Regge-Wheeler-Zerilli formalism and metric reconstruction

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26th Capra Meeting on Radiation Reaction in General Relativity
Niels Bohr Institute, Copenhagen

3 July 2023

Introduction and Motivation

- Linearised Einstein field equation

$$\mathcal{E}[h] = 8\pi T$$

- Instead, often solve scalar field equations for invariant master variables, but sometimes we need full h
- Teukolsky reconstruction formalism is elegant
- RWZ reconstruction is opaque

RWZ equations

- Scalar wave equations

$$(-\partial_t^2 + \partial_{r^*}^2 - V_{odd}^{lm})\Psi_{odd}^{lm} = S_{odd}^{lm}$$
$$(-\partial_t^2 + \partial_{r^*}^2 - V_{even}^{lm})\Psi_{even}^{lm} = S_{even}^{lm}$$

- Gauge invariant master functions

$$\Psi_{odd}^{lm}(t,r) = \frac{2r}{(\ell-1)(\ell+2)}\left(\partial_r h_t^{lm} - \partial_t h_r^{lm} - \frac{2}{r}h_t^{lm}\right)$$
$$\Psi_{even}^{lm}(t,r) = \frac{2r}{\ell(\ell+1)}\left(K^{lm} + \frac{1}{\Lambda}(f^2 h_{rr}^{lm} - rf\partial_r K^{lm})\right)$$

where $\Lambda = \ell(\ell+1) - 2 + 6M/r$

RWZ reconstruction in literature

$$h_t^{lm} = \frac{f}{2}\partial_r(r\Psi_{odd}^{lm}) - \frac{1}{(\ell-1)(\ell+2)} r^2 f P_{lm}^t$$

$$h_r^{lm} = \frac{r}{2f}\partial_t \Psi_{odd}^{lm} + \frac{1}{(\ell-1)(\ell+2)} \frac{r^2}{f} P_{lm}^r$$

$$K^{lm} = (f\partial_r + A)\Psi_{even}^{lm} - \frac{4r^2 f^2}{\ell(\ell+1)\Lambda} Q_{lm}^{tt}$$

$$h_{rr}^{lm} = \left[\frac{\Lambda\ell(\ell+1)}{4rf^2} + (\frac{r}{f}\partial_r - \frac{\Lambda}{2f^2})(f\partial_r + A)\right]\Psi_{even}^{lm} - (\frac{r}{f}\partial_r - \frac{\Lambda}{2f^2})\frac{4r^2 f^2}{\ell(\ell+1)\Lambda} Q_{lm}^{tt}$$

$$h_{tr}^{lm} = \left[r\partial_t\partial_r + rB\partial_t\right]\Psi_{even}^{lm} - \frac{2r^2}{\ell(\ell+1)} Q_{lm}^{tr} - \frac{4r^3 f}{\Lambda\ell(\ell+1)}\partial_t Q_{lm}^{tt}$$

$$h_{tt}^{lm} = f^2 h_{rr}^{lm} + f Q_{lm}^{\sharp}$$

where

$$A = \frac{2}{r\Lambda}[(\frac{1}{2}\ell(\ell+1) - 1)\frac{1}{2}\ell(\ell+1) + \frac{3M}{r}(\frac{1}{2}\ell(\ell+1) - 1 + \frac{2M}{r})]$$

$$B = \frac{2}{rf\Lambda}[(\frac{1}{2}\ell(\ell+1) - 1)(1 - \frac{3M}{r}) - \frac{3M^2}{r^2}]$$

Vacuum reconstruction: Teukolsky (CCK)

- Operator identity (Wald)

$$\hat{\mathcal{O}}\psi_0 = \mathcal{S}$$
$$\implies \hat{\mathcal{O}}\hat{\mathcal{T}}[h] = \hat{\mathcal{S}}\hat{\mathcal{E}}[h]$$
$$\implies \hat{\mathcal{O}}\hat{\mathcal{T}} = \hat{\mathcal{S}}\hat{\mathcal{E}}$$

- For Hertz potential Φ, if

$$\hat{\mathcal{O}}^\dagger\Phi = 0$$

then solution to $\mathcal{E}[h] = 0$ is

$$h_{\alpha\beta} = \Re\hat{\mathcal{S}}^\dagger_{\alpha\beta}\Phi$$

- Circularity condition

$$\psi_0 = \hat{\mathcal{T}}\hat{\mathcal{S}}^\dagger\Phi$$

RW equation in 4D

- Some formulations in math literature
- Choose to write manifold $\mathcal{M} = \underbrace{\mathcal{M}^2}_{a=t,r} \times \underbrace{S^2}_{A=\theta,\phi}$

$$\hat{O}\Psi = S$$

where $\hat{O} := (\Box + \frac{8M}{r^3})$ and

$$\Psi = \sum_{lm} (\ell - 1)\ell(\ell + 1)(\ell + 2)\frac{1}{r}\Psi^{lm}_{odd}(t, r)Y_{lm}(\theta, \phi) := \hat{\Psi}[h]$$

$$S = \sum_{lm} (\ell - 1)\ell(\ell + 1)(\ell + 2)\frac{1}{r}S^{lm}_{odd}(t, r)Y_{lm}(\theta, \phi) := \hat{S}[T]$$

- Note: $\hat{O}^\dagger = \hat{O}$

Odd-parity: vacuum

- Circularity condition from $\Psi = \hat{\Psi}[h]$

$$\begin{aligned}\Psi &= \hat{\Psi}\hat{S}^{\dagger}\Phi \\ &= -8(r^2 D^2 \hat{O} - \mathcal{D})\Phi \\ &= 8\mathcal{D}\Phi\end{aligned}$$

- Purely angular relation, where

$$\mathcal{D} := D^2(D^2 + 2)$$

- Reconstruct metric from $h = S^{\dagger}\Phi$

$$h_{aB} = -4\epsilon_a{}^b \epsilon_B{}^A D_A \nabla_b (r^2 \Phi)$$

Odd-parity: operator symmetries

- Operator relations

$$\hat{\Psi}^\dagger = \frac{1}{2} r^{-2} \hat{S}^\dagger (r^2 \cdot) = \frac{1}{2} r^{-4} \hat{S}(r^4 \cdot) = r^{-6} \hat{\Psi}(r^6 \cdot)$$

  very useful
- Odd-parity analogous with electromagnetic ($s = 1$) perturbations in Kerr

Odd-parity: nonvacuum

- Reconstructed metric includes corrector tensor (GHZ)

$$h_{\alpha\beta} = S^{\dagger}_{\alpha\beta}\Phi + x_{\alpha\beta}$$

- Circularity condition and Einstein equation

$$\Psi = 8\mathcal{D}\Phi - 8r^2 D^2 S_{\Phi} + \hat{\Psi} x$$
$$8\pi T = \hat{\Psi}^{\dagger} S_{\Phi} + \mathcal{E} x$$

- Freedom to demand $\Psi = 8\mathcal{D}\Phi$, now have

$$8r^2 D^2 S_{\Phi} = \hat{\Psi} x$$
$$\mathcal{D}(8\pi T) = \frac{1}{8}\hat{\Psi}^{\dagger}\hat{S}(8\pi T) + \mathcal{D}\mathcal{E} x$$

Odd-parity: nonvacuum (continued)

- Conditions on x

$$\hat{\Psi} x = 8 r^2 D^2 S_\Phi$$

$$\mathcal{D}\mathcal{E} x = \mathcal{D}(8\pi T) - \frac{1}{8}\hat{\Psi}^\dagger \hat{S}(8\pi T)$$

- Satisfied by any solution to

$$(D^2 + 1) x_{aB} = 16\pi (r^2 T_{aB} + f_{aB})$$

- Solution for x_{aB} is unique up to a $\ker[D^2 + 1]$; an $\ell = 1$ vector harmonic

Summary

- Solved: Odd-parity vacuum and non-vacuum
- Next steps: Ideas and progress on even-parity
- Next steps: Further analogy with Teukolsky reconstruction
  - Bridge between mathematics and GW physics communities

Conclusions

- Operator algebraic formulation
  - provides geometric insight
  - clarifies structure
  - helps facilitate general derivations
- Foundation could enable extensions
  - to studies in modified gravity theories
  - to 2nd order GSF

Thank you