Assessing the importance of first post-adiabatic terms for LISA data analysis of EMRIs and IMRIs

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Intro

Two-time scale expansion orbital phase ($q = \mu/M$)

$$\Phi_{\text{GW}} = \underbrace{q^{-1}\mathcal{C}^{(0)}}_{\text{adiabatic}} + \underbrace{q^{-1/2}\mathcal{C}^{(1/2)}}_{\text{resonances}} + \underbrace{q^{0}\mathcal{C}^{(1)}}_{\text{post-1-adiabatic}} + \mathcal{O}(q)$$

$\mathcal{C}^{(0)}$ contains leading dissipative self-force terms
$\mathcal{C}^{(1)}$ contains sub-leading self force plus secondary spin terms

For circular equatorial in Schwarzschild we (almost have) all $\mathcal{C}^{(1)}$![1]

What is the impact of 1PA terms for LISA data analysis?

We assessed impact 1PA effect with a Fisher matrix approach

[1] Wardell+, arXiv:2112.12265

Outline of the presentation

1. The 1PA Waveform
2. Parameter estimation
3. Conclusions and future perspective

The 1PA Waveform

Post-adiabatic fluxes

We consider circular, equatorial orbits with spin aligned.

Small corrections due secondary spin χ and 2SF

$$E = E^0 + \sigma E^1_{spin} + qE_{2SF} \qquad \Omega(r) = \Omega^0(r)$$

with $\sigma := S/(\mu M) = \chi q$.

Gravitational wave fluxes

$$\mathcal{F}(r,\Omega) = \mathcal{F}^0(r,\Omega^0) + \sigma \mathcal{F}^1_{spin}(r,\Omega^0) + q\mathcal{F}^1_{2SF}(r,\Omega^0)$$

$\mathcal{F}^0$ and $\mathcal{F}^1_{spin}$ computed with Teukolsky formalism[2].
$\mathcal{F}^1_{spin}$ transformed in fixed frequency parametrization
$\mathcal{F}^1_{2SF}$ computed here [3] while E_{2SF} here [4]

[2] Piovano+ , Phys. Rev. D 104, 124019 (2021)
[3] Wardell+, Phys. Rev. Lett. 127, 151102 (2021)
[4] Pound+, Phys. Rev. Lett. 124, 021101 (2020)

Radiation reaction equations

The orbital phase ϕ_p and frequency Ω satisfy [5]

$$\frac{\mathrm{d}\phi_p}{\mathrm{d}t} = \Omega \qquad \frac{\mathrm{d}\Omega}{\mathrm{d}t} = q[F_0(\Omega(t)) + qF_1(\Omega(t))]$$

Assuming $\phi_p(\Omega) = q^{-1}\tilde{\phi}_0(\Omega) + \tilde{\phi}_1(\Omega)$, we get

$$\frac{\mathrm{d}\tilde{\phi}_0}{\mathrm{d}\Omega} = \frac{\Omega}{F_0(\Omega)} \qquad \frac{\mathrm{d}\tilde{\phi}_1}{\mathrm{d}\Omega} = -\frac{\Omega F_1(\Omega)}{F_0^{\Omega}(\Omega)^2}$$

A similar result can be obtain for $\tilde{t}(\Omega) = q^{-1}\tilde{t}_0(\Omega) + \tilde{t}_1(\Omega)$

$$\frac{\mathrm{d}\tilde{t}_0}{\mathrm{d}\Omega} = \frac{1}{F_0(\Omega)} \qquad \frac{\mathrm{d}\tilde{t}_1}{\mathrm{d}\Omega} = -\frac{F_1(\Omega)}{F_0(\Omega)^2}$$

[5] Wardell+, arXiv:2112.12265

Frequency domain waveform with the SPA

We use the Stationary Phase Approximation (SPA) for frequency-domain waveform

$$\tilde{h}_\alpha(f) = \frac{\mu}{D} \sum_{\ell m} \left[\mathcal{A}^0_{\alpha,\ell m}(\tilde{t}(f)) + q\mathcal{A}^1_{\alpha,\ell m}(\tilde{t}(f)) \right] \sqrt{\frac{2\pi}{q|mF_0(f)|}} e^{-i(\tilde{\Phi}_m(f) \pm \pi/4)}$$

$$\tilde{\Phi}_m(f) = 2\pi f (q^{-1}\tilde{t}_0(f) + \tilde{t}_1(f) - t_0) - m(q^{-1}\tilde{\phi}_0(f) + \tilde{\phi}_1(f) - \phi_0) - \phi^{\text{Dop}}(\tilde{t}(f))$$

$\alpha = I, II$ for the two LISA channels (long-wavelength approximation)

$\phi^{\text{Dop}}(\tilde{t}(f))$ is the Doppler shift

Excellent agreement with Fast-Fourier Transform (FFT)

Parameter estimation

Error estimate with Fisher Information matrices

In the high SNR limit

$$\Sigma_{ij} = (\Gamma^{-1})_{ij} \qquad \text{covariance matrix}$$

$$\Gamma_{ij} = \sum_{\alpha=I,II} \left(\frac{d\tilde{h}_\alpha}{dx^i} \,\middle|\, \frac{d\tilde{h}_\alpha}{dx^j} \right)_{\vec{x}=\vec{x}_0} \qquad \text{Fisher matrix}$$

$$(p_\alpha | q_\alpha) = 4\mathrm{Re} \int_{f_{\min}}^{f_{\max}} \frac{df}{S_n(f)} \tilde{p}^*_\alpha(f) \tilde{q}_\alpha(f) \qquad \text{scalar product}$$

10 parameters for circular equatorial orbits with spins (anti)aligned

- intrinsic: masses $(\ln M, \ln \mu)$, spin χ, r_0
- extrinsic: ϕ_0, angles $(\vartheta_S, \varphi_S, \vartheta_K, \varphi_K)$, distance $\ln D$

Biases Δx^i_{th} à la Cutler-Vallisneri [6]

$$\Delta x^i_{\mathrm{th}} \approx \left(\Gamma^{-1}(\vec{x})\right)^{ij} \left(\frac{\partial h_{\mathrm{app}}}{\partial x^j} \,\middle|\, \sum_{\ell m} \left[\Delta \mathcal{A}_{\ell m} + i \mathcal{A}_{\ell m} \Delta \tilde{\Phi}_m \right] e^{-i \tilde{\Phi}_m(f)} \right)$$

[6] Phys. Rev. D 76, 104018

Semi-analytic derivatives of waveforms with the SPA

$$\frac{\partial \tilde{h}(f;\vec{x})}{\partial x^j} = \sum_{\ell m}\left[\frac{\partial \mathcal{A}_{\ell m}(f;\vec{x})}{\partial x^j} + i\mathcal{A}_{\ell m}(f;\vec{x})\frac{\partial \tilde{\Phi}_m(f;\vec{x})}{\partial x^j}\right]e^{-i\tilde{\Phi}_m(f;\vec{x})}$$

Implicit dependence on $\vec{x} = (\ln M, \ln \mu, \chi)$ in $\tilde{t}(f;\vec{x})$, $\phi(\tilde{t}(f;\vec{x});\vec{x})$, $r(\tilde{t};\vec{x})$

By the theorem of the implicit functions

$$\frac{\partial \tilde{t}(f;\vec{x})}{\partial x^i} = -\frac{1}{\dot{\Omega}(t;\vec{x})}\frac{\partial \Omega(t;\vec{x})}{\partial x^i}\bigg|_{t=\tilde{t}(f;\vec{x})}$$

$\phi(\tilde{t}(f;\vec{x});\vec{x})$ and $r(\tilde{t};\vec{x})$ are solutions of ODEs like

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \mathscr{G}(y(t);\vec{x}) \qquad y(0) = y_0$$

then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial y(t;x^i)}{\partial x^i}\right) = \frac{\partial \mathscr{G}(y(t),x^i)}{\partial y}\frac{\partial y(t;x^i)}{\partial x^i} + \frac{\partial \mathscr{G}(y(t),x^i)}{\partial x^i} \qquad \frac{\partial y(0;x^i)}{\partial x^i} = 0$$

Statistical errors 0PA

Masses: $M = 10^6 M_\odot, \mu = 10 M_\odot$. $\chi = 0$. 2 years observation

$\mu[M_\odot]$	$\ln M$	$\ln \mu$	$\hat{r}_0$	ϕ_0	$\Delta\Omega_S$	$\Delta\Omega_K$	$\ln D$
10	-5.1	-4.8	-5.5	-0.81	-3.0	-1.9	-1.4
100	-5.3	-5.2	-6.1	-1.1	-3.4	-2.7	-1.8
1000	-5.2	-5.3	-6.6	-1.6	-4.2	-3.6	-2.3

Table: $\log_{10}$ statistical errors 0PA waveforms

Adding second order fluxes does not affect the statistical errors

Statistical errors 1PA

Masses: $M = 10^6 M_\odot, \mu = 10 M_\odot$. $\chi = 1$. 2 years observation

$\mu[M_\odot]$	prior	$\ln M$	$\ln \mu$	χ	r_0/M	ϕ_0	$\Delta\Omega_S$	$\Delta\Omega_K$	$\ln D$
10	no	-2.8	-3.1	2.6	-3.8	-0.79	-3.0	-1.9	-1.4
	yes	-5.0	-4.8	0	-5.5	-0.81	-3.0	-1.9	-1.4
100	no	-3.1	-3.3	1.3	-4.7	-1.0	-3.4	-2.7	-1.8
	yes	-4.4	-4.6	0	-5.9	-1.2	-3.4	-2.7	-1.8
1000	no	-3.7	-3.9	-0.39	-5.9	-1.5	-4.2	-3.5	-2.3
	yes	-3.8	-4.0	-0.42	-5.9	-1.5	-4.2	-3.5	-2.3

Table: $\log_{10}$ statistical errors 1PA waveforms with and without a prior on χ.

Errors on the source parameters with prior

$$\sigma_{x_i}^2 = \Sigma_{ii} = [(\Gamma + \Gamma_0)^{-1}]_{ii} \quad (\Gamma_0)_{ij} = 1/\sigma_0 \delta_{i\chi}\delta_{\chi j}$$

Removing second order fluxes does not affect the statistical errors

Biases on the parameters - 0PA vs 1PA

Ratio $\mathcal{R}$ systematic over statistical errors

$$\mathcal{R} = \frac{|\Delta x^i_{\rm th}|}{\sigma_{x_i}}$$

0PA waveform (approximation) vs 1PA waveform ("true" model, $\chi = 1$)

Masses: $M = 10^6 M_\odot, \mu = 10 M_\odot$. 2 years observation

$\mu[M_\odot]$	M	μ	r_0/M	ϕ_0	ϑ_S	φ_S	ϑ_K	φ_K	D
10	0.29	-0.27	1.3	-1.7	-1.2	-1.7	-1.6	-2.4	-1.7
100	1.5	1.2	3.1	0	0	-0.72	-0.34	0.17	- 0.18
1000	2.5	2.2	4.8	1.5	1.0	1.3	0.8	1.7	1.1

Table: $\log_{10} \mathcal{R}$.

Biases on the parameters - 0PA vs 1PA- Golden EMRIs

What about a lucky (for us) EMRIs with SNR 100?

Ratio $\mathcal{R}$ systematic over statistical errors

$$\mathcal{R} = \frac{|\Delta x^i_{\text{th}}|}{\sigma_{x_i}}$$

0PA waveform (approximation) vs 1PA waveform ("true" model, $\chi = 1$)

Masses: $M = 10^6 M_\odot, \mu = 10 M_\odot$. 2 years observation

$\mu[M_\odot]$	$\ln M$	$\ln \mu$	r_0/M	ϕ_0	ϑ_S	φ_S	ϑ_K	φ_K	$\ln D$
10	0.86	0.30	1.9	-1.1	-0.58	-1.0	-1.0	-1.9	-1.1

Table: $\log_{10} \mathcal{R}$ with SNR=100.

Conclusions and future perspective

Conclusions

We assessed importance 1PA terms for EMRIs and IMRIs

- 2SF do not affect statistical errors on the parameters
- secondary spin spoiled precision on the masses...
- ...unless we impose a prior
- neglecting 1PA induce large biases for IMRI
- 1PA marginal relevant only for golden EMRIs (SNR $= 100$)

Caveat1: we use a Fisher matrix approach
CAVEAT2: we considered circular, equatorial orbits with spins aligned in Schwarzschild.

We can say nothing about the general case...

...but these results might extend for circular, equatorial orbits in Kerr

On the bright side: we see some biases already for this simple model!

Future work - short term

TO DO list to complete this work in progress

- Careful check with Ollie Burke's results
- Learn how to use few [7]
- Do an MCM analysis with secondary spin using few [8]
- Implement frequency-domain Fisher matrix in few
- Push Fisher matrices code to higher mass-ratio ($q \sim 10^{-2}$)

[7] https://bhptoolkit.org/FastEMRIWaveforms/html/index.html
[8] https://bhptoolkit.org/FastEMRIWaveforms/html/index.html

Final notes and acknowledgments

- Thanks to Ollie Burke for enlightening discussion on data analysis for LISA.
  Check his thesis for a fantastic introduction on the topic
- this work ~~was sponsor by~~ uses the BHPToolkit
  https://bhptoolkit.org/
  ~~Click to support my Youtube channel!~~
  Version 1.0.0 is out for Teukosky, SpinWeightedSpheroidalHarmonics and more!!!
  New features!!! New packages!!! ~~New bugs!!!~~
  See Niels's talk on Friday
- Feel free to contact me at gabriel.piovano@ucd.ie

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Thank you for you attention!

Backup slides

Biases on the parameters - 0PA vs 0PA+2SF

Ratio $\mathcal{R}$ systematic over statistical errors

$$\mathcal{R} = \frac{|\Delta x^i_{\rm th}|}{\sigma_{x_i}}$$

0PA waveform (approximation) vs 0PA+2SF waveform (“true” model)

Masses: $M = 10^6 M_\odot, \mu = 10 M_\odot$. 2 years observation

$\mu[M_\odot]$	M	μ	r_0/M	ϕ_0	ϑ_S	φ_S	ϑ_K	φ_K	D
10	0.16	-0.17	1.3	-1.7	-1.2	-1.6	-1.6	-2.3	-1.7
100	1.3	1.3	3.1	0	-0.027	- 0.65	-0.37	0.13	- 0.20
1000	2.4	2.3	4.8	1.5	1.0	1.3	0.78	1.7	1.1

Table: $\log_{10} \mathcal{R}$