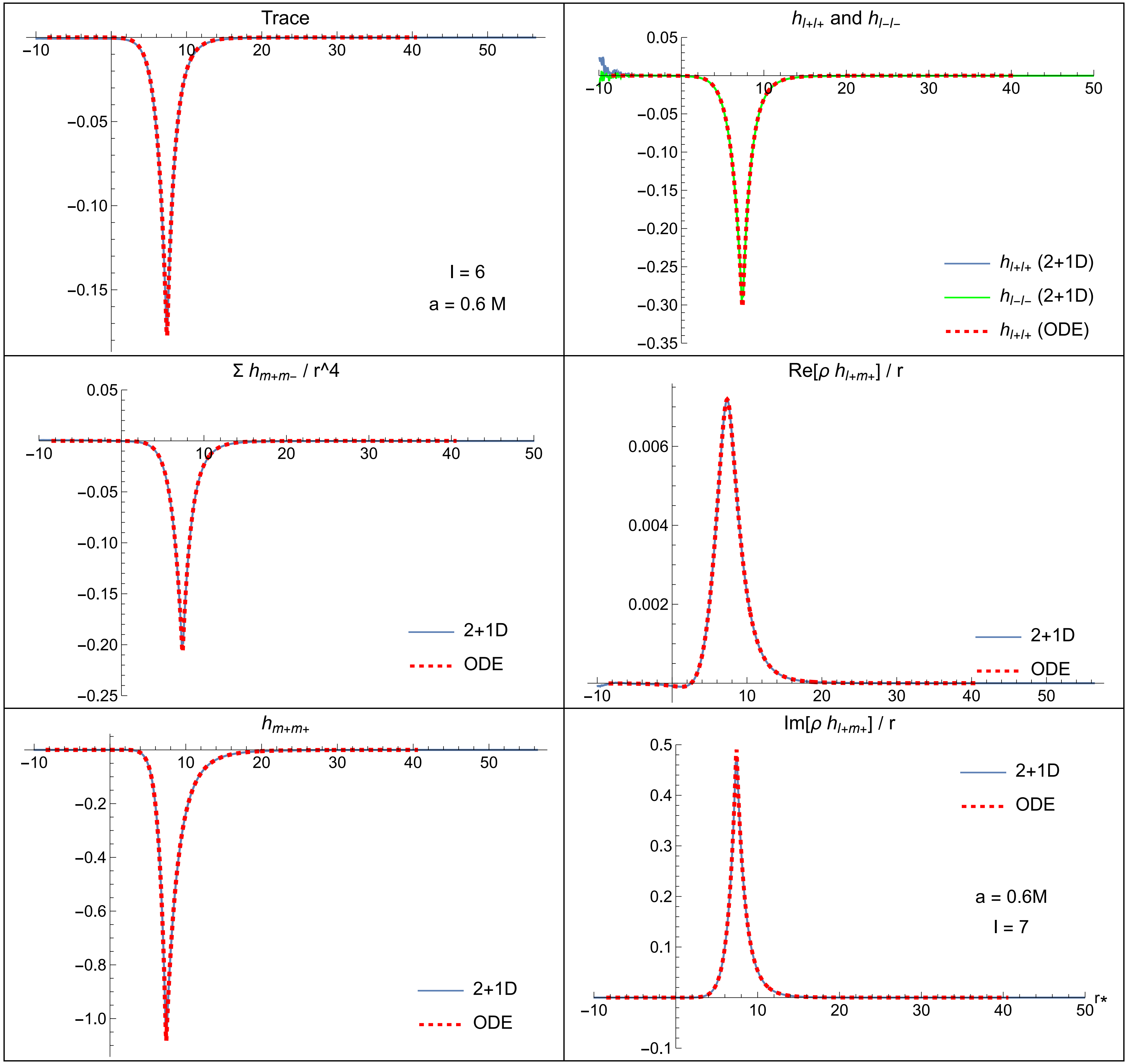
$$\ell = 0,\ 1$$

Results: m = 0 static sector



$\ell = 2, 3$

Results: m = 0 static sector



$\ell = 6, 7$

Conclusions

- We have found the Lorenz-gauge metric perturbation for a particle on a circular equatorial orbit of Kerr, via a sum over ℓm-modes in the frequency domain.
- It was sufficient to "glue together" vacuum perturbations & completion pieces at $r = r_0$. After projection this yields 4 equations + 16 consistency checks per $(\ell, \ell + 1)$.
- The metric perturbation is formed from (differential operators acting on) one-variable Teukolsky functions and auxiliary scalars that satisfy **ODEs**.
- We have successfully validated the MP against the 2+1D time domain data.
- The Mathematica code will be publicly released in due course.
- Extension to eccentric orbits should be feasible using the **extended homogeneous solutions** approach.
- Work is in progress (Wardell and Kavanagh) on obtaining Lorenz-gauge solutions for general (conserved) stress-energy tensors.

Extra slides

Solving the linearised Einstein equation with sources

Theorem by S Aksteiner, L Andersson and T Backdahl, *Phys. Rev. D* **99**, 044043 (2019)

Theorem IV.1. Let $\dot{g}_{ABA'B'}$ be a real solution to the linearized Einstein equations with linearized Weyl curvature $\vartheta\Psi_{ABCD}$ and linearized source $\vartheta\Phi_{ABA'B'}$ on a vacuum background of Petrov type D. Furthermore, let $\hat{\phi}_{ABCD} = \kappa_1^4(\mathcal{K}^1\mathcal{P}^2\vartheta\Psi)_{ABCD}$ be the modified linearized Weyl spinor and let

$$\mathcal{M}_{ABA'B'} = (\mathcal{C}^\dagger\mathcal{C}^\dagger_{(4,0)}\hat{\phi})_{ABA'B'}. \tag{55}$$

Then we have

gauge terms

$\partial_t h_{ab}$

$$\mathcal{M}_{ABA'B'} = \frac{1}{2}\nabla_{AA'}\mathcal{A}_{BB'} + \frac{1}{2}\nabla_{BB'}\mathcal{A}_{AA'} + \frac{1}{2}\Psi_2\kappa_1^3(\mathcal{L}_\xi\dot{g})_{ABA'B'} + (\mathcal{N}\vartheta\Phi)_{ABA'B'}, \tag{56}$$

1 on Kerr

where the complex one form $\mathcal{A}_{AA'}$ and the source term $(\mathcal{N}\vartheta\Phi)_{ABA'B'}$ are given by

$$(\mathcal{N}\vartheta\Phi)_{ABA'B'} = -(\mathcal{C}^\dagger(\kappa_1^4\mathcal{K}^1\mathcal{P}^{1/2}\mathcal{C}\vartheta\Phi))_{ABA'B'} + (\mathcal{C}^\dagger(\kappa_1^4\mathcal{K}^1\mathcal{P}^{3/2}\mathcal{C}\vartheta\Phi))_{ABA'B'} - 3\Psi_2\kappa_1^4(\mathcal{K}^1\vartheta\Phi)_{ABA'B'}.$$

Can be converted to a tensorial equation in (e.g.) GHP formalism

The components of $\hat{\phi}_{ABCD}$ in a principal dyad are

$$\begin{pmatrix}\hat{\phi}_0\\ \hat{\phi}_1\\ \hat{\phi}_2\\ \hat{\phi}_3\\ \hat{\phi}_4\end{pmatrix} = \begin{pmatrix}\kappa_1^4\vartheta\Psi_0\\ 0\\ 0\\ 0\\ -\kappa_1^4\vartheta\Psi_4\end{pmatrix}.$$

Note sign

AAB on Kerr

$$-i\omega\, h_{ab}^{(\mathrm{AAB})} = H_{ab}^{-} + N_{ab}$$

- $h_{ab}^{(\mathrm{AAB})}$ is a solution to the **sourced** equations (for $\omega \neq 0$).
- H_{ab}^{-} is made from second-derivative operators on the Weyl scalars Ψ_0 and $-\Psi_4$.
- The Weyl scalars satisfy the **sourced** Teukolsky equations.
- No Hertz potentials needed, because the inversion is straightforward ($\pounds_T = \partial_t$)
- N_{ab} is made from second-derivative operators on the trace-free stress-energy tensor.
- Hence N_{ab} has distributions on the worldline.
- $h_{ab}^{(\mathrm{AAB})}$ is **not** in Lorenz gauge.

“Lorenz-ification”: AAB -> Lorenz

- Apply the **vacuum** method globally:

    1. Transform the two parts of H^{-}_{ab} (radiation -> Lorenz), using the preceding method.

    2. Restore a trace part, using the preceding method.

- The MP is then in Lorenz gauge **except for on the worldline**.

- Now make a gauge transformation to global Lorenz gauge:

$$\Box\xi^a_{(s=1)} = -J^a, \qquad \nabla_a\xi^a = 0, \qquad \nabla_a J^a = 0,$$

$$J^a = -\left(H_{-}^{ab}{}_{;b} + h^{(s=2);b}_{ab}\right) - N^{ab}{}_{;b} + \Box\xi_{(s=0)}$$

Electromagnetism in Lorenz gauge,
with a source on the worldline.

Q. How to solve the EM-type equations for ξ?
A. Apply the **EM circularity relation** (a rewriting of Green and Toomani's method, Capra 24):

A circularity relation with sources

$$H^{ab} = \frac{2i}{\omega} F^{c[a} h^{b]}{}_c$$

Potential of the potential $\quad H^{ab}$

↓ Vacuum field eq. + Bianchi identity

Potential $\quad A^a \equiv \nabla_b H^{ab} + \frac{i}{\omega} h^a{}_b J^b \quad$ (not Lorenz gauge)

↓ Bianchi identity only

Field $\quad F_{ab} = \nabla_a A_b - \nabla_b A_a$

N.B. On this slide h_{ab} is the principal tensor / conformal Killing-Yano tensor.