Effective actions of compact objects in GR

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Background: Effective actions are useful to treat compact objects as effective point particles

Effective action provides systematic way to include finite size effects to the point particle

$$S_{pp} = -\int m\,\mathrm{d}\tau + c_E \int E_{ab}E^{ab}\,\mathrm{d}\tau + c_B \int B_{ab}B^{ab}\,\mathrm{d}\tau + \dots$$

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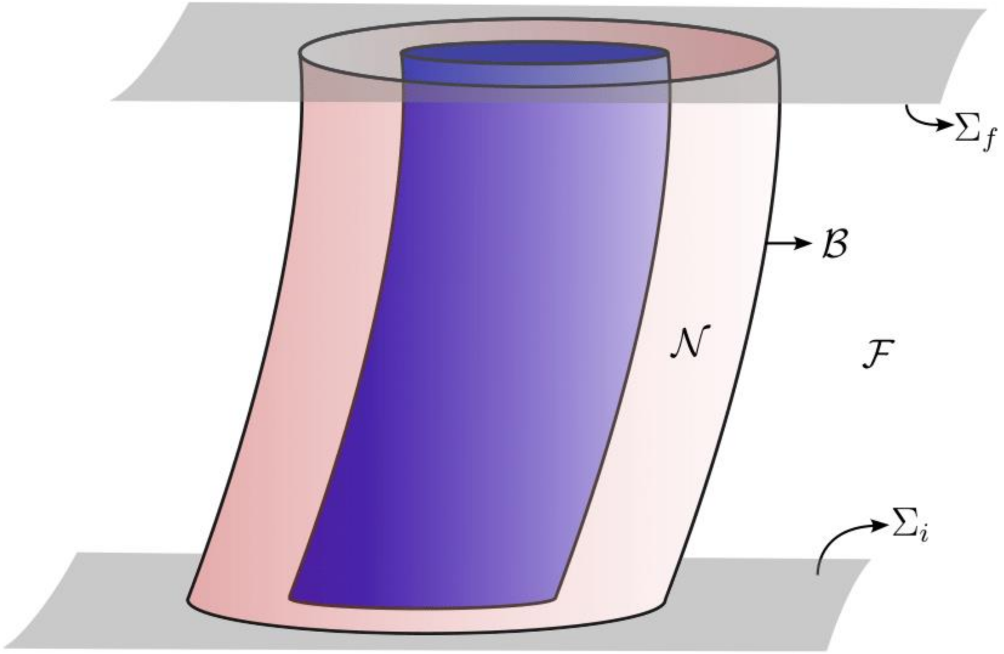
How is the full GR solution related to the point particle action?

Goal: Derive and effective action for compact object from first principles

1. Derive effective action for a scalar field in Minkowski space
2. Derive Effective action for GR
3. Derive Effective action for Black holes
4. Hidden symmetries?

We can split spacetime into two regions and get and effective action for the outer fields

$$S[\phi] = -\frac{1}{2}\int \partial_\mu\phi\partial^\mu\phi \,\mathrm{d}^4x + \int J\phi \,\mathrm{d}^4x \,,$$



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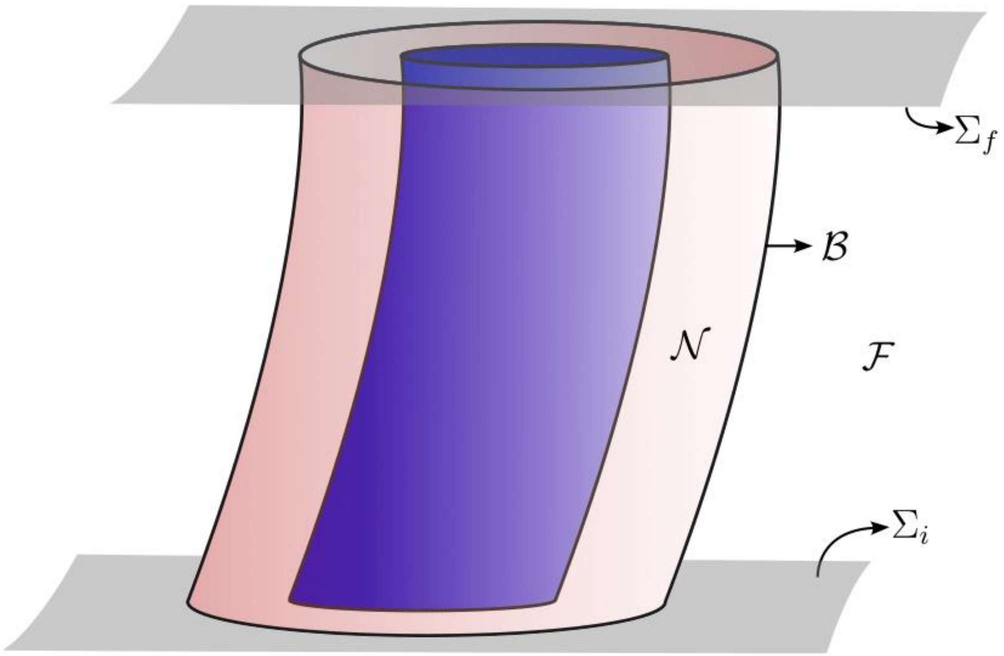

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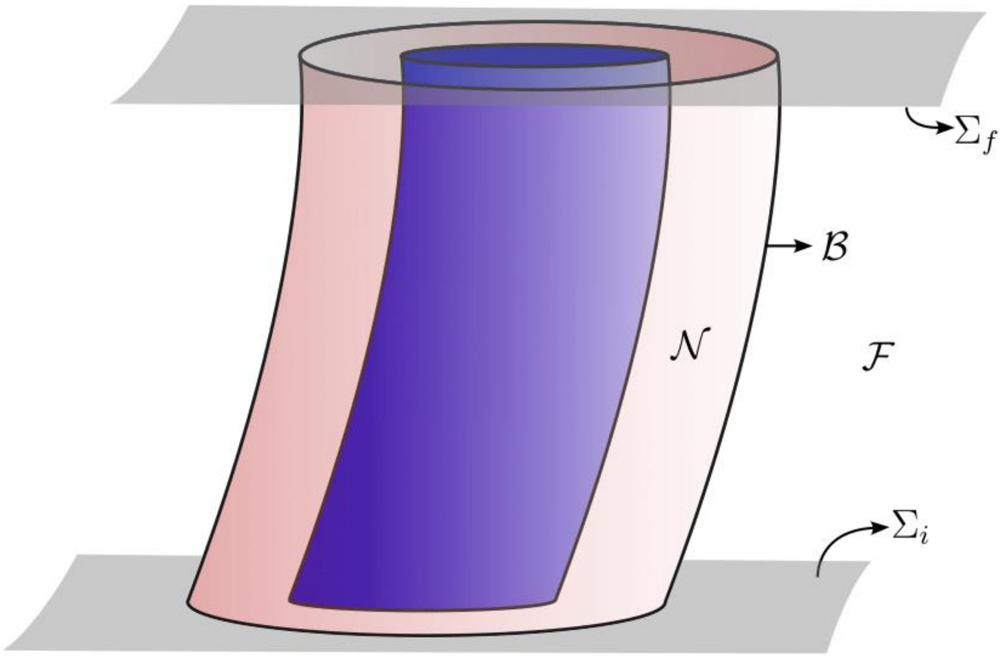

We can minimize $S[\phi]$ with the following procedure:

1. fix $\phi^{\mathcal{B}} = \phi\big|_{\mathcal{B}}$ and calculate $S_{\min} = \min\left(S_{\mathcal{N}}\right)$
2. $S_{\text{eff}} = S_{\mathcal{F}} + S_{\min}[\phi^{\mathcal{B}}]$

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While the value of the action does depend on the detailed structure of the body, for periodic systems this structure only affects an irrelevant constant

Example: A static linear dipole

$$\phi_{\mathcal{B}} = \phi_{\circ}(t) + r_{\mathcal{B}}\tilde{E}_a(t)\Omega^a + r_{\mathcal{B}}^2\tilde{E}_{ab}(t)\Omega^a\Omega^b + \dots$$

Spherical harmonic components

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$$S_{\min}[\phi] = \int \mathrm{d}t \, \tilde{E}_a \tilde{E}^a \left(\frac{4\pi r_{\mathcal{B}}^3}{6} + \sum_{n=1}^{\infty} (-1)^n \frac{\chi^n}{(4\pi r_{\mathcal{B}}^3)^{n-1}} L^{3n} \right)$$

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We send $r_{\mathcal{B}} \to 0$, but we must always have $r_{\mathcal{B}} > L$

Limit is formal

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Point particle limit of example:

$$S_{pp}[\phi] = -\frac{\chi L^3}{2} \int \mathrm{d}t \, E_a E^a$$

Divergent number

As $r_{\mathcal{B}} \to 0$, the moments $\tilde{E}_a \to E_a = \partial_a \phi$

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Only finite $r_{\mathcal{B}}$ with $r_{\mathcal{B}} \ll \mathcal{R}$ is physical. $r_{\mathcal{B}} \to 0$ limit is formal, introducing infinities that need to be regularized

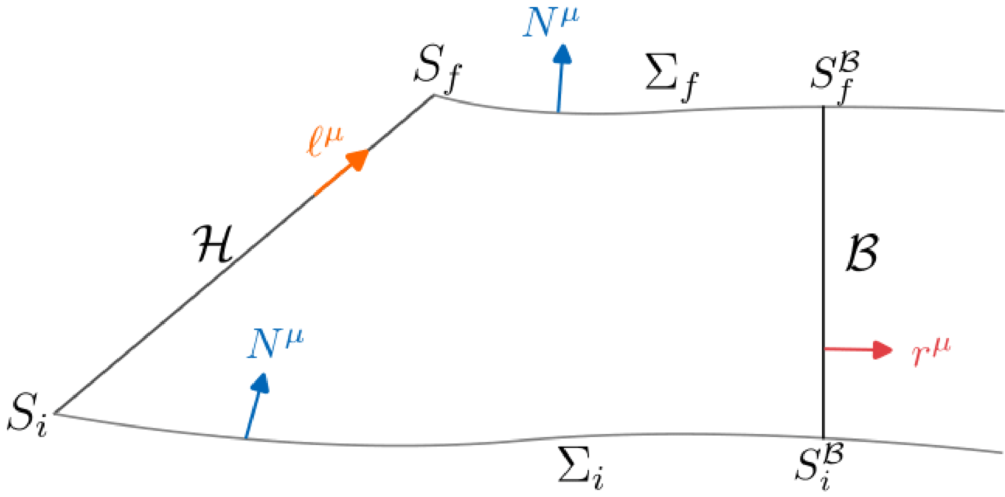
Effective action for GR

In GR, we need to add many boundary terms

$$S_{EH}[g_{\mu\nu}] = S_{EH}^{\mathcal{N}}[g_{\mu\nu}] + \frac{1}{8\pi}\int_{\mathcal{B}} K\mathrm{d}^3V + \frac{1}{8\pi}\int_{S_f^{\mathcal{B}}} \eta d^2V - \frac{1}{8\pi}\int_{S_i^{\mathcal{B}}} \eta d^2V$$
$$- \frac{1}{8\pi}\int_{\mathcal{H}} \kappa_{(\ell)}\mathrm{d}\lambda\,\mathrm{d}^2V + \frac{1}{8\pi}\int_{S_f} a\,\mathrm{d}^2S - \frac{1}{8\pi}\int_{S_i} a\,\mathrm{d}^2V$$
$$+ S_{EH}^{\mathcal{F}}[g_{\mu\nu}] - \frac{1}{8\pi}\int_{\mathcal{B}} K\mathrm{d}^3V - \frac{1}{8\pi}\int_{S_f^{\mathcal{B}}} \eta d^2V - \frac{1}{8\pi}\int_{S_i^{\mathcal{B}}} \eta d^2V$$

['16 Lehner, Myers, Poisson, Sorkin]

Conceptually same as scalar field!



$$\ell^\mu \nabla_\mu \ell^\nu = \kappa_{(\ell)} \ell^\nu \qquad \ell^\mu \nabla_\mu \lambda = 1$$

$$\eta = \sinh^{-1}(\hat{r}^\mu N_\mu) \qquad a = \log(-\ell^\mu N_\mu)$$

Effective action for static black holes

In vacuum the action is simple enough to evaluate directly. No trickery required to make it a boundary term.

Vacuum Einstein Equations + Periodicity:

$$S_{\min}[g_{\mu\nu}] = -\frac{1}{8\pi}\int_{\mathcal{H}} \kappa_{(\ell)}\,\mathrm{d}^2V\mathrm{d}\lambda + \frac{1}{8\pi}\int_{\mathcal{B}} K\,\mathrm{d}^3V$$

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Affine parameter of KVF

$$\frac{1}{8\pi}\int_{\mathcal{B}} K \, \mathrm{d}^3 V = -\frac{1}{2}\int M \mathrm{d}\tau + \frac{1}{8\pi}\int_{\mathcal{B}} {}^2\pi \, \mathrm{d}^3 V$$

2D extrinsic curvature

Static effective action

Gibbons-Hawking York term of
the far-zone metric
(not evaluated on a solution)

$$S_{\text{eff}}[g_{\mu\nu}] = \frac{1}{16\pi}\int_{\mathcal{F}} R\mathrm{d}^4V - \int M\mathrm{d}\tau + \frac{1}{8\pi}\int_{\mathcal{B}} \left({}^2\pi[h^{\mathcal{B}}_{ab}] - K\right) d^3V$$

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UV limit: shrink the tube as much as possible

$$\lim_{r\to r_{\mathcal{H}}}\int_{\mathcal{B}} {}^2\pi\,\mathrm{d}^3V = \int_{\mathcal{H}} \theta_{(\ell)}\mathrm{d}^2V\mathrm{d}\tau = 0$$

The point particle limit is obtained by perturbing the length scale of the horizon around 0

$$S_{\text{eff}}[g_{\mu\nu}] = \frac{1}{16\pi}\int_{\mathcal{F}} R \mathrm{d}^4 V - \int M \mathrm{d}\tau$$

No approximations!

Only assumed existence of static symmetry.

No tidal terms at any order

Conclusions

- The point-particle action is a formal limit of the physical effective action.
- Other methods of regularizing the point particle action , if they are indeed equivalent to finite $r_{\mathcal{B}}$ ‘regularization’, could be used for more efficient computation.
- If the null generator of the horizon is a killing symmetry, the effective action is the associated komar charge
- The vanishing of the static coefficients for a non-rotating black hole come from the static killing symmetry.

Backup slides

The variation of the action depends only on the boundary for periodic perturbations

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\begin{aligned}
\delta S^{\mathcal{N}}[g_{\mu\nu}] = &-\frac{1}{16\pi}\int_{\mathcal{B}}(K_{ab}-Kh_{ab})\delta h^{ab}\,\mathrm{d}^3V \\
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Calculation of effective action will follow exact procedure of scalar field