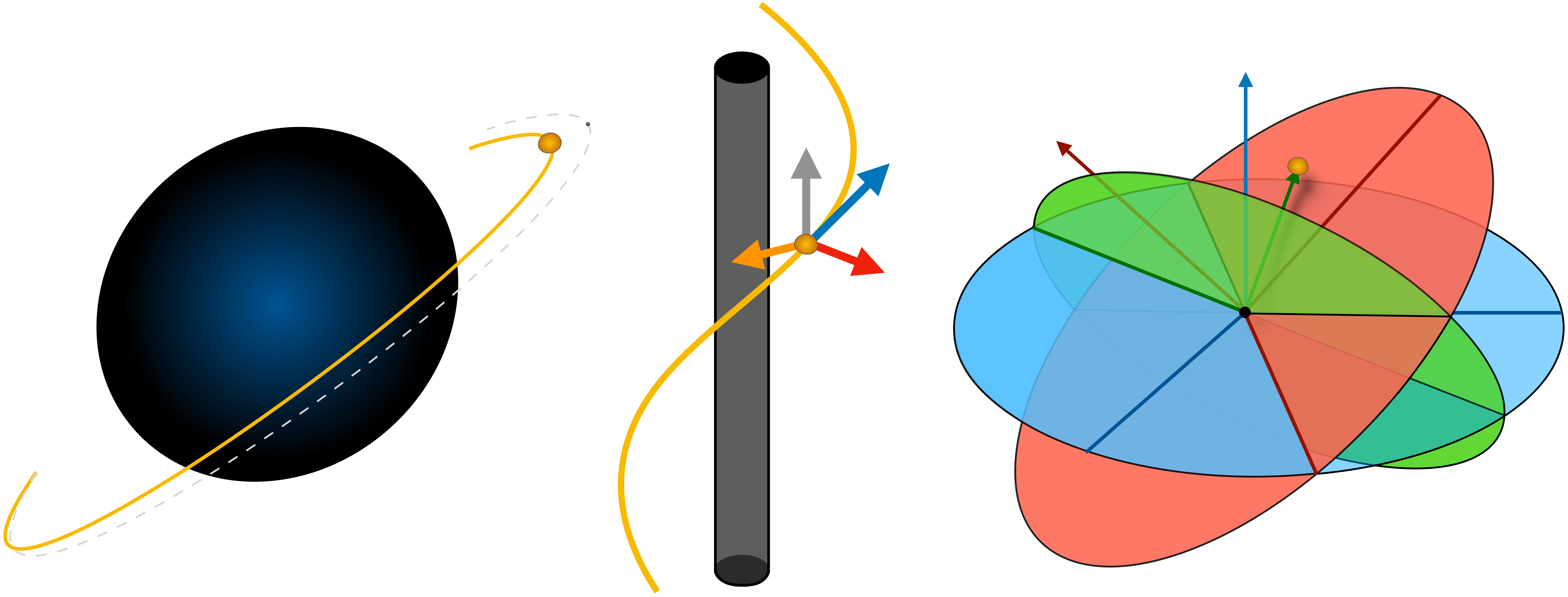
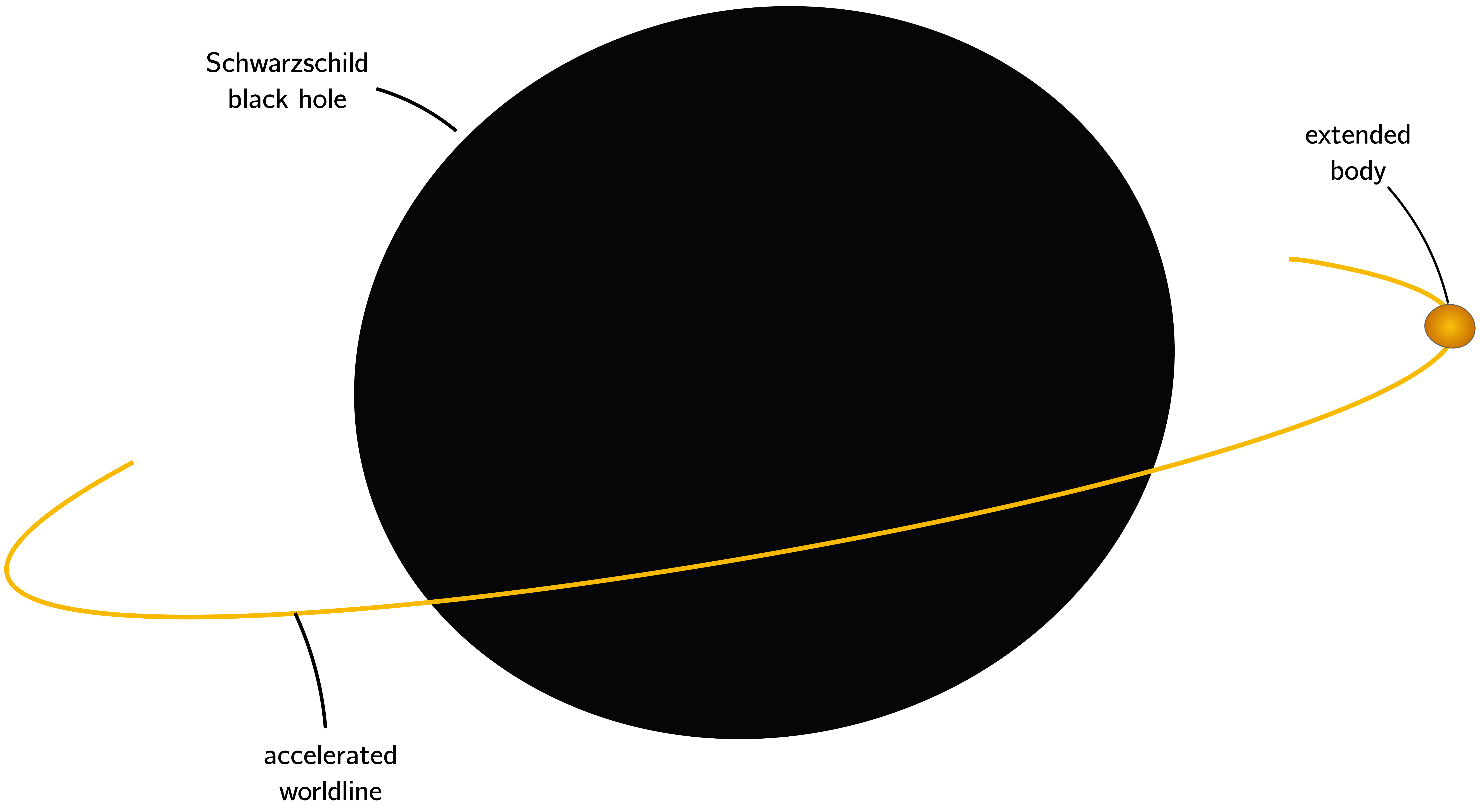
Symplectic mechanics of a *Spinning particle* in the *Schwarzschild spacetime*



General geometric setup

General context - EMRIs



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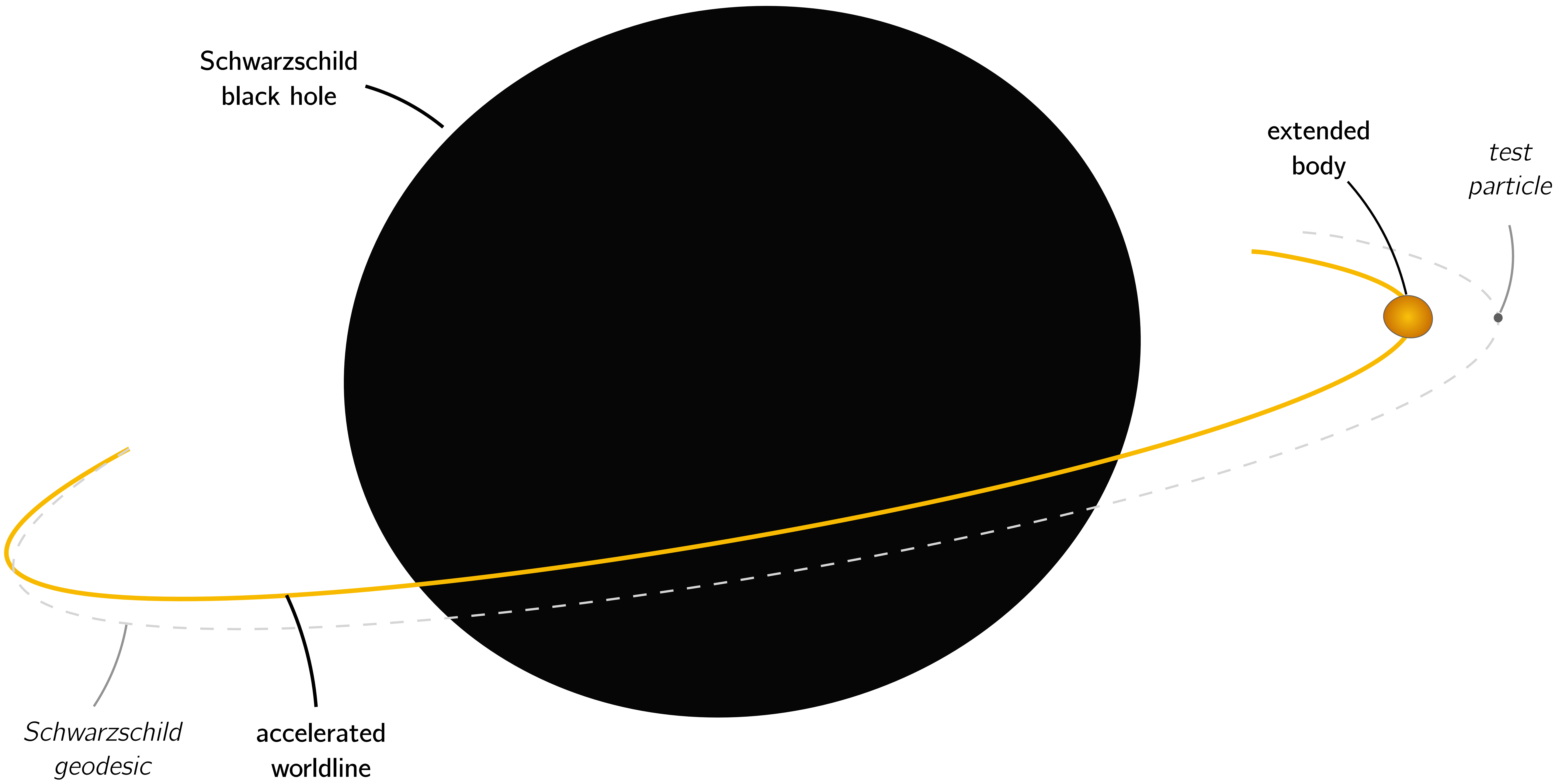

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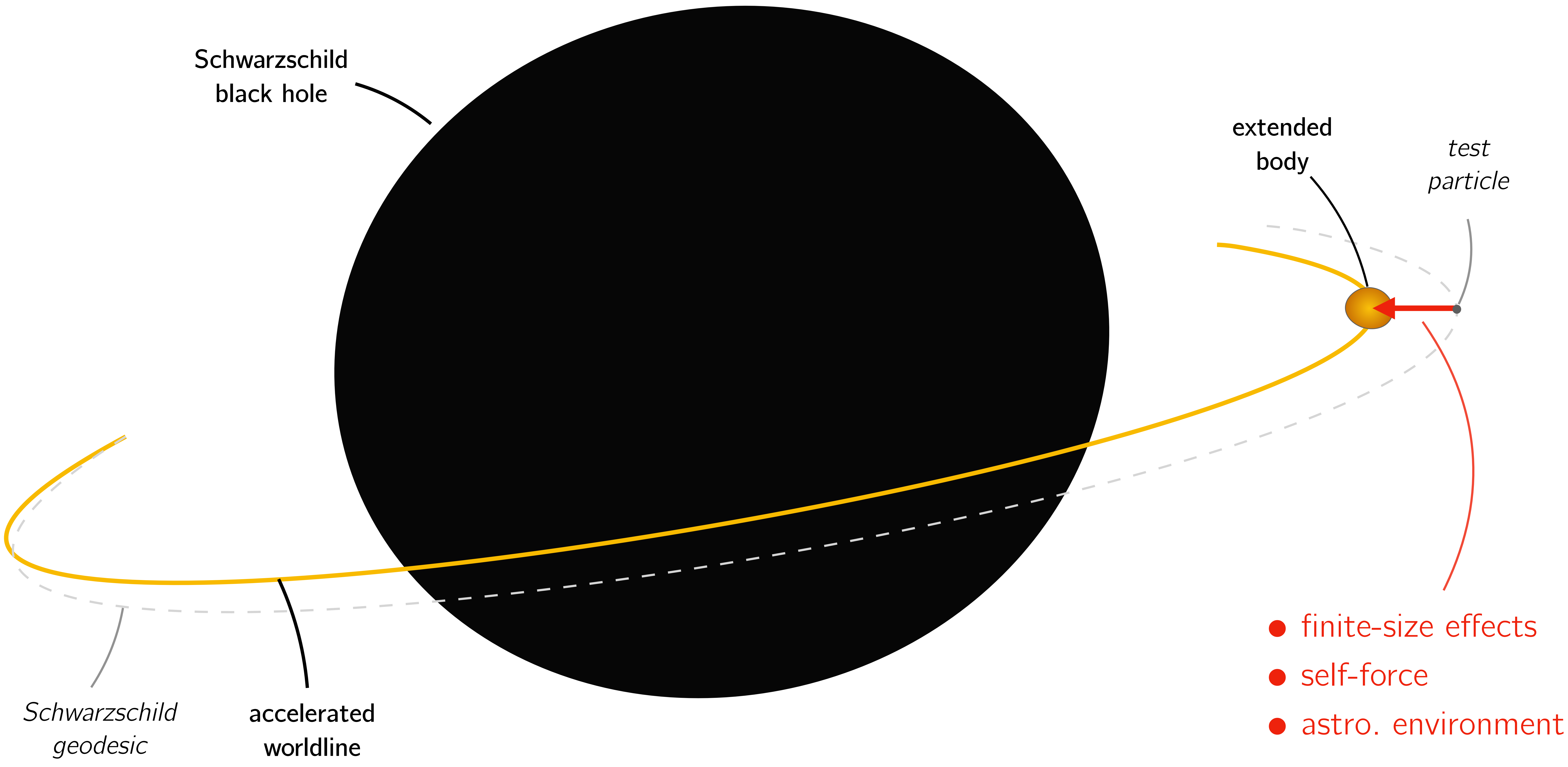
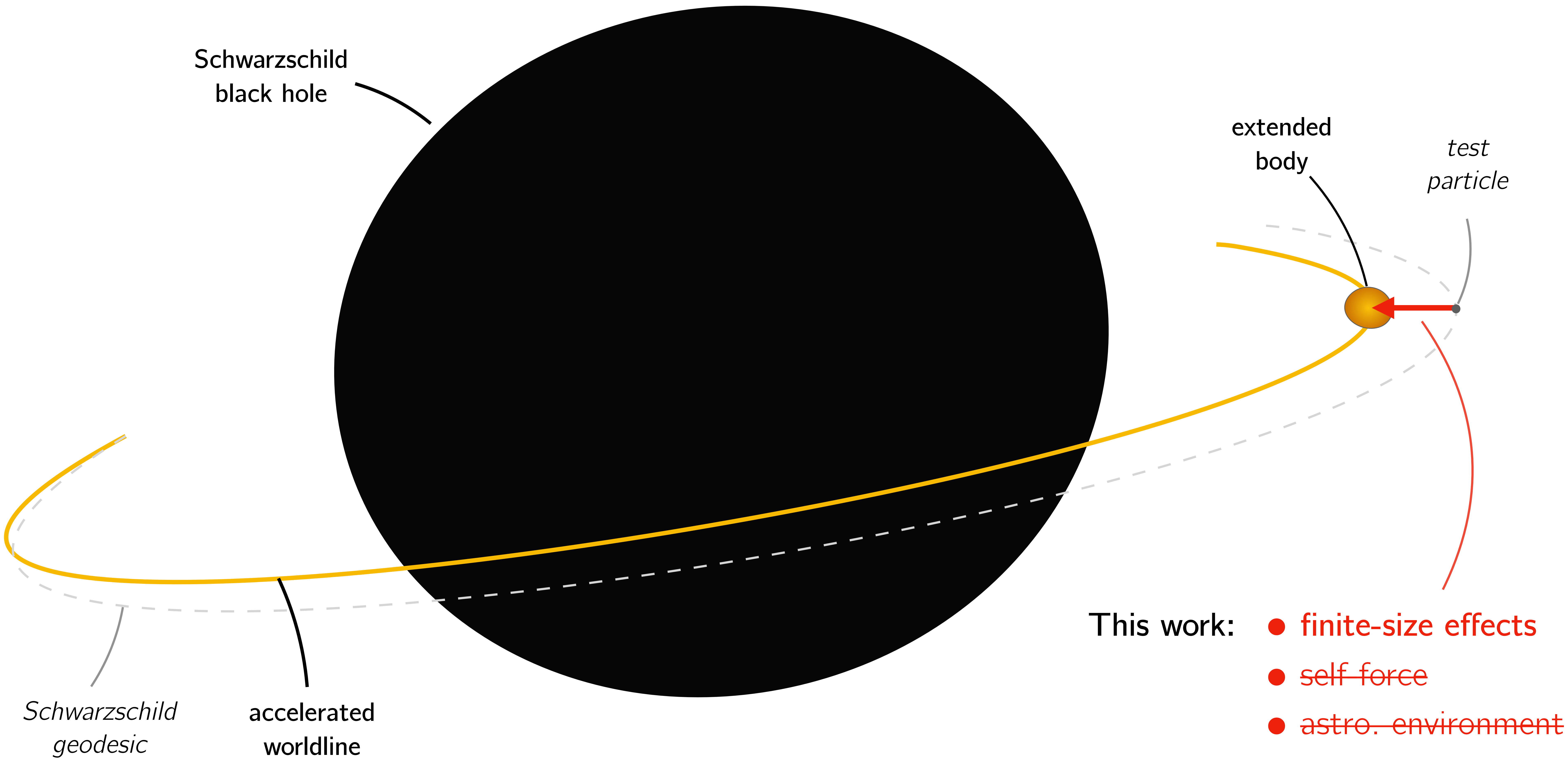


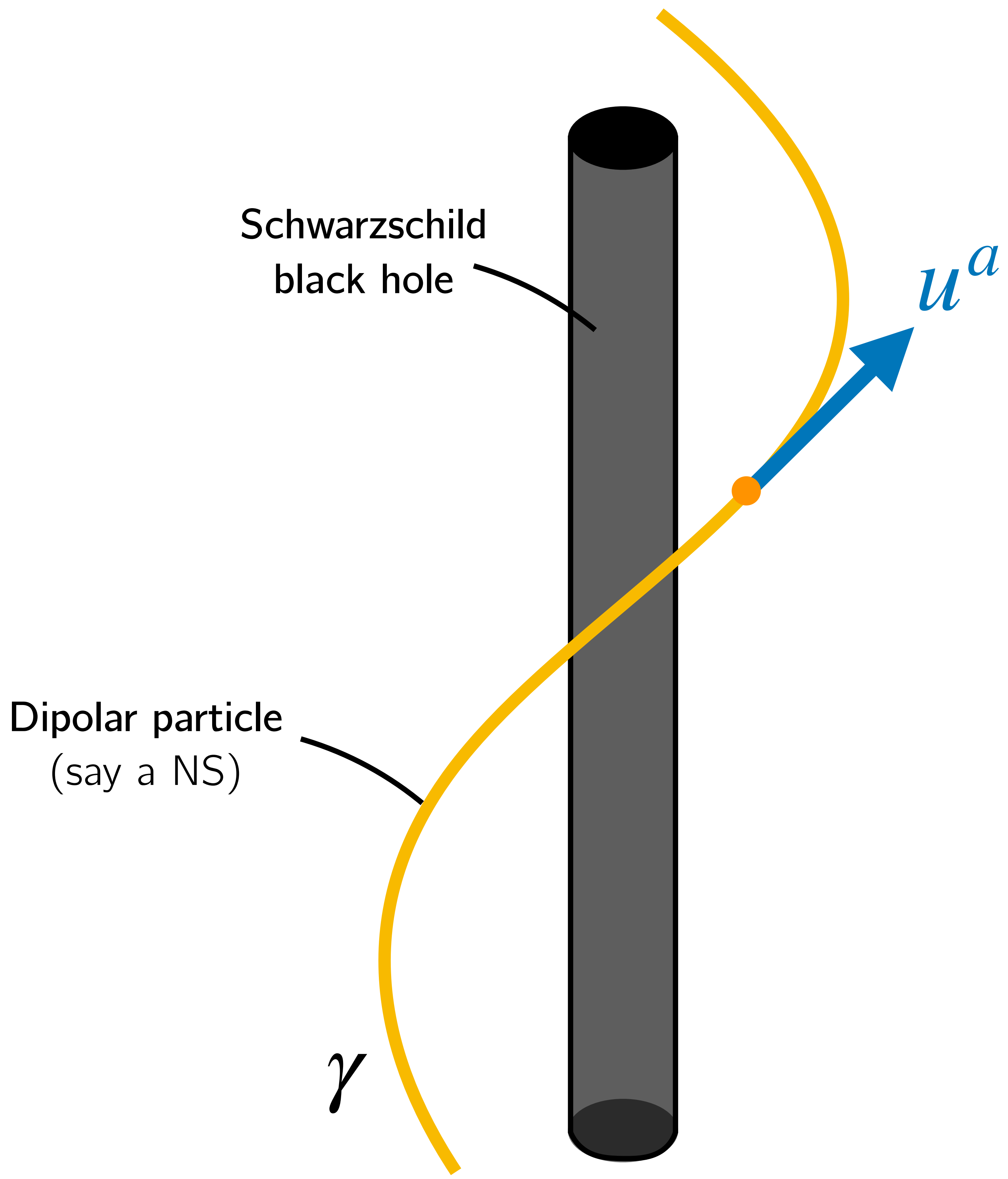
- finite-size effects
- self-force
- astro. environment

General context - EMRIs



This work:
- finite-size effects
- ~~self-force~~
- ~~astro. environment~~

The secondary as a particle with multipoles



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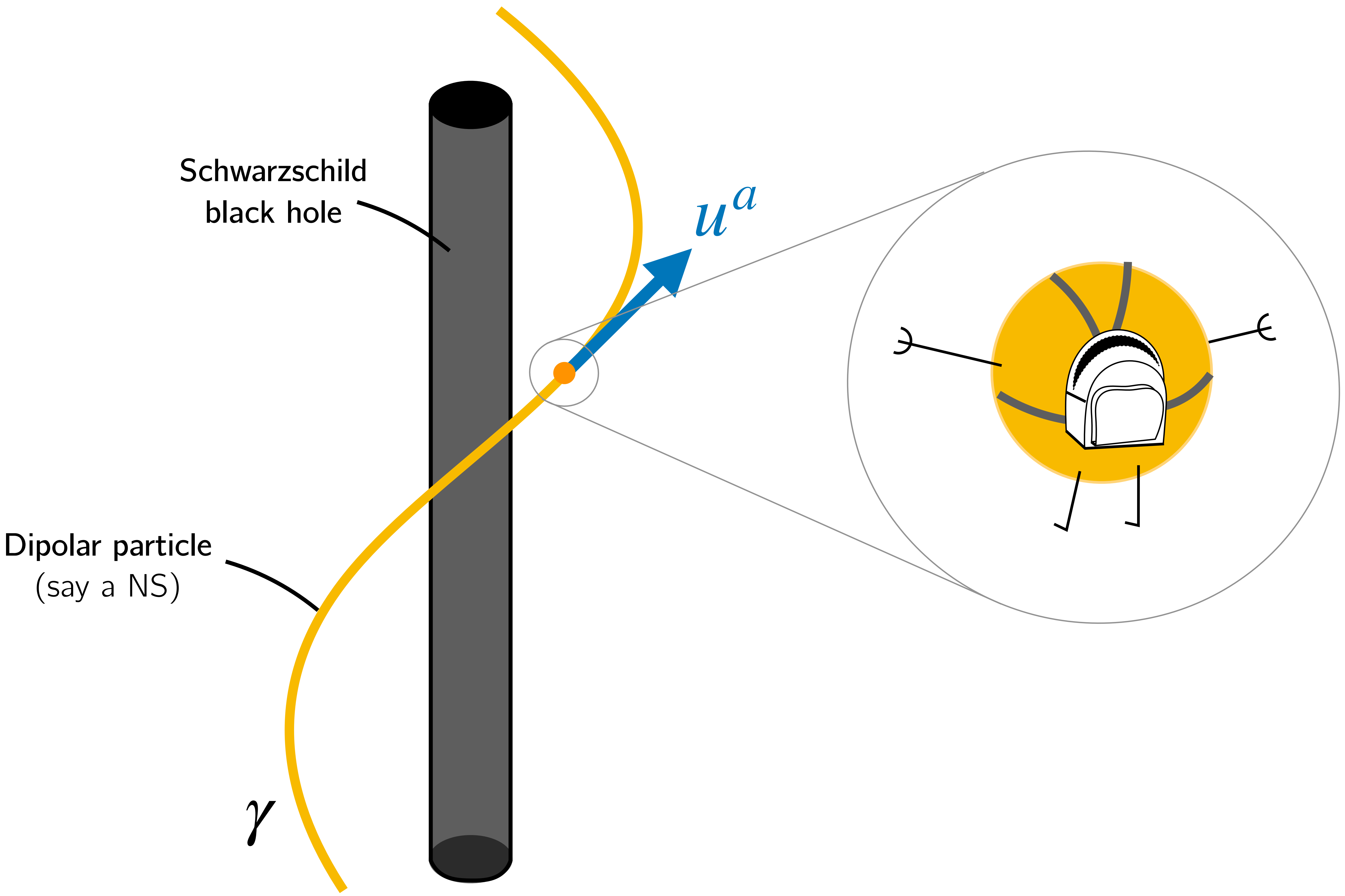

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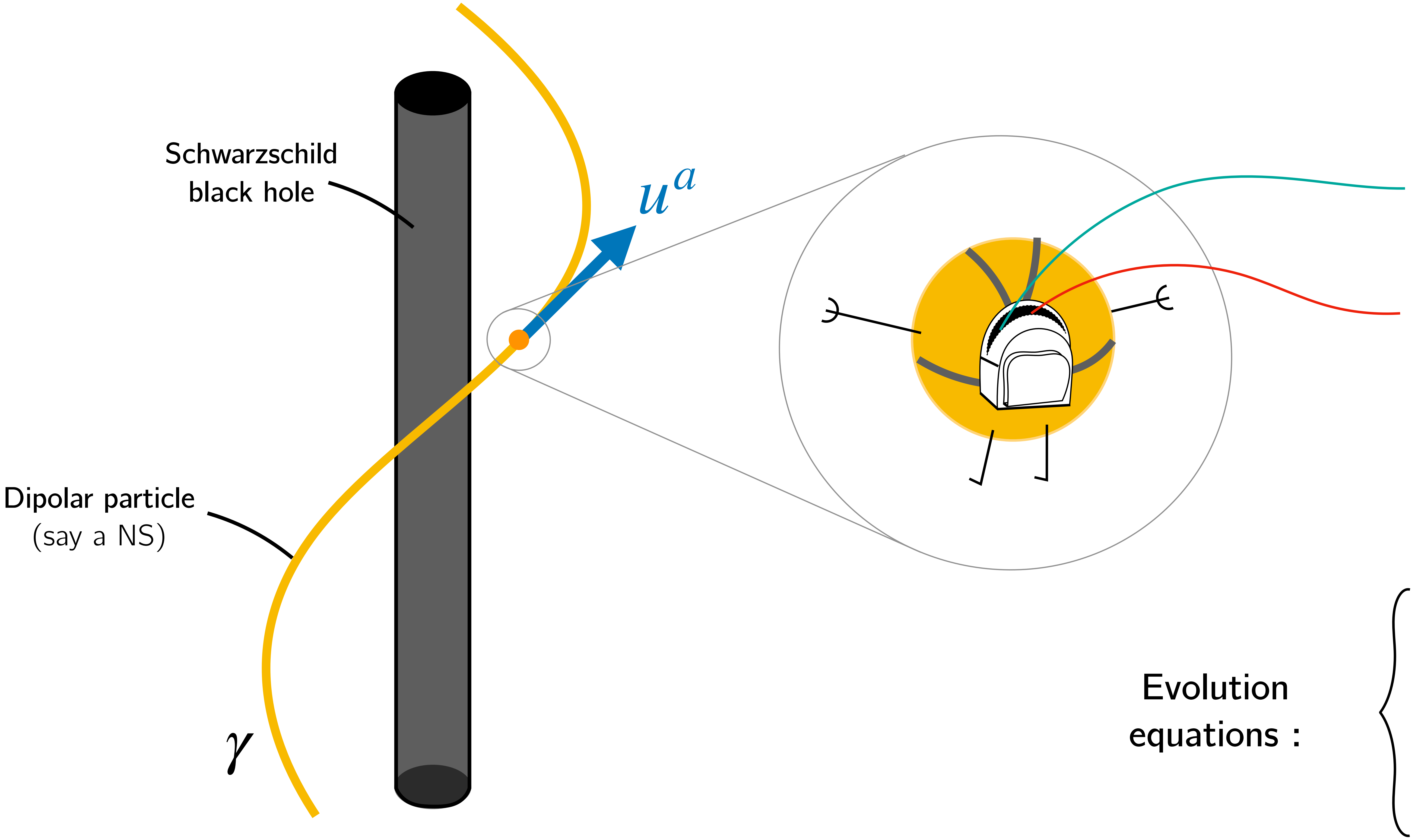


Two multipoles

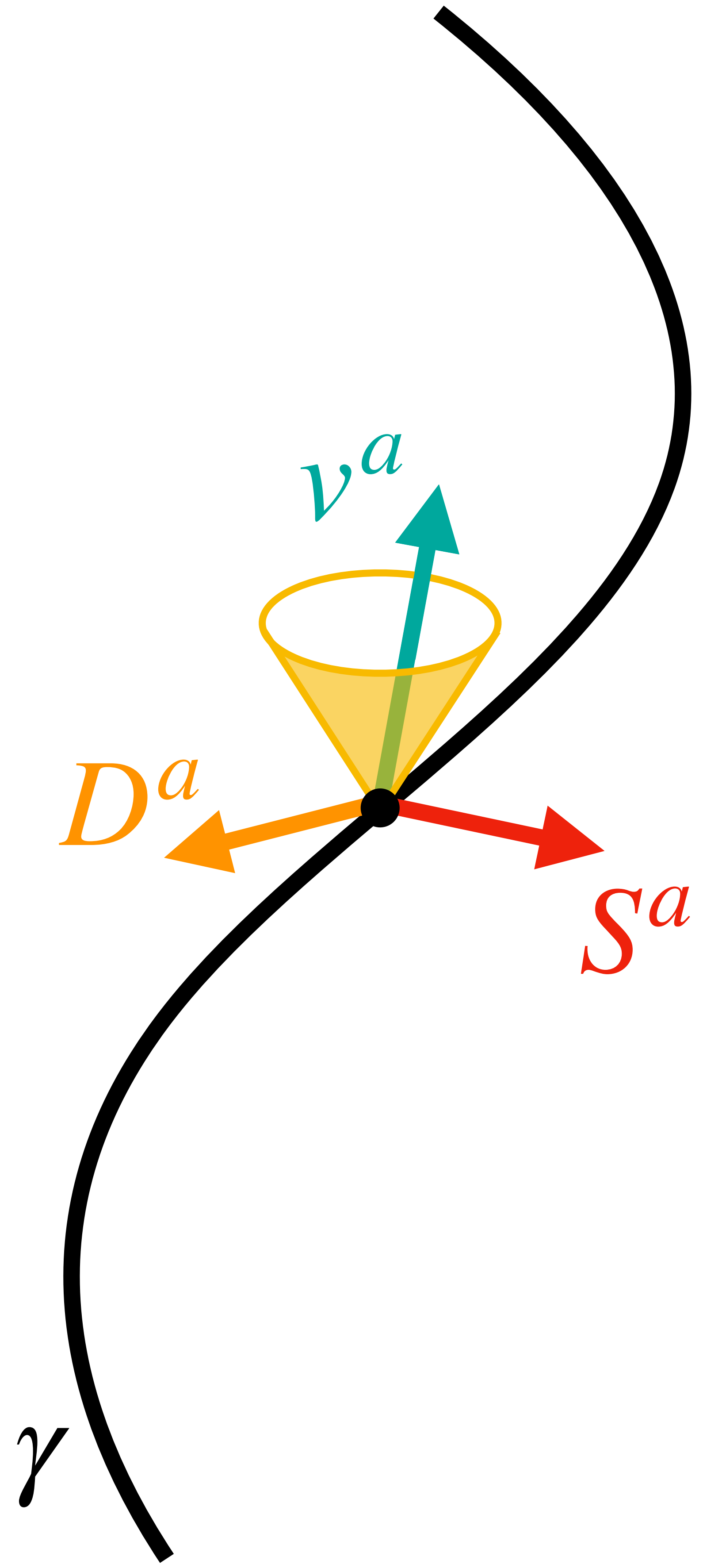
p^a linear momentum

S^{ab} angular momentum

Evolution equations :

$$\begin{cases} \dot{p}^a = \frac{1}{2} R_{bcd}{}^{a} S^{bc} u^d \\ \dot{S}^{ab} = 2 p^{[a} u^{b]} \end{cases}$$

Hodge decomposition of the spin tensor



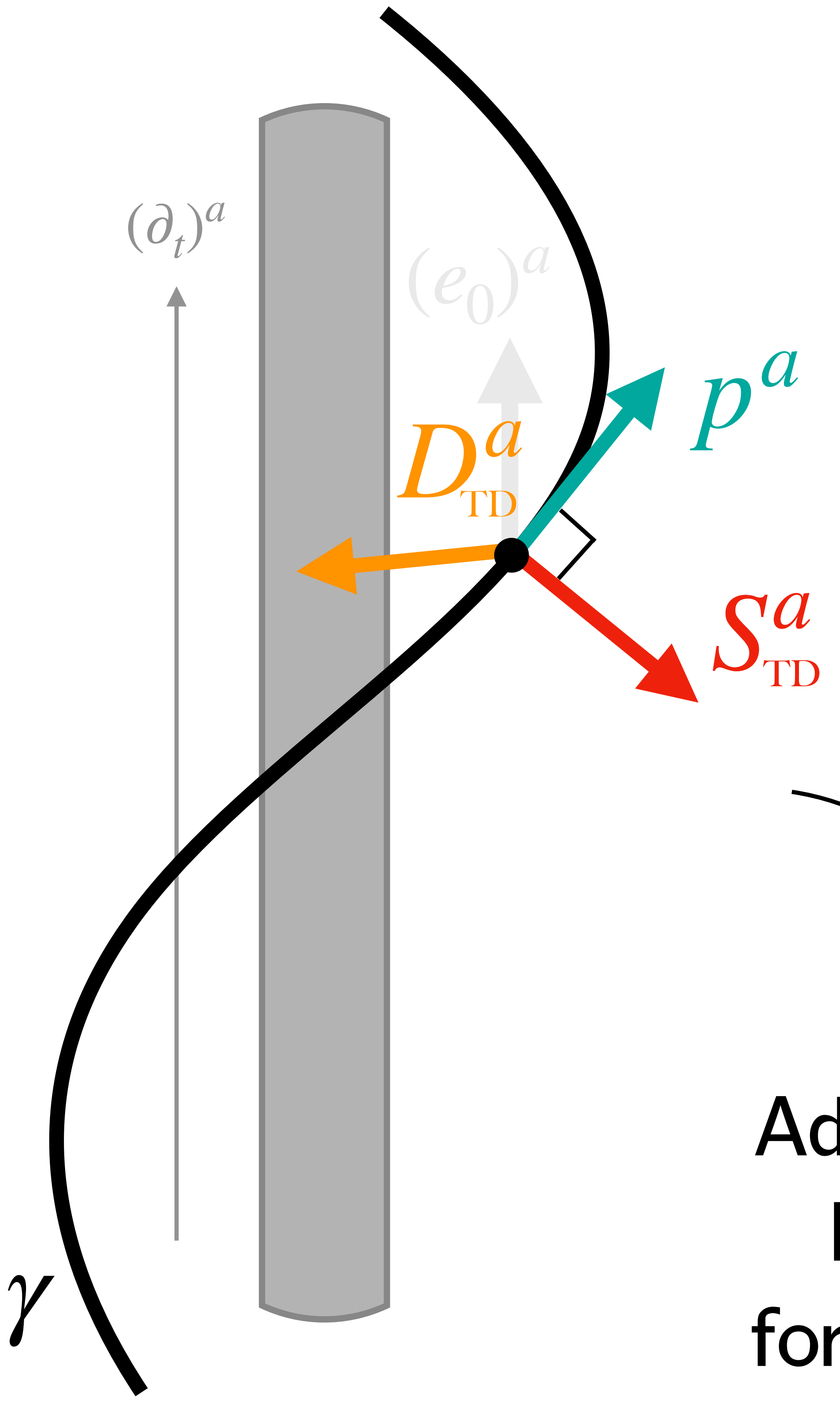
$$\left\{ \begin{array}{ll} S^{ab} & \text{spin tensor} \\ v^a & \text{time-like direction} \end{array} \right. \Leftrightarrow \left\{ \begin{array}{ll} D^a & \text{mass dipole} \\ S^a & \text{spin vector} \end{array} \right.$$

Spin supplementary condition (SSC):

$D^a = 0$ for some v^a

Hodge decomposition of the spin tensor

- wrt 4-momentum direction



Adapted to MPTD formulation

Tulcyjew-Dixon SSC:

$$D^a_{\text{TD}} = 0$$

Hodge decomposition of the spin tensor

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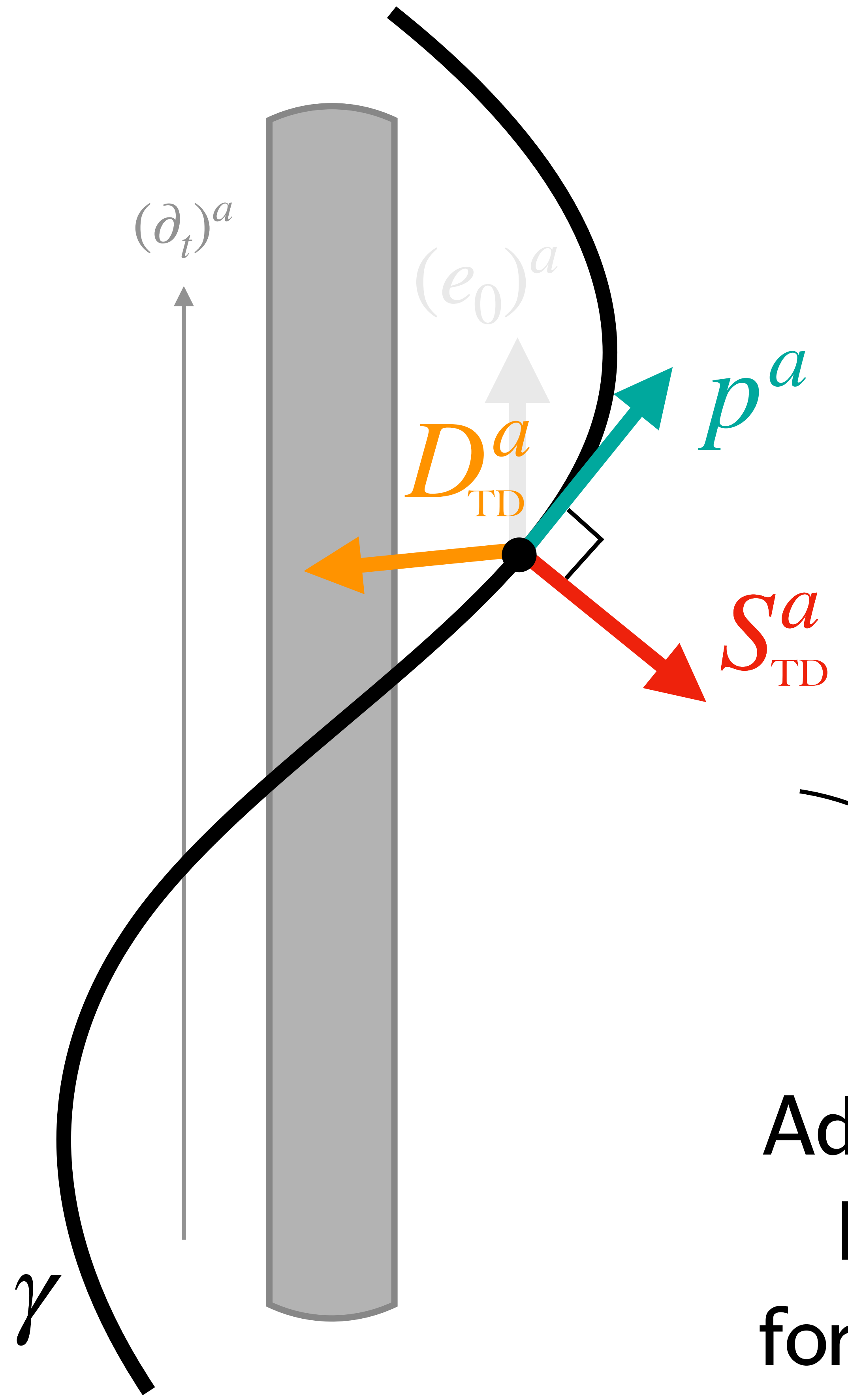

Adapted to
MPTD
formulation

Tulcyjew-Dixon SSC:

$$D^a_{\text{TD}} = 0$$

Usual recipe

- Evolution equations:

$$\begin{cases} \dot{p}^a = B^a{}_b S^b \\ \dot{S}^a_{\text{TD}} = 0 \\ p^a = \mu u^a \end{cases}$$

Hodge decomposition of the spin tensor

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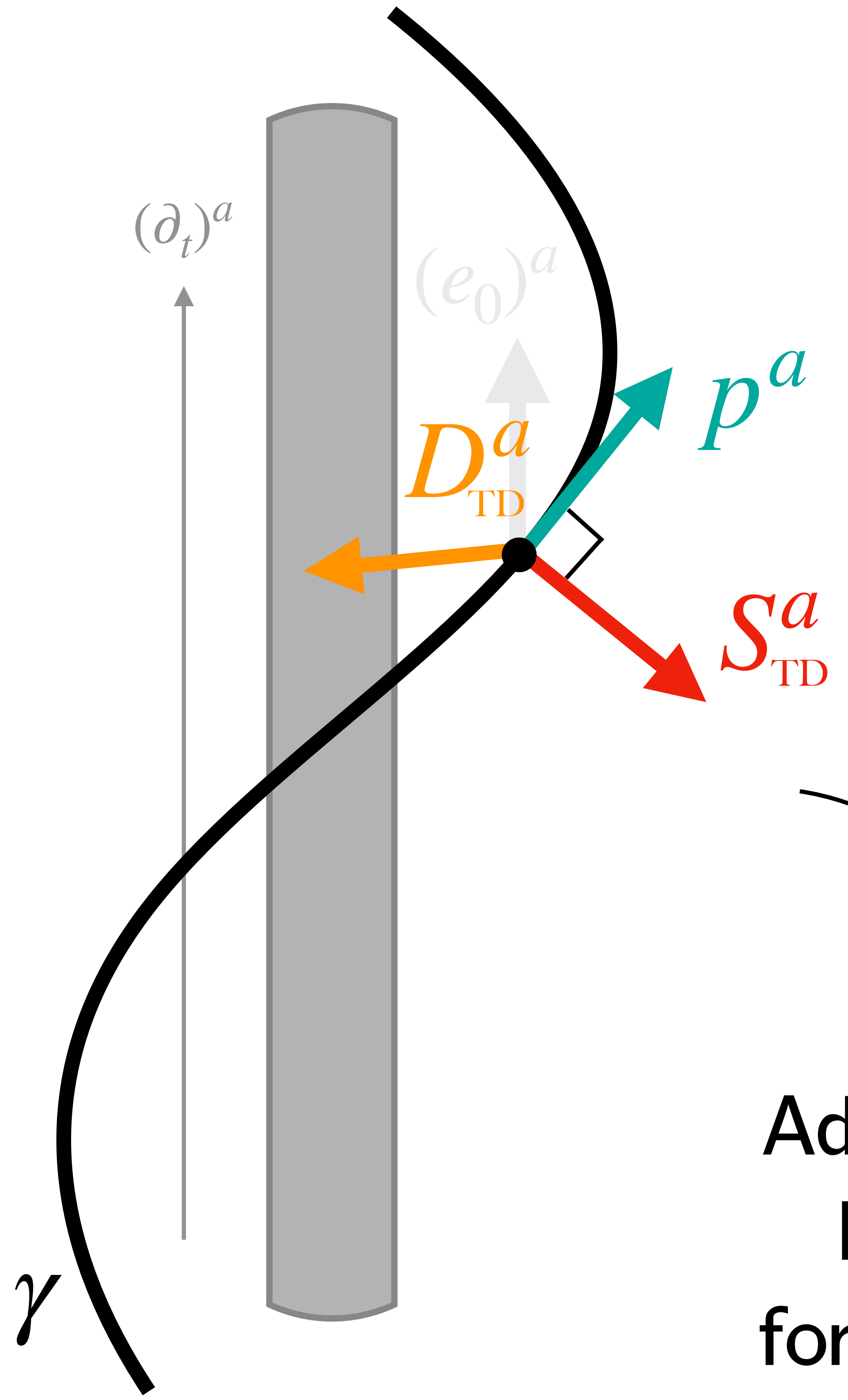

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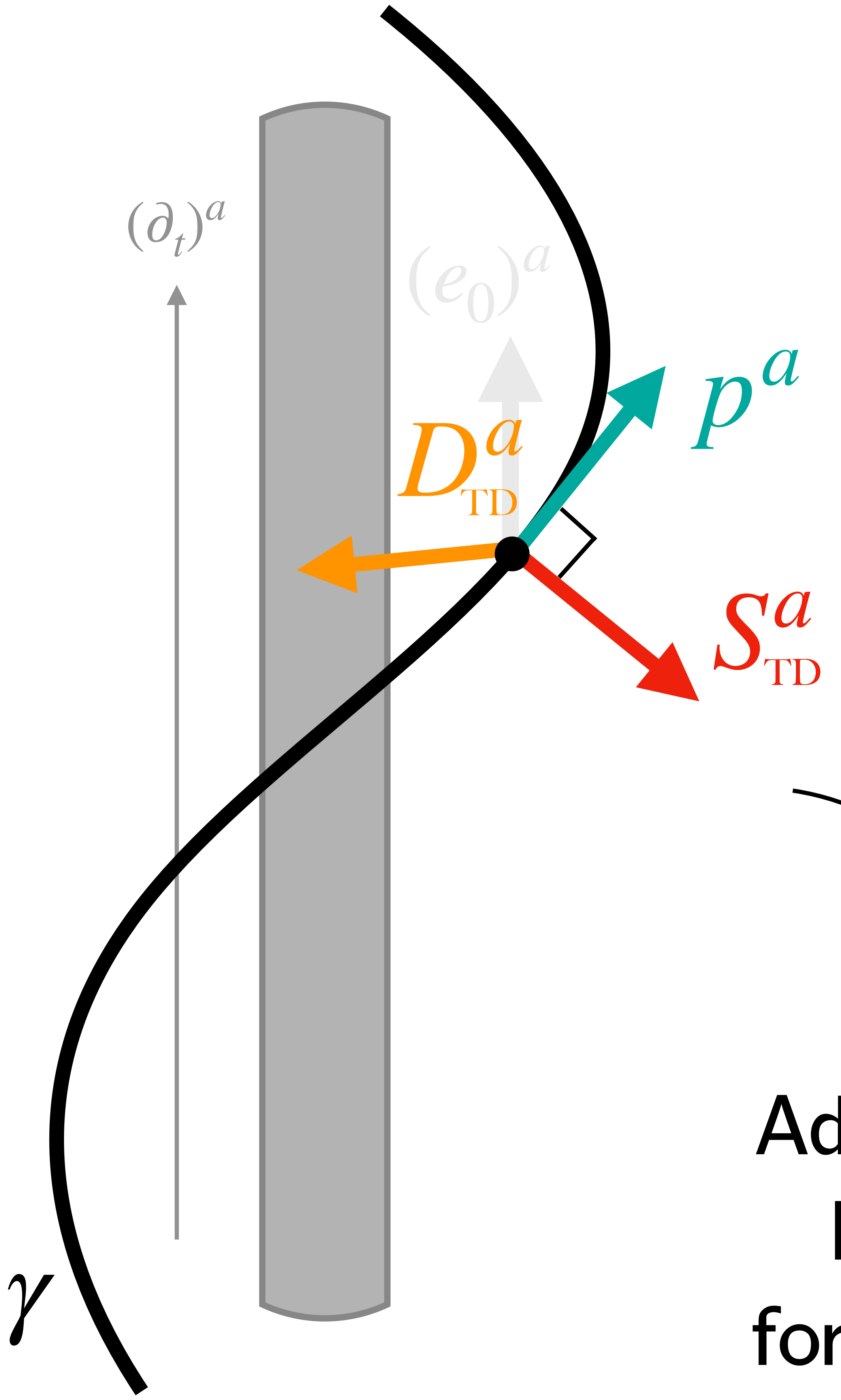
- Parallel-transported tetrad

Hodge decomposition of the spin tensor

- wrt 4-momentum direction



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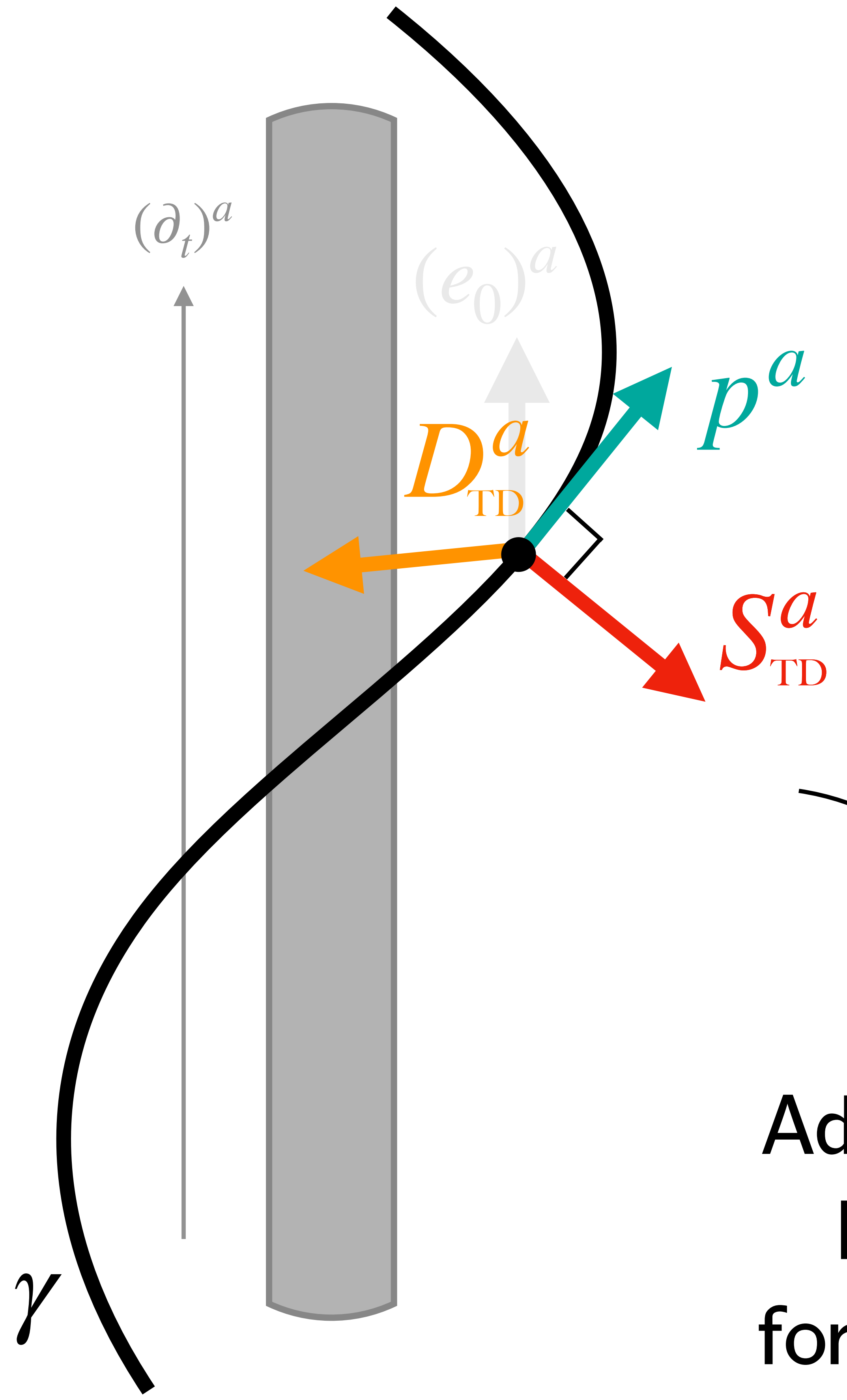
- Parallel-transported tetrad
- Special config. assumption

- quasi-circular
- quasi-aligned
- quasi-equatorial

Hodge decomposition of the spin tensor

- wrt 4-momentum direction



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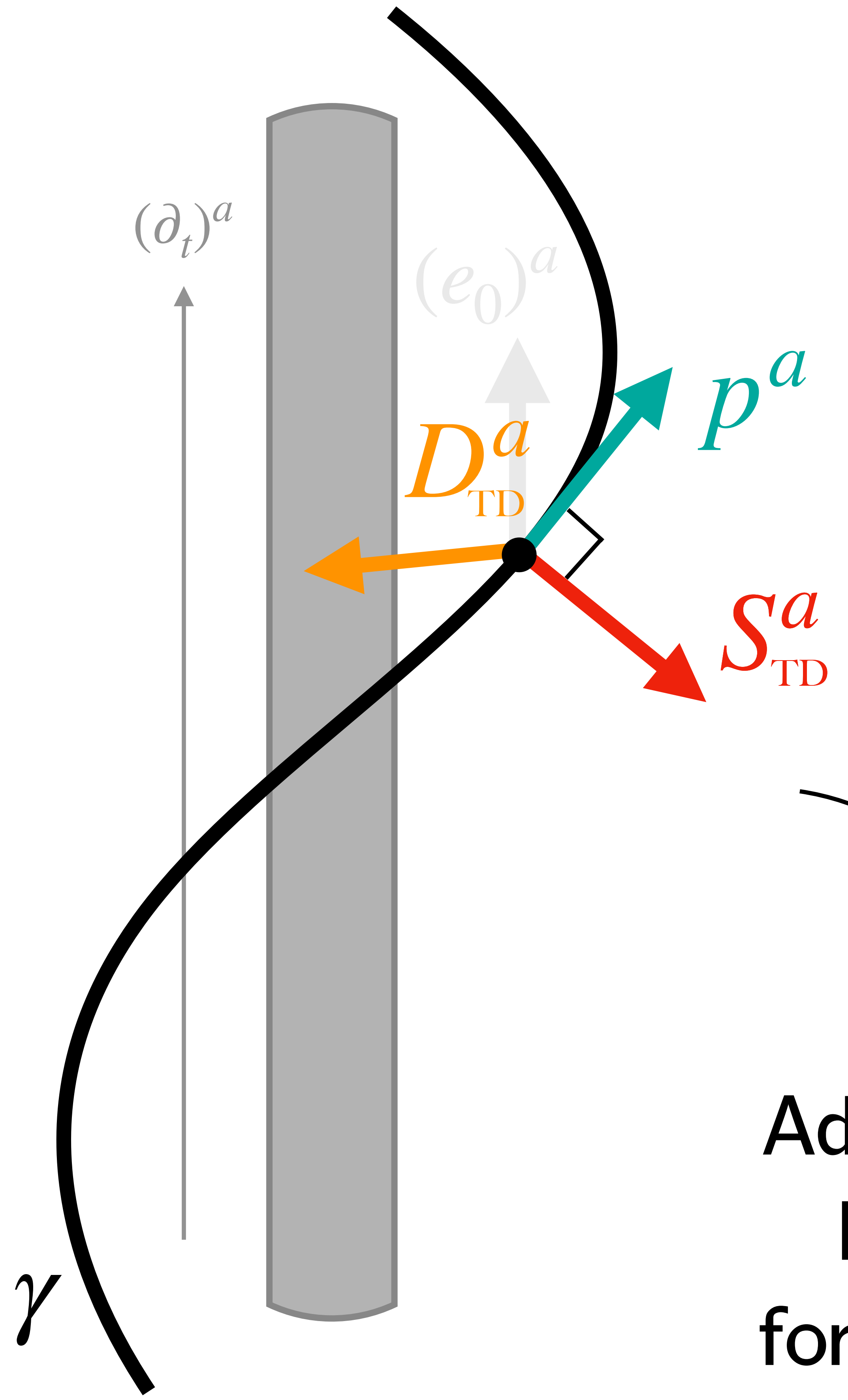
Can we do better?

Hodge decomposition of the spin tensor

- wrt 4-momentum direction



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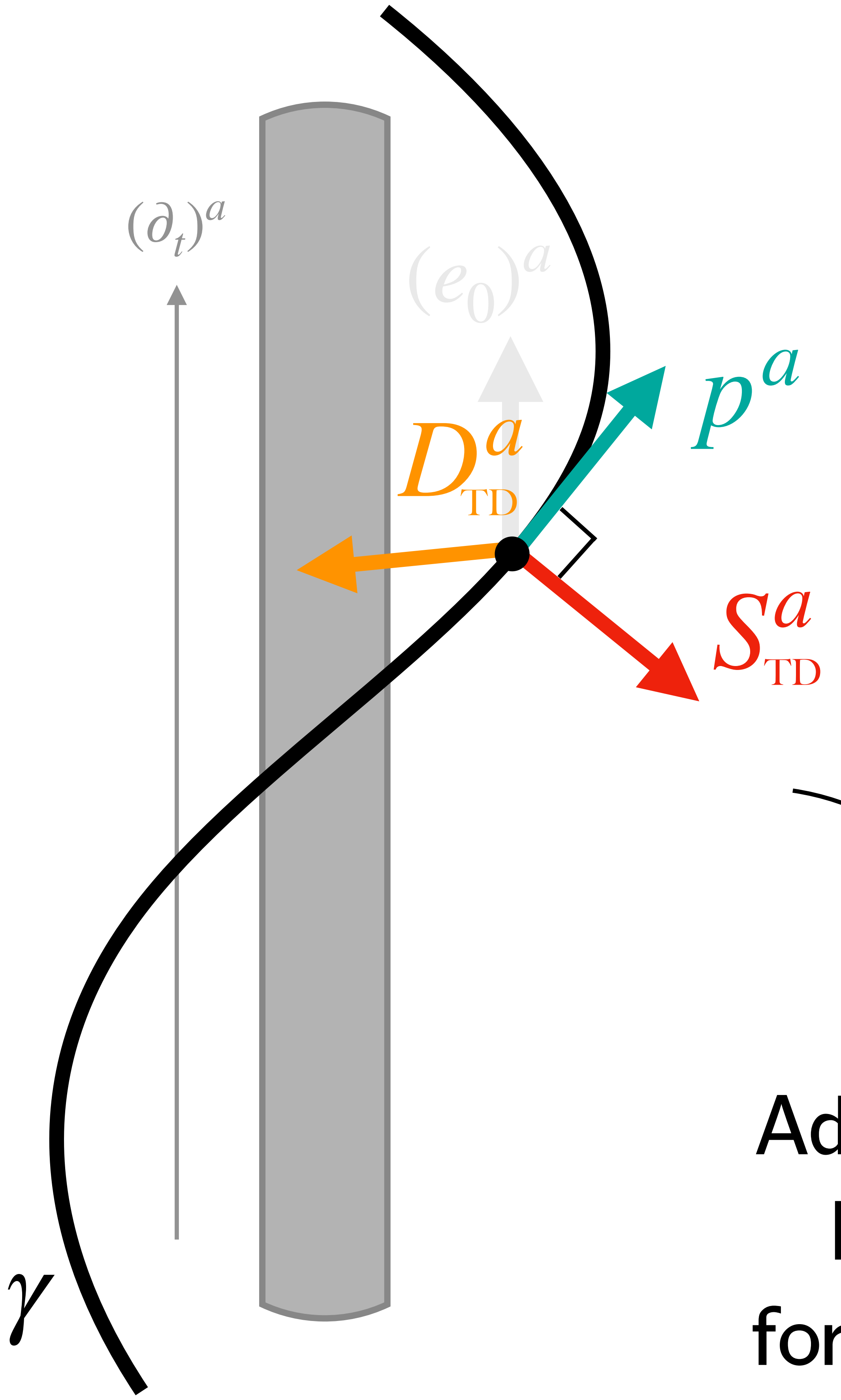


Tulcyjew-Dixon SSC:

$$D^a_{\rm TD} = 0$$

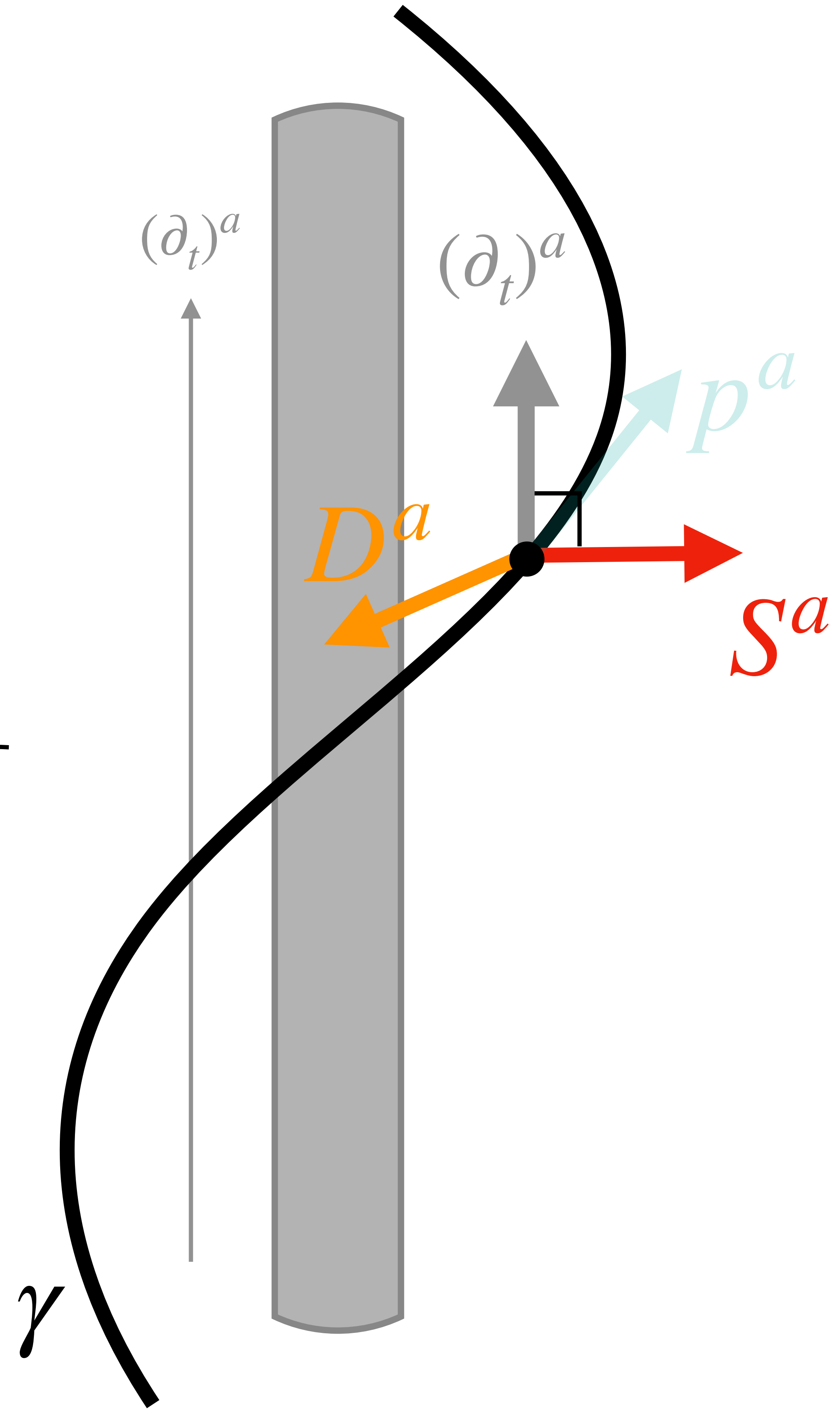
$$\Downarrow$$

$$S^a_{\rm TD} = F(S^a)$$

Adapted to
MPTD
formulation

Adapted to
Hamiltonian
formulation

- wrt Killing time-like direction



Turning all this into a Hamiltonian system

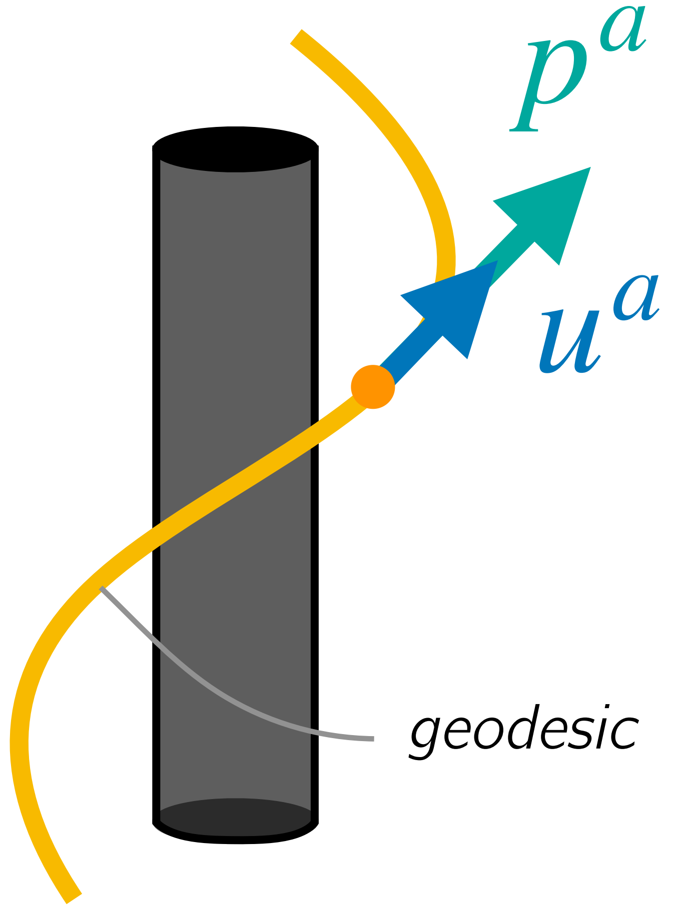
1. Geodesic case

Hamiltonian formulation of geodesic motion

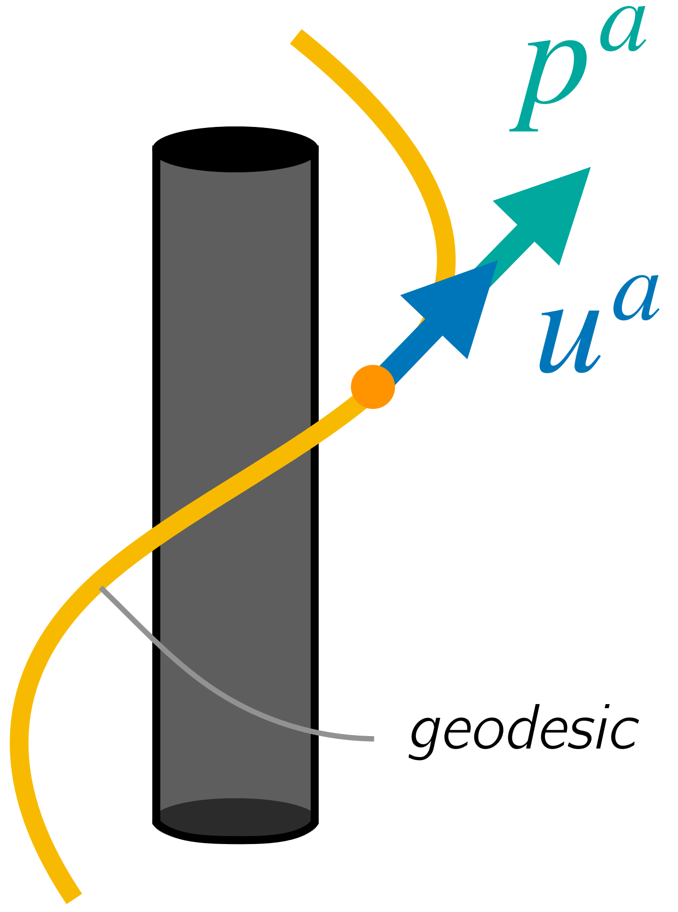
- MP formulation

$$\left.\begin{aligned}\dot{p}^a &= \frac{1}{2}R_{bcd}{}^{a}S^{bc}u^d \\ \dot{S}^{ab} &= 2p^{[a}u^{b]}\end{aligned}\right\}$$ Evolution equations

No spin: $$\left\{\begin{aligned}\dot{p}^a &= 0 \\ S^{ab} &= 0 \\ p^a &= \mu u^a\end{aligned}\right.$$



Hamiltonian formulation of geodesic motion



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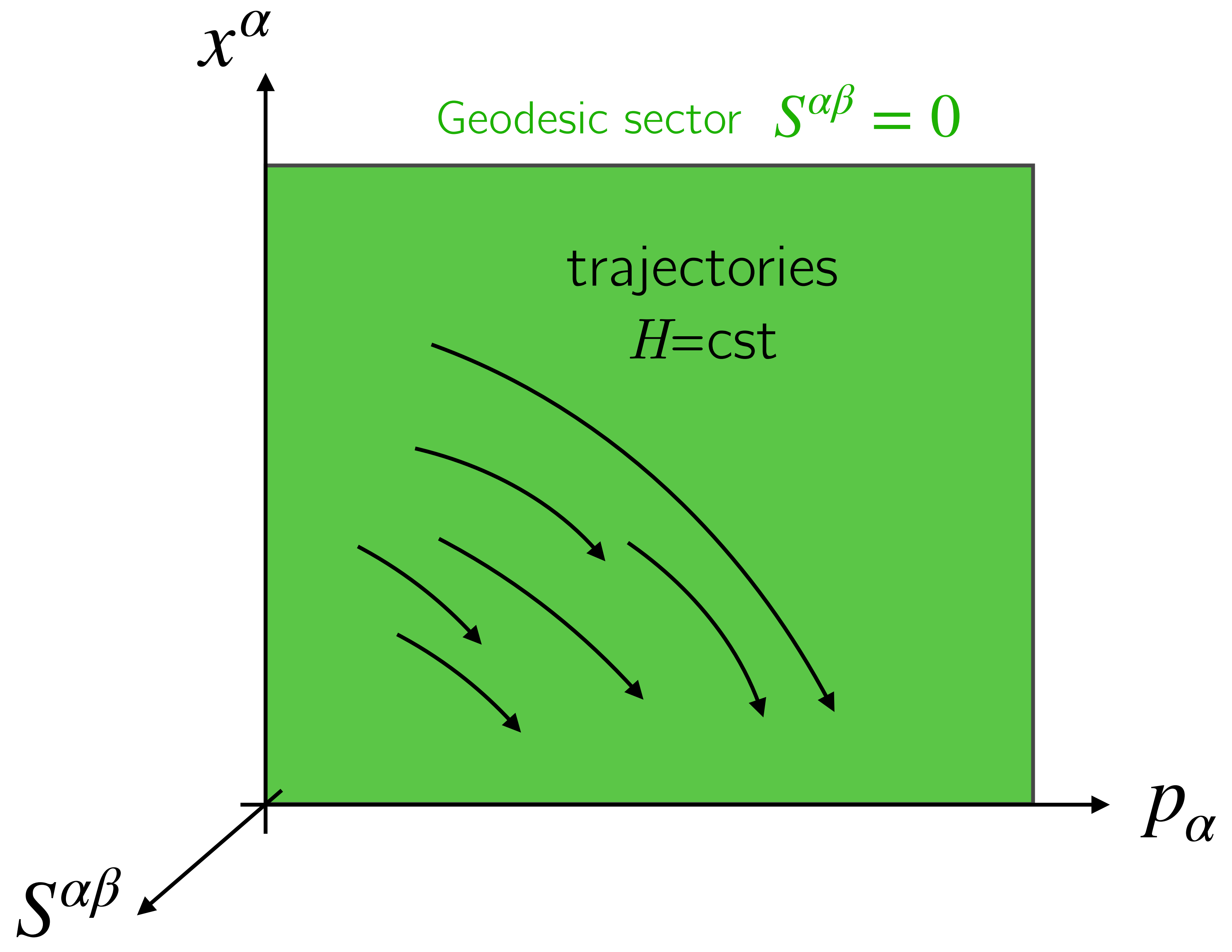
Hamiltonian formulation

- Phase space: $(x^\alpha, p_\beta) \in \mathbb{R}^4 \times \mathbb{R}^4$
- Symplectic structure: canonical $\{x^\alpha, p_\beta\} = \delta^\alpha_\beta$
- Hamiltonian: $H = \frac{1}{2}g^{\alpha\beta}p_\alpha p_\beta$



Example in Schwarzschild

$f = 1 - \frac{2M}{r}$

- Hamiltonian $H(t, p_t, r, p_r, \theta, p_\theta, \phi, p_\phi) = -\frac{p_t^2}{2f} + \frac{f p_r}{2} + \frac{1}{2r^2}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right)$

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- Killing invariants: $C_k = p_a k^a$

  - energy: $E = -p_t$
  - norm of ang. mom. $J^2 = p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}$
  - component of ang. mom. $J_z = p_\phi$

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- Euclidean interpretation
  - conserved angular momentum $\overrightarrow{J}$
  - invariant plane ⊥ $\overrightarrow{J}$
  - motion is confined within plane

Hamiltonian reduction

- Canonical transformation

$$(\theta, p_\theta, \phi, p_\phi) \mapsto (\underset{\substack{\swarrow \\ \text{eccentric} \\ \text{anomaly}}}{\psi}, \underbrace{J, \omega, J_z}_{\substack{\text{invariant} \\ \text{plane}}})$$

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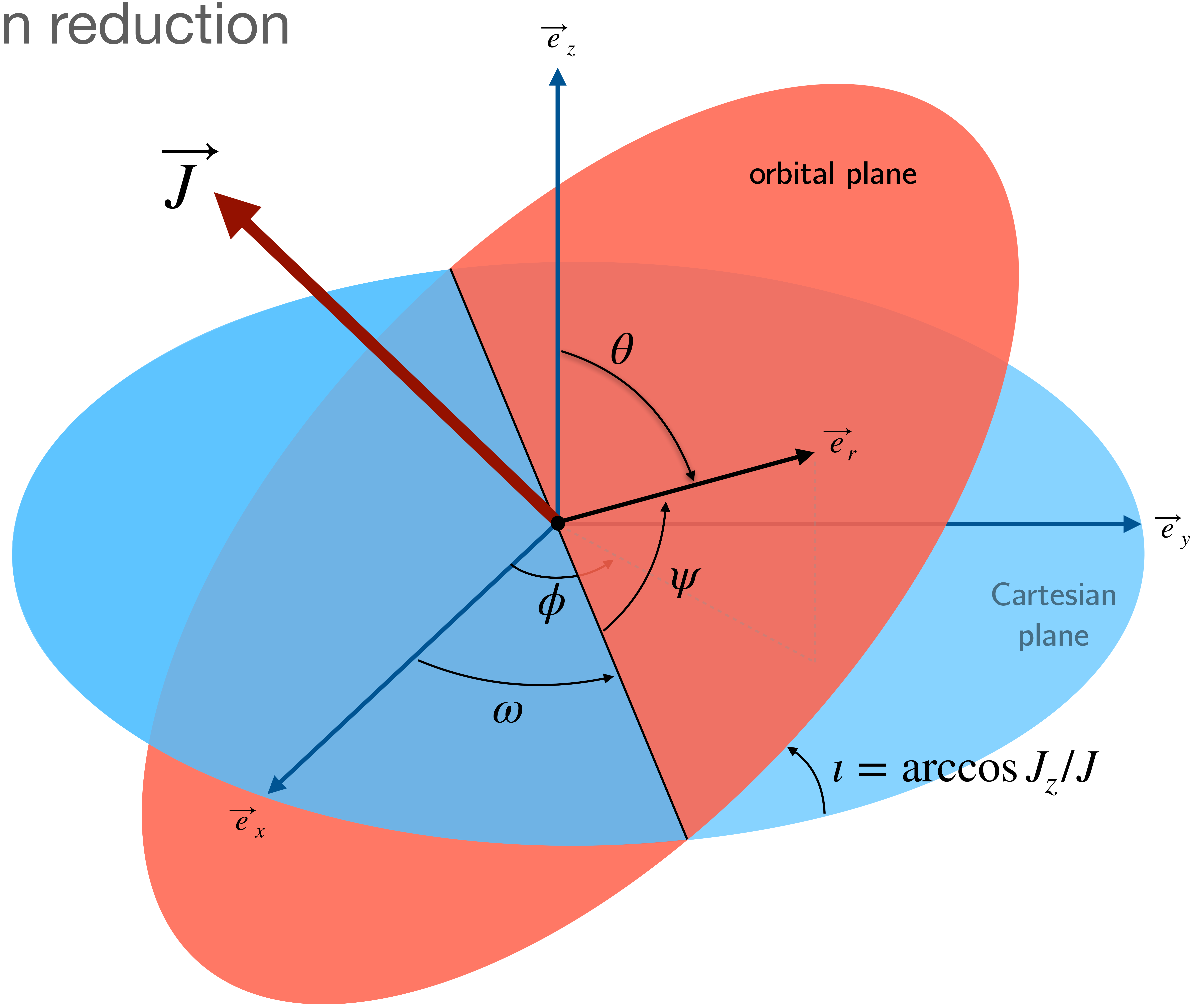

Hamiltonian reduction

- Canonical transformation

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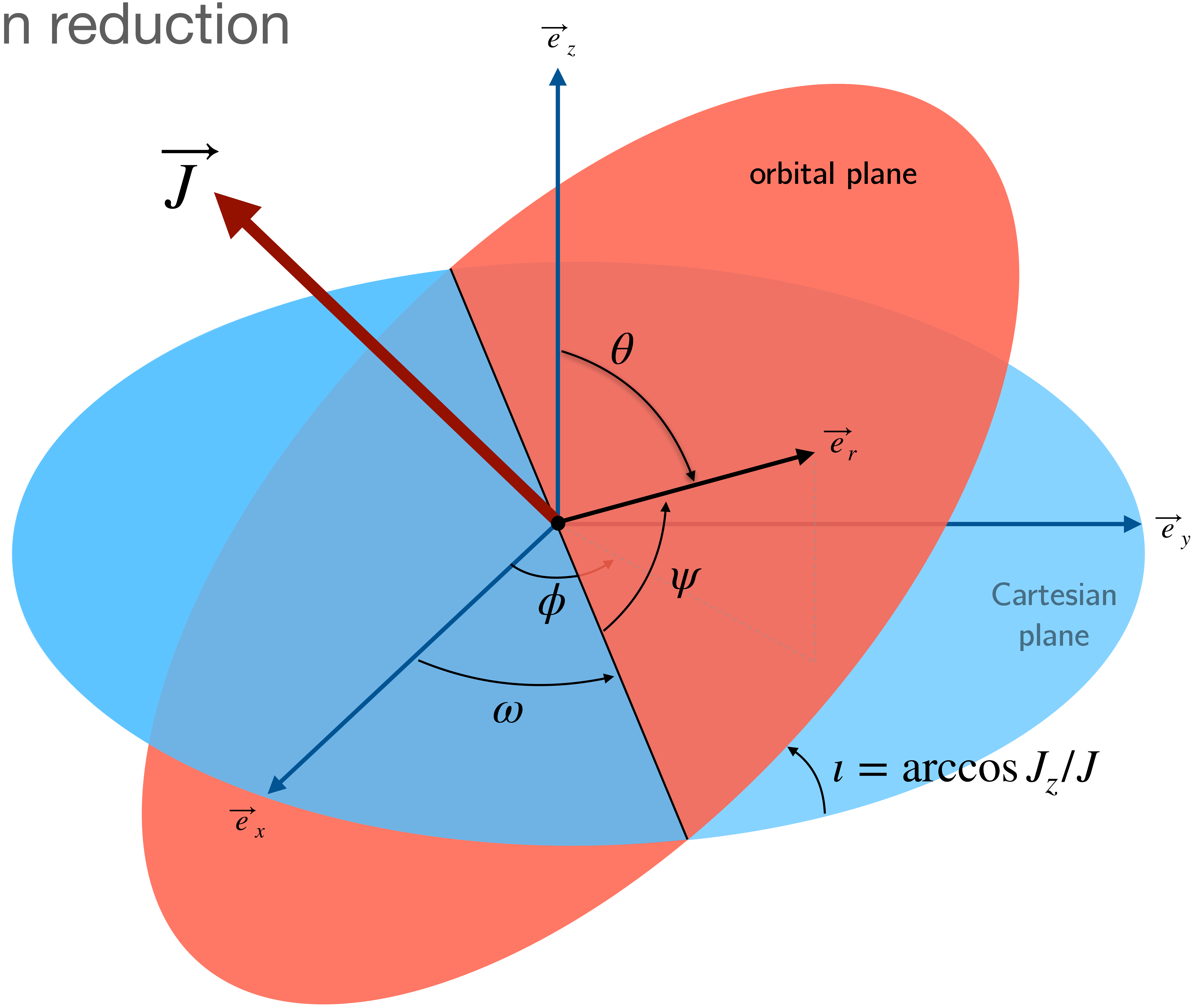
$\swarrow$ eccentric anomaly

- Reduced Hamiltonian

$$H(r, p_r) = -\frac{E^2}{2f} + \frac{f p_r^2}{2} + \frac{J^2}{2r^2}$$

1 dof with analytic solutions



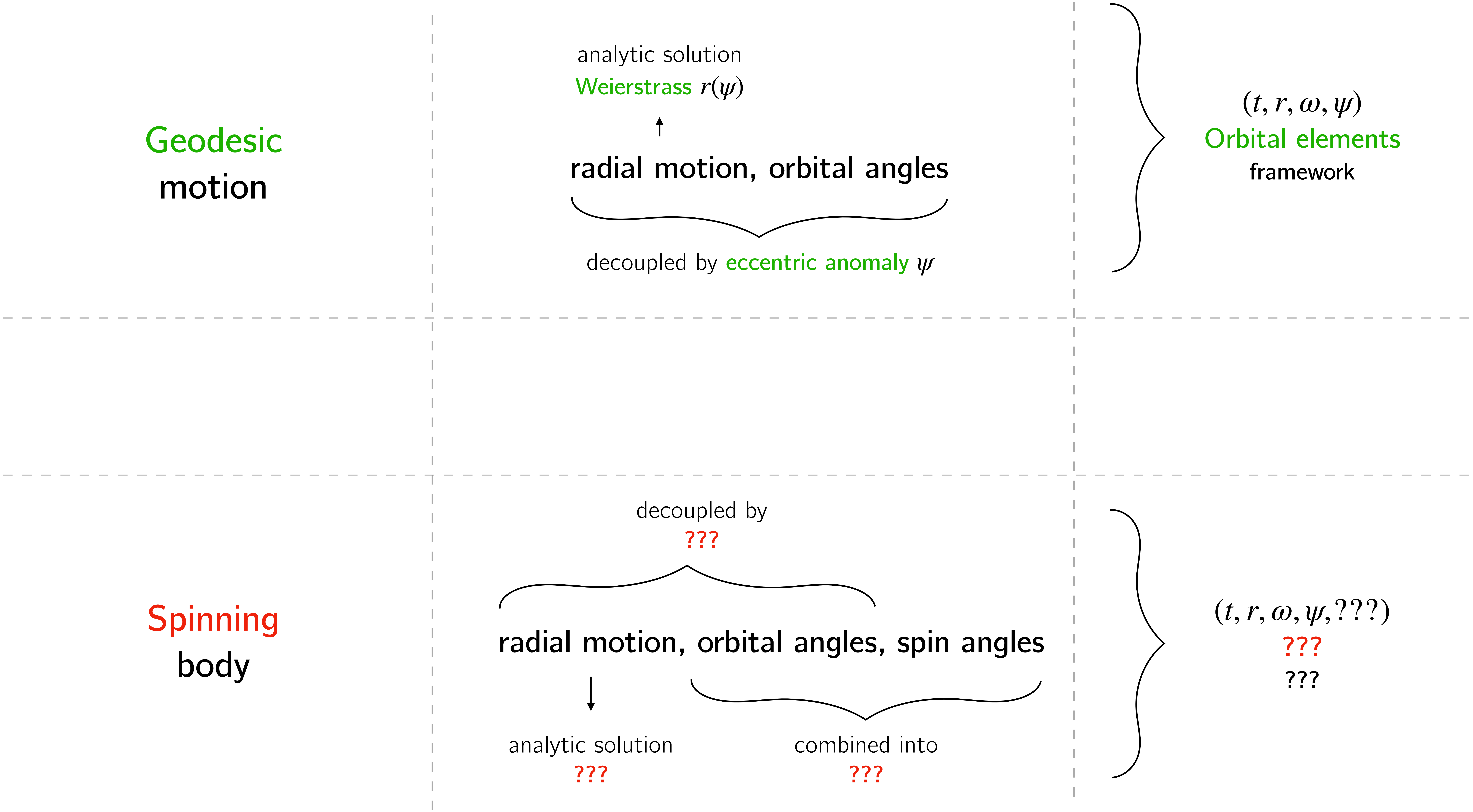
Geodesic
motion

analytic solution
Weierstrass $r(\psi)$
↑

radial motion, orbital angles

decoupled by eccentric anomaly ψ

(t, r, ω, ψ)
Orbital elements
framework



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Orbital elements

framework

Spinning body

decoupled by

???

radial motion, orbital angles, spin angles

↓

analytic solution

???

combined into

???

$(t, r, \omega, \psi, ???)$

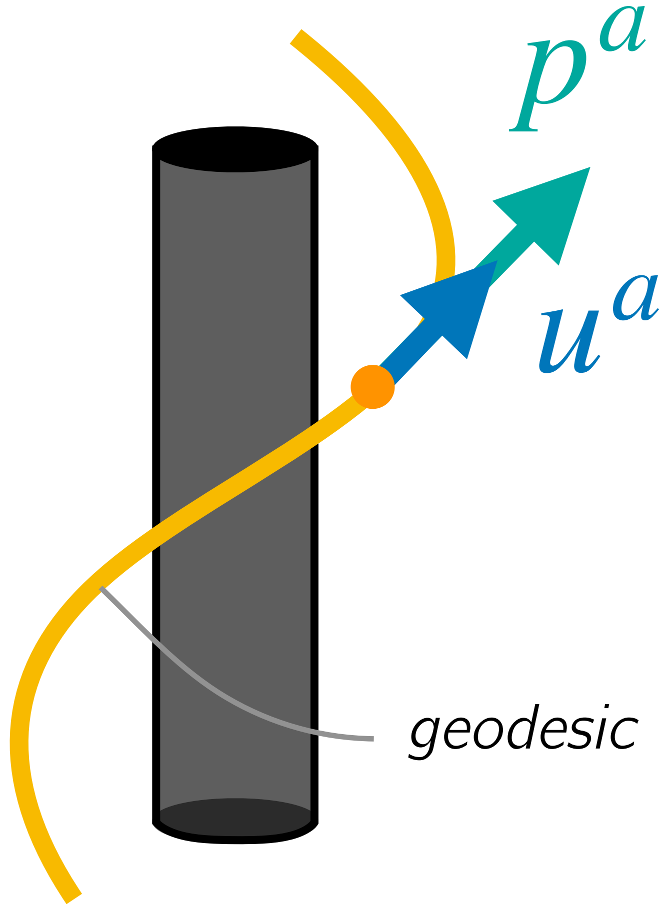
???

???

Turning all this into a Hamiltonian system

2. Spinning case

Hamiltonian formulation of geodesic motion



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Evolution equations
+
Tulczyjew-Dixon SSC

To linear order:

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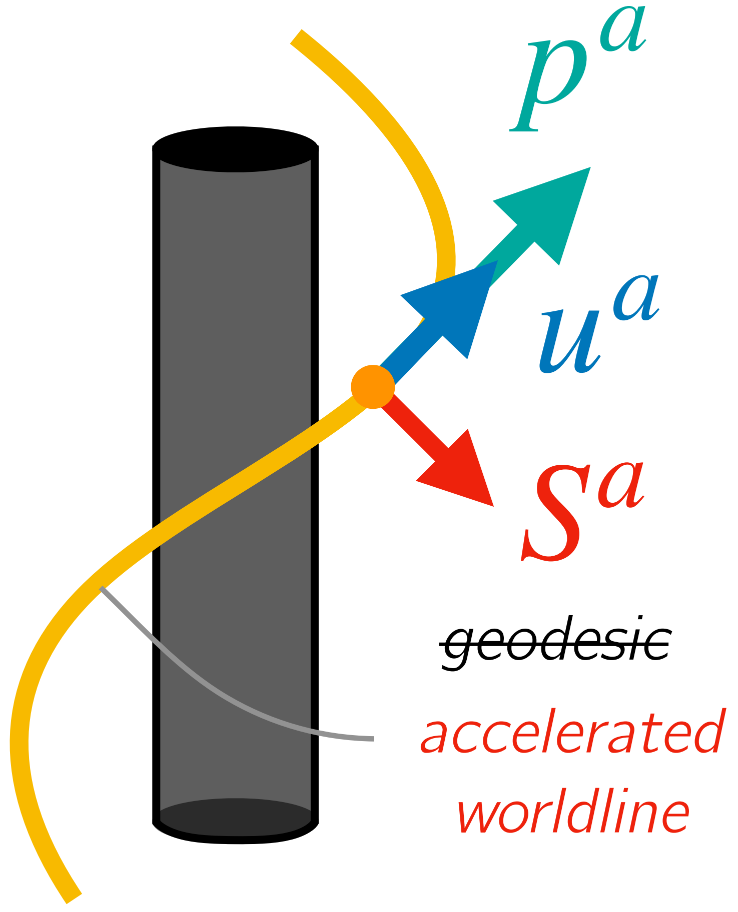
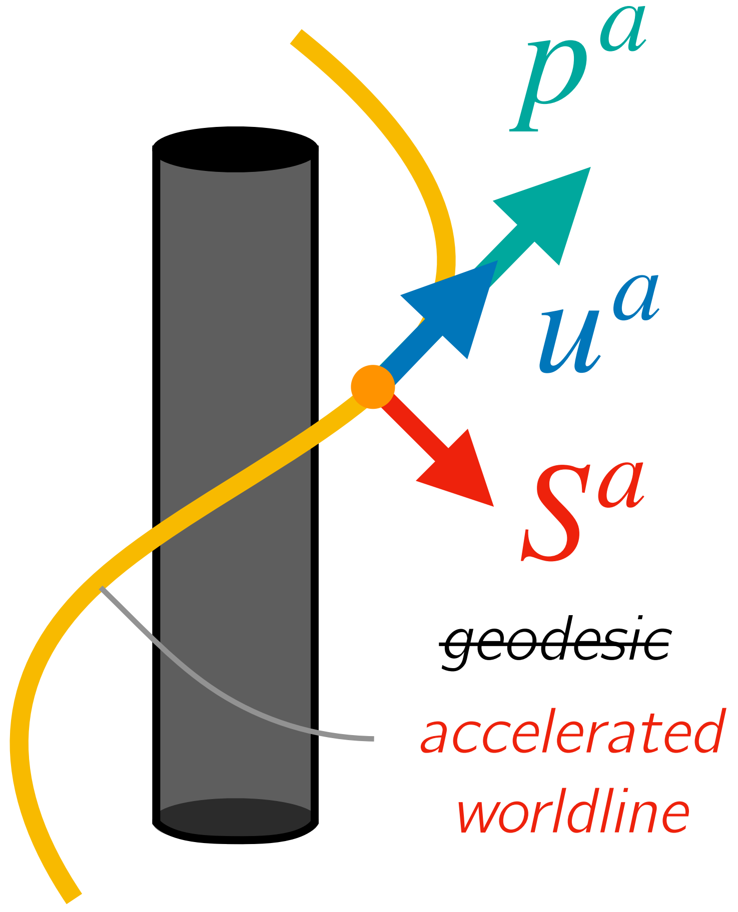


Hamiltonian formulation of spinning motion



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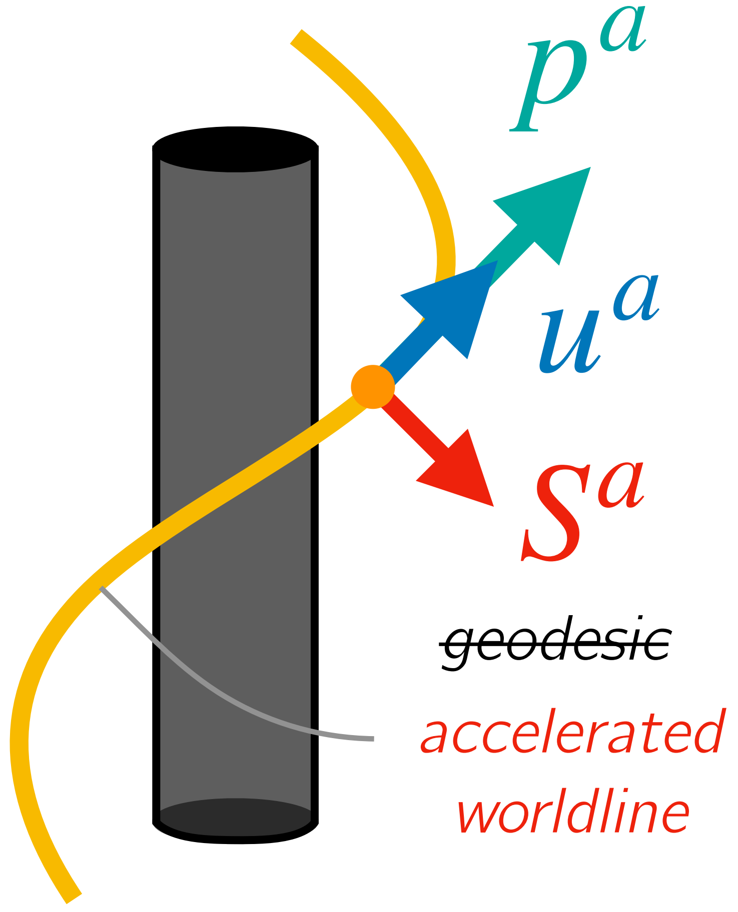
Hamiltonian formulation

- Phase space: $(x^\alpha, p_\alpha, S^{\alpha\beta}) \in \mathbb{R}^4 \times \mathbb{R}^4 \in \mathbb{R}^6$
- Symplectic structure: ~~canonical~~
- Hamiltonian: $H = \frac{1}{2} g^{\alpha\beta} p_\alpha p_\beta$
- Phase space constraint: $p_\alpha S^{\alpha\beta} = 0$

Hamiltonian formulation of spinning motion



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Needs fixing !

↓

cf. Paper I
arXiv:2210.03866

- Phase space: $(x^\alpha, \pi_\alpha, \sigma, \pi_\sigma) \in \mathbb{R}^4 \times \mathbb{R}^4 \in \mathbb{R}^2$
- Symplectic structure: canonical
- Hamiltonian: $H(x^\alpha, \pi_\alpha, \sigma, \pi_\sigma)$

Application:
a spin in Schwarzschild

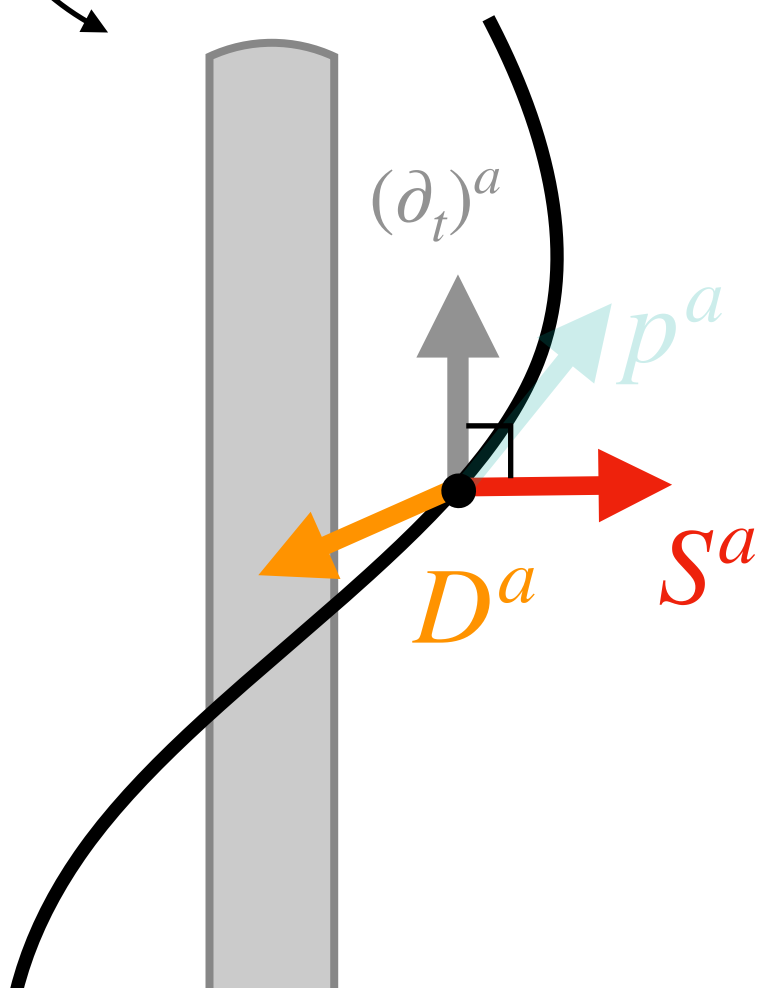
Example in Schwarzschild

- Hamiltonian: $H(t, \pi_t, r, \pi_r, \theta, \pi_\theta, \phi, \pi_\phi, \overbrace{\sigma, \pi_\sigma})$

canonical coord. for $\vec{S}$

$$H = -\frac{\pi_t^2}{2f} + \frac{f\pi_r^2}{2} + \frac{1}{2r^2}\left(\pi_\theta^2 + \frac{\pi_\phi^2}{\sin^2\theta} - \frac{2\pi_\phi\cos\theta}{\sin^2\theta}S^1\right) + \frac{\pi_\phi\sqrt{f}}{r^2\sin\theta}S^2 - \frac{\sqrt{f}\pi_\theta}{r^2}S^3 + \frac{M\pi_t}{r^2 f}D^1$$

functions of (σ, π_σ)



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functions of (σ, π_σ)

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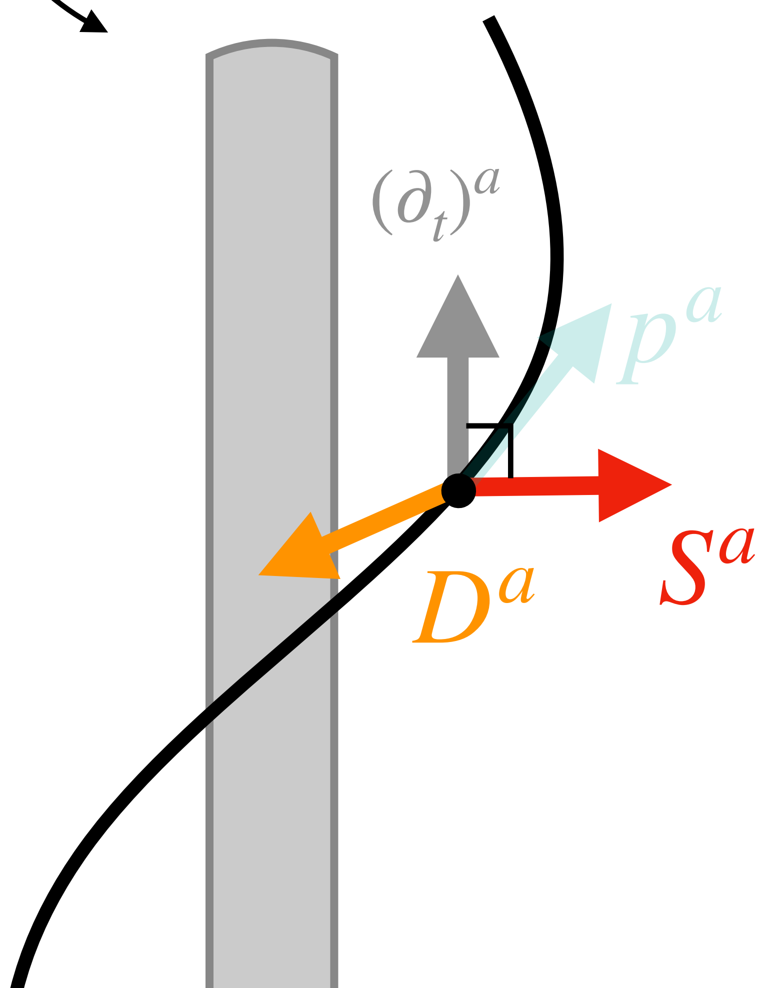
  - component of ang. mom. $J_z = \pi_\phi$
  - Rüdiger invariant. $\mathscr{K} = rD^1(\sigma, \pi_\sigma)$



Hamiltonian reduction

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$$(\theta, \pi_\theta, \phi, \pi_\phi, \sigma, \pi_\sigma) \mapsto (\underbrace{\psi}_{\text{eccentric anomaly}}, \underbrace{J, \omega, J_z}_{\text{invariant plane}}, \underbrace{s, \pi_s}_{\text{combine spin+orbital}})$$

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"Andoyer" variables

(1860's)

cf. lunar problem in classical mechanics

Hamiltonian reduction

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eccentric anomaly

invariant plane

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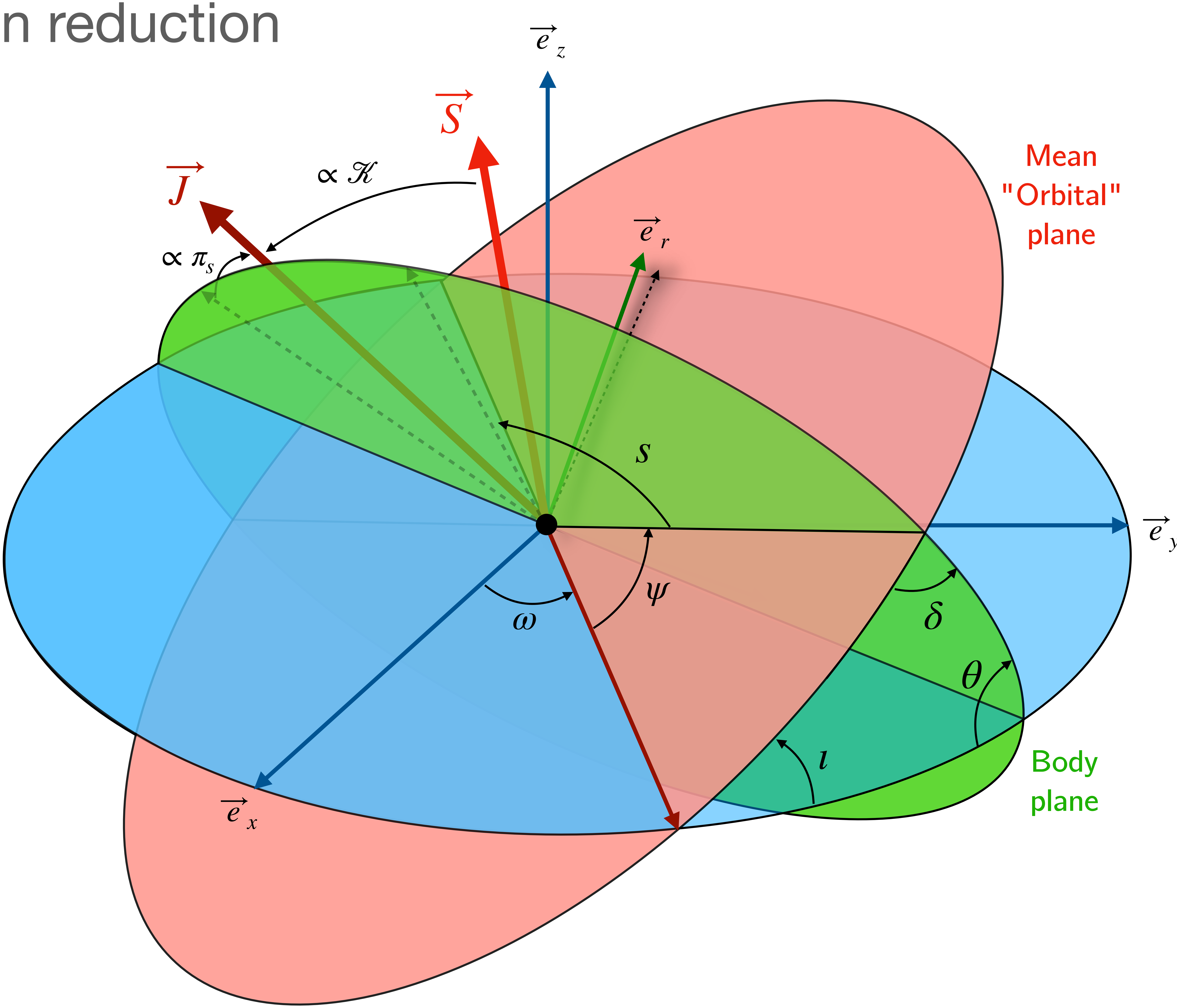

After the Hamiltonian reduction

Orbital motion **decoupled** from **spin** motion **!**

After the Hamiltonian reduction

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- Reduced Hamiltonian for (t, r, ψ, ω)

$$H(r, \pi_r) = -\frac{E^2}{2f} + \frac{f\pi_r^2}{2} + \frac{J^2}{2r^2} - \frac{ME\mathscr{K}}{r^3 f}$$

- **1 dof** Hamiltonian
- same **analytic solutions**
- **spin** affects orbit via $(\mathscr{K}, J)$

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  - **1 dof** Hamiltonian
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- Hill equation for (s, π_s)

$$\frac{d^2F(\psi)}{d\psi^2} + \left(1 - \frac{3M}{r(\psi)}\right) F(\psi) = 0$$

  - parametric **harm. oscil.**
  - **same equation** for s and π_s
  - **quasi-periodic** solutions

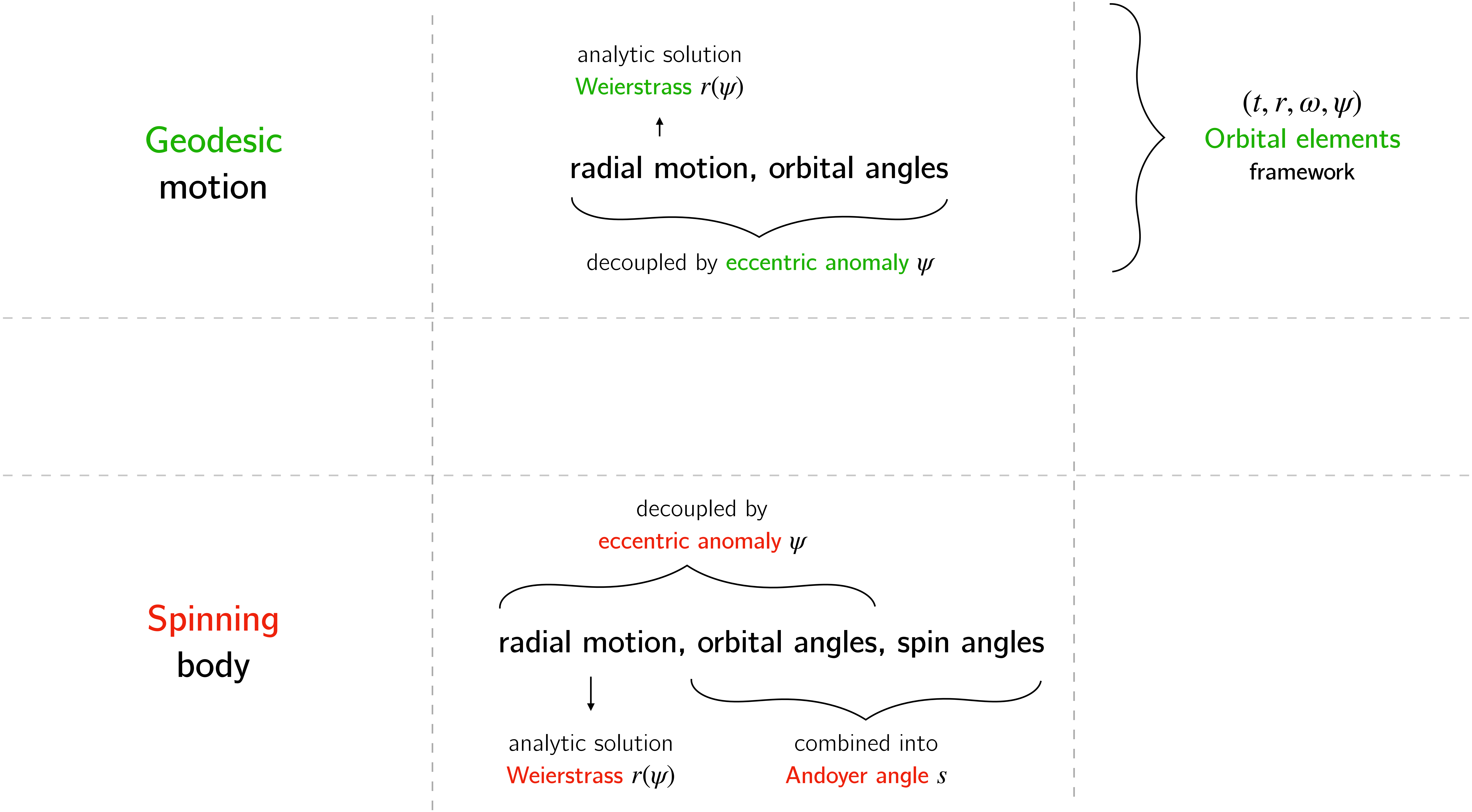
Geodesic
motion

analytic solution
Weierstrass $r(\psi)$
↑
radial motion, orbital angles

decoupled by eccentric anomaly ψ

(t, r, ω, ψ)
Orbital elements
framework

Spinning
body



Geodesic motion

analytic solution
Weierstrass $r(\psi)$

radial motion, orbital angles

decoupled by eccentric anomaly ψ

(t, r, ω, ψ)
Orbital elements framework

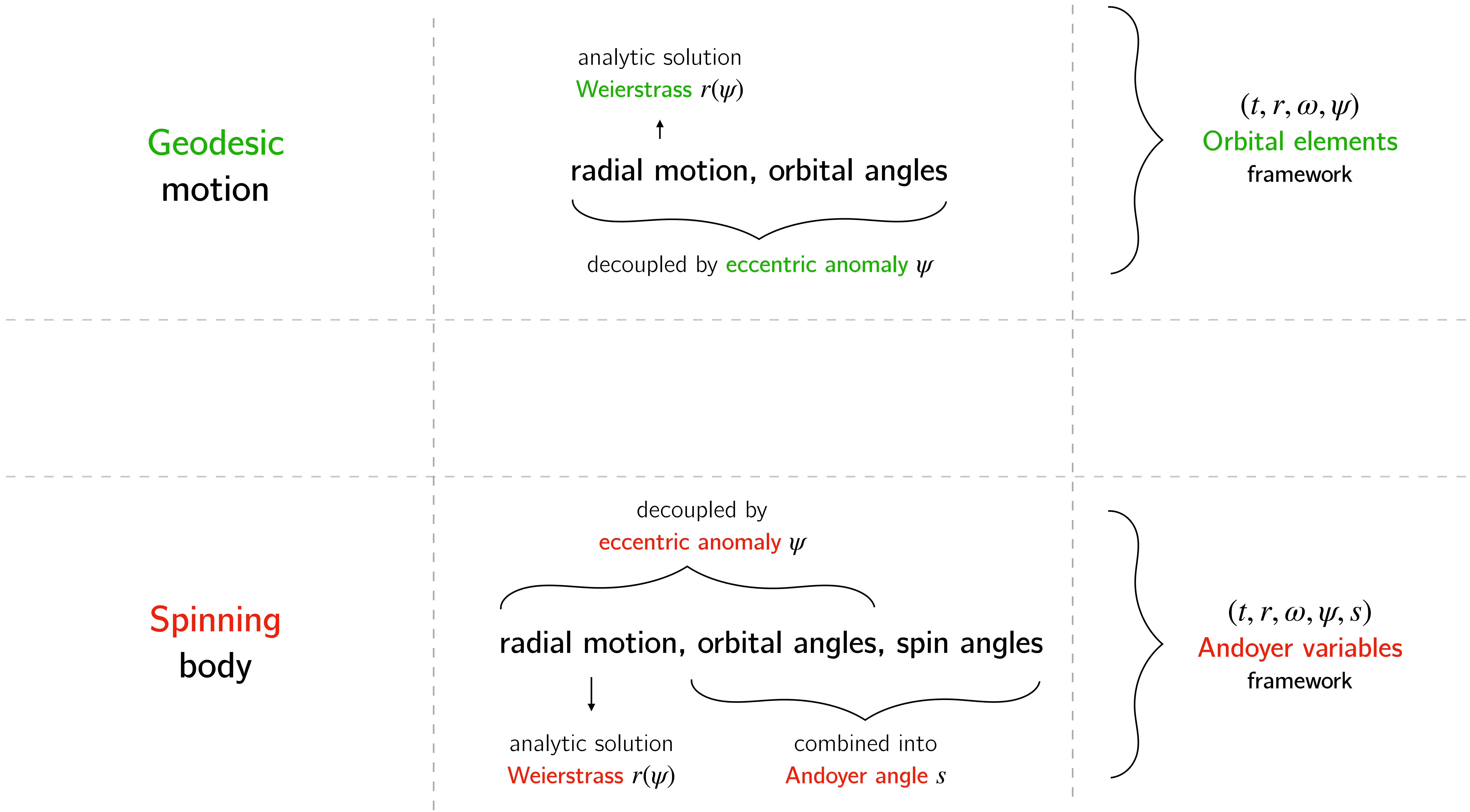
Spinning body

decoupled by
eccentric anomaly ψ

radial motion, orbital angles, spin angles

analytic solution
Weierstrass $r(\psi)$

combined into
Andoyer angle s



Geodesic motion

analytic solution

Weierstrass $r(\psi)$

↑

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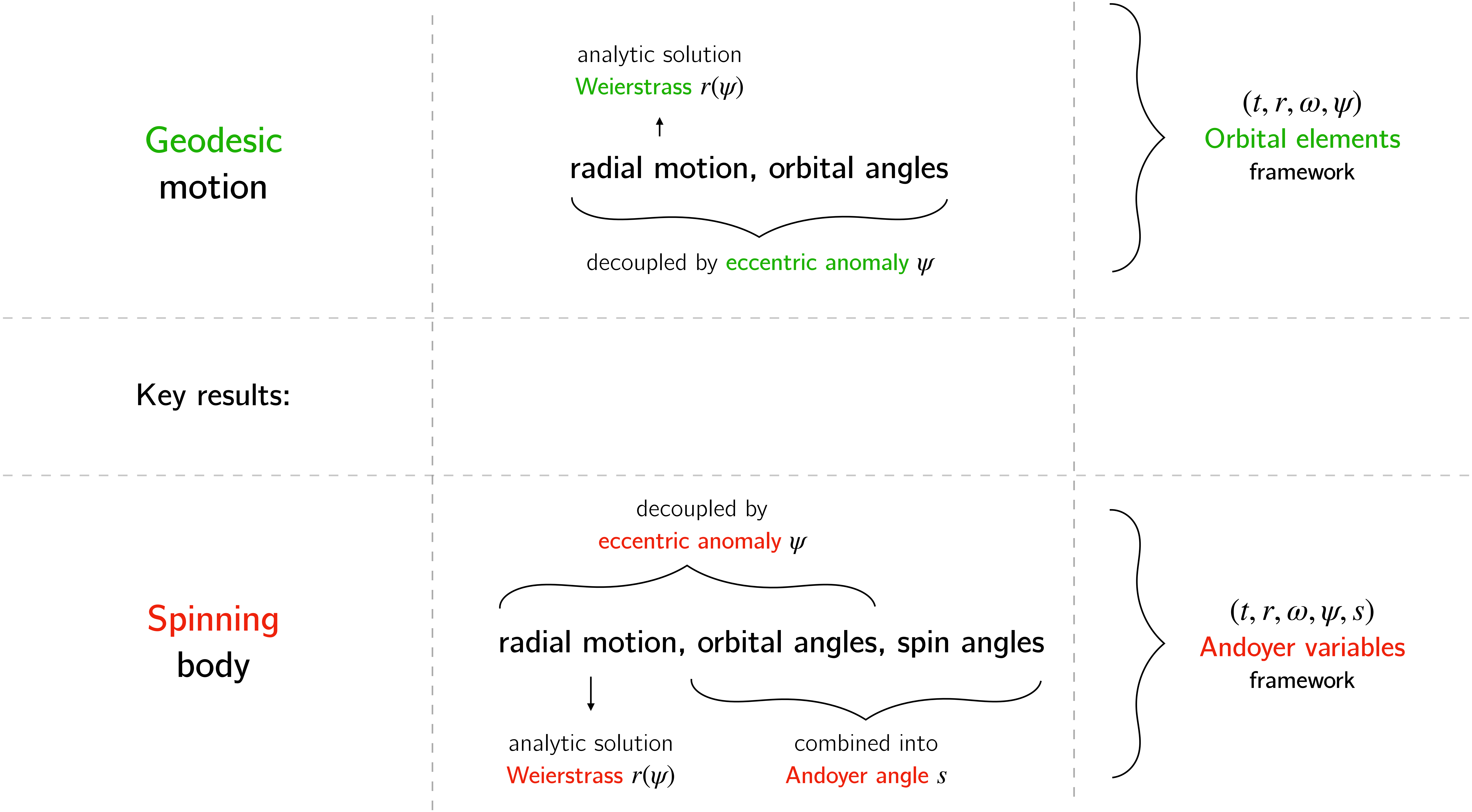
Andoyer variables

framework



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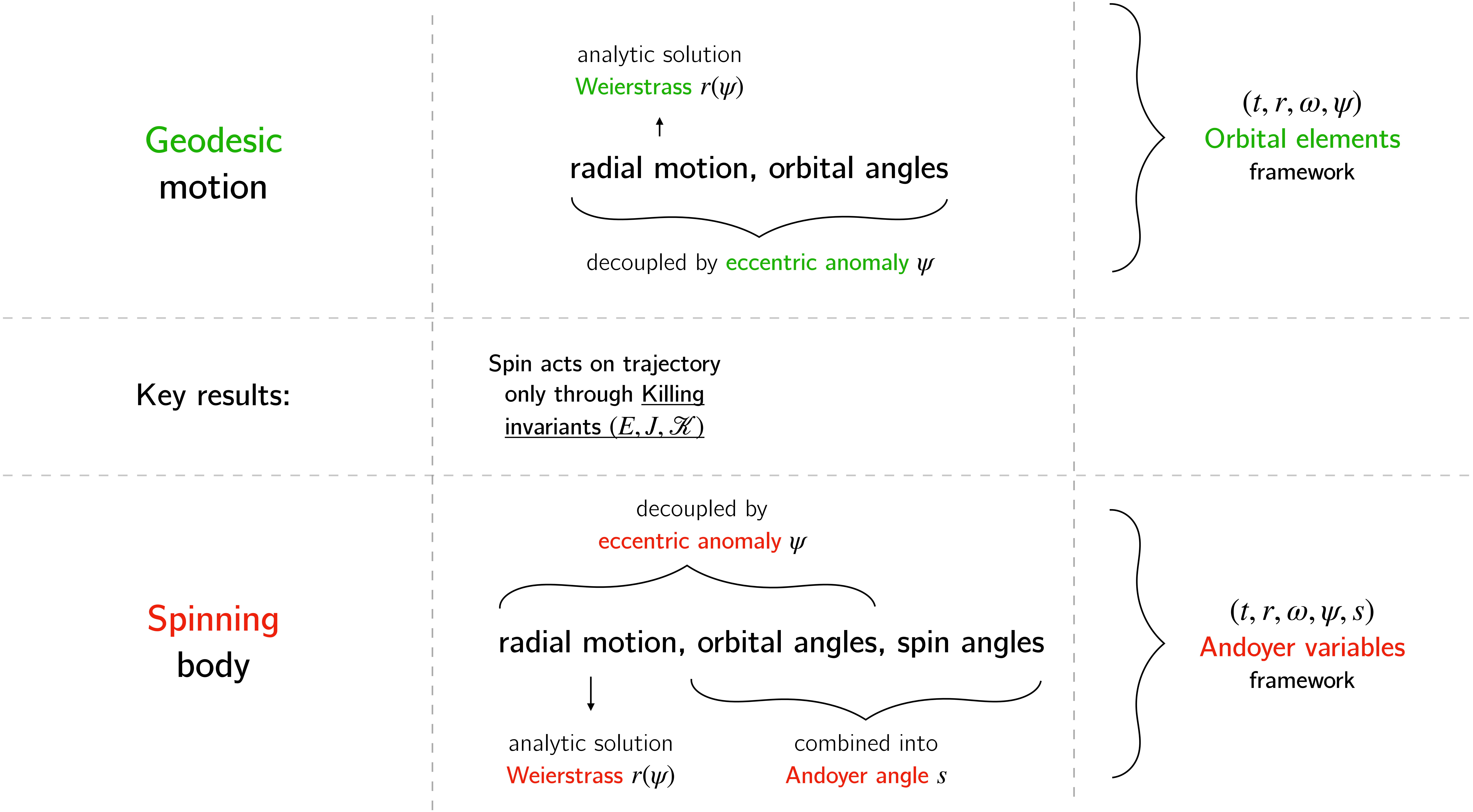

Geodesic motion

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Orbital elements framework

Key results:

Spin acts on trajectory only through <u>Killing invariants $(E, J, \mathscr{K})$</u>

Spinning body

decoupled by eccentric anomaly ψ

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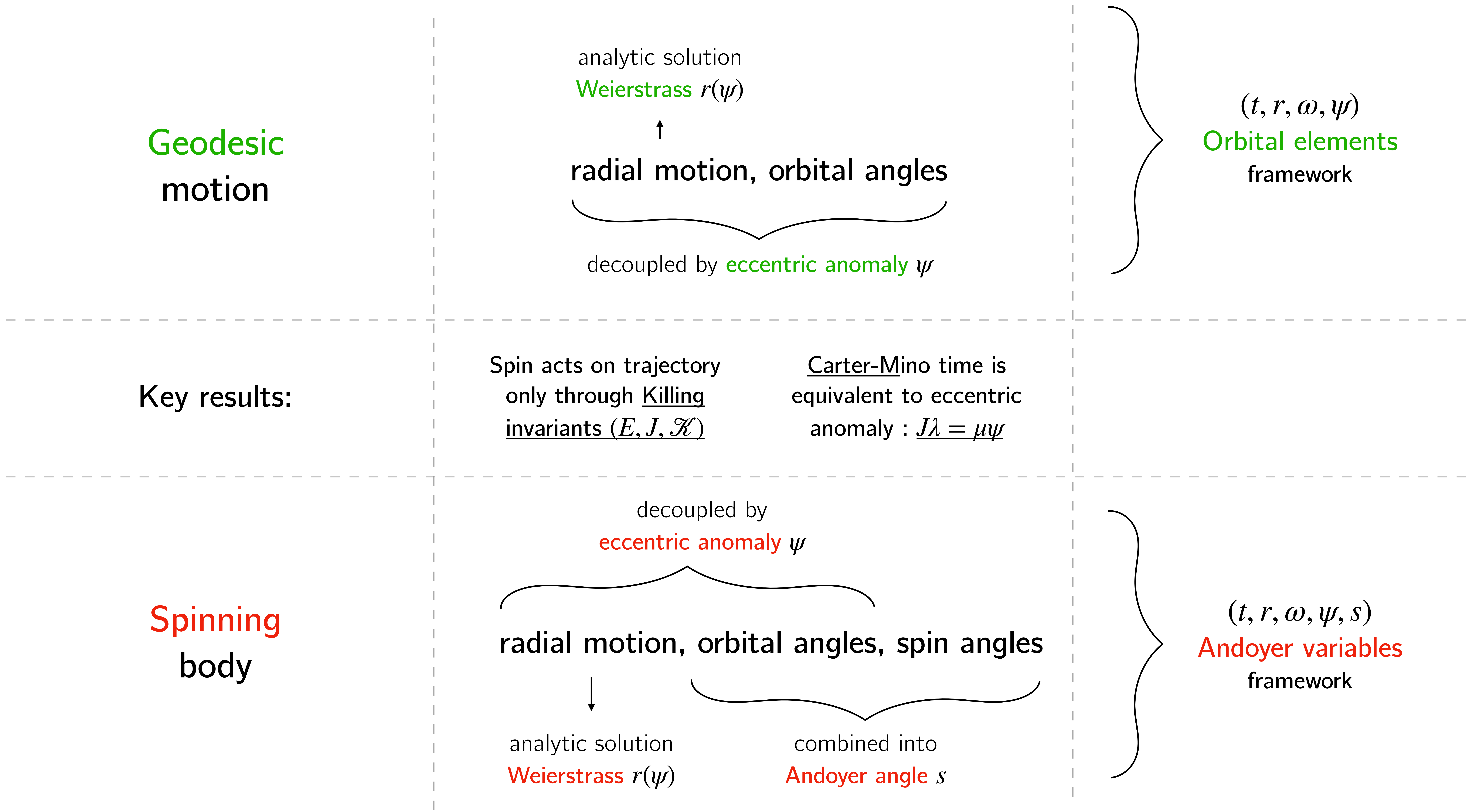
Andoyer variables framework



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Carter-Mino time is equivalent to eccentric anomaly : $J\lambda = \mu\psi$

Spinning body

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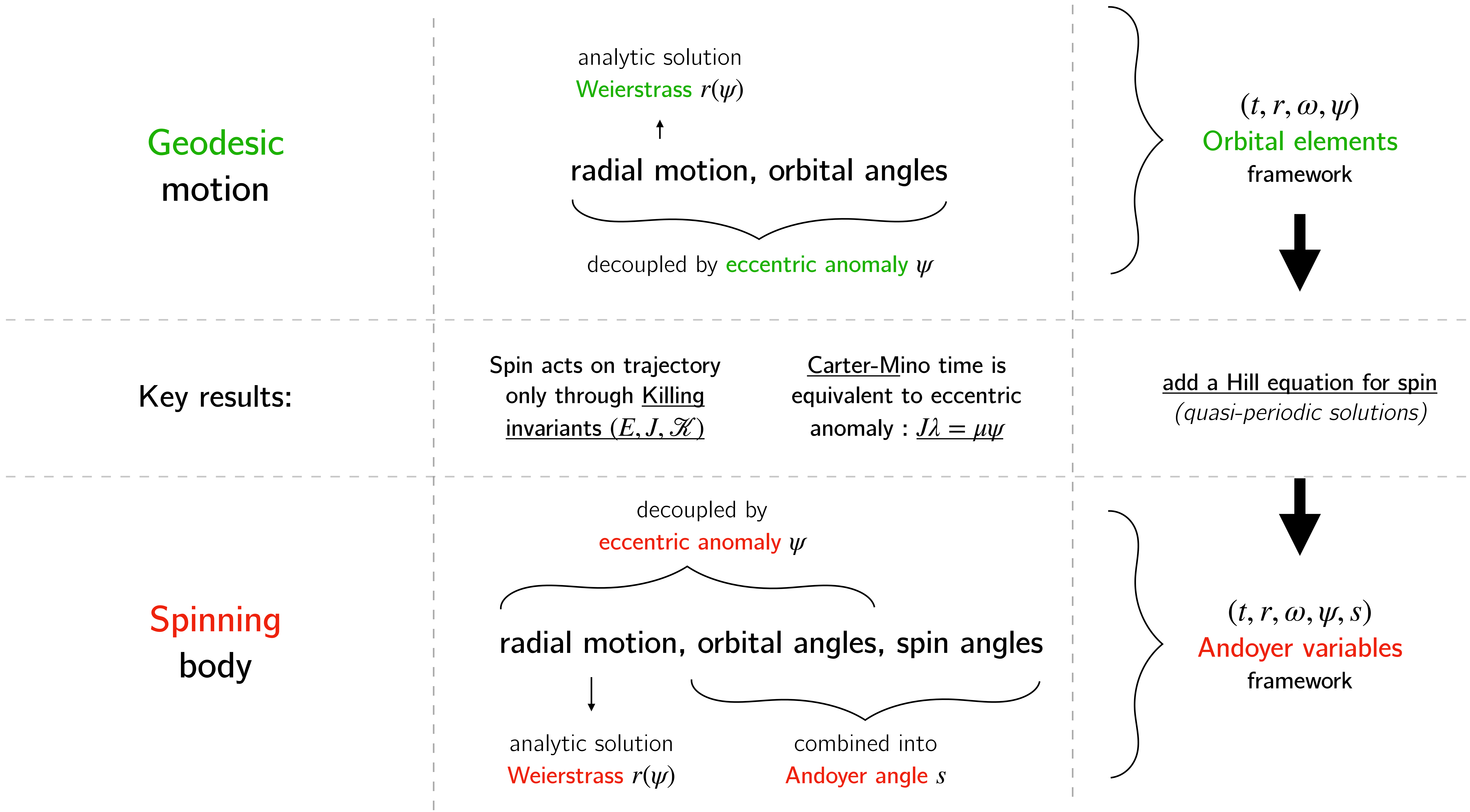

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Key results:

Spin acts on trajectory only through Killing invariants $(E, J, \mathscr{K})$

Carter-Mino time is equivalent to eccentric anomaly : $J\lambda = \mu\psi$

add a Hill equation for spin
(quasi-periodic solutions)

Spinning body

decoupled by
eccentric anomaly ψ

radial motion, orbital angles, spin angles

analytic solution
Weierstrass $r(\psi)$

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To conclude

Recap'

- Unified Hamiltonian framework for a spin around Schwarzschild
- New Andoyer variables decouple radial (analytic) from rotational (semi-analytic)
- Works for <u>any spin configuration</u> (mis-aligned)

  and <u>any orbital configuration</u> (non-planar, eccentric, non-equatorial)
- Can do everything like geodesics: ISCO, separatrix, resonances, etc

To conclude

Recap'

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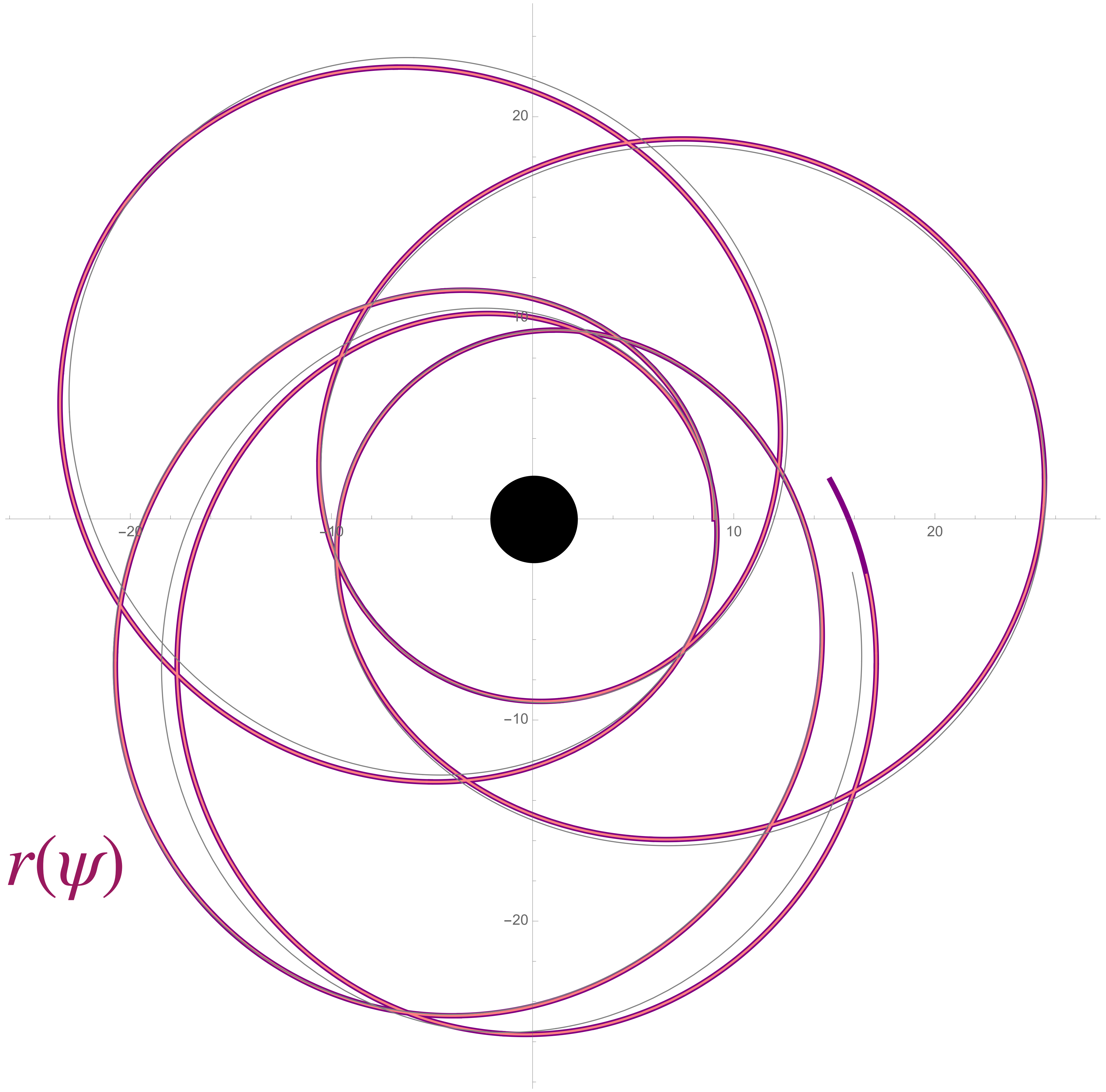
- Works for any spin configuration (mis-aligned)

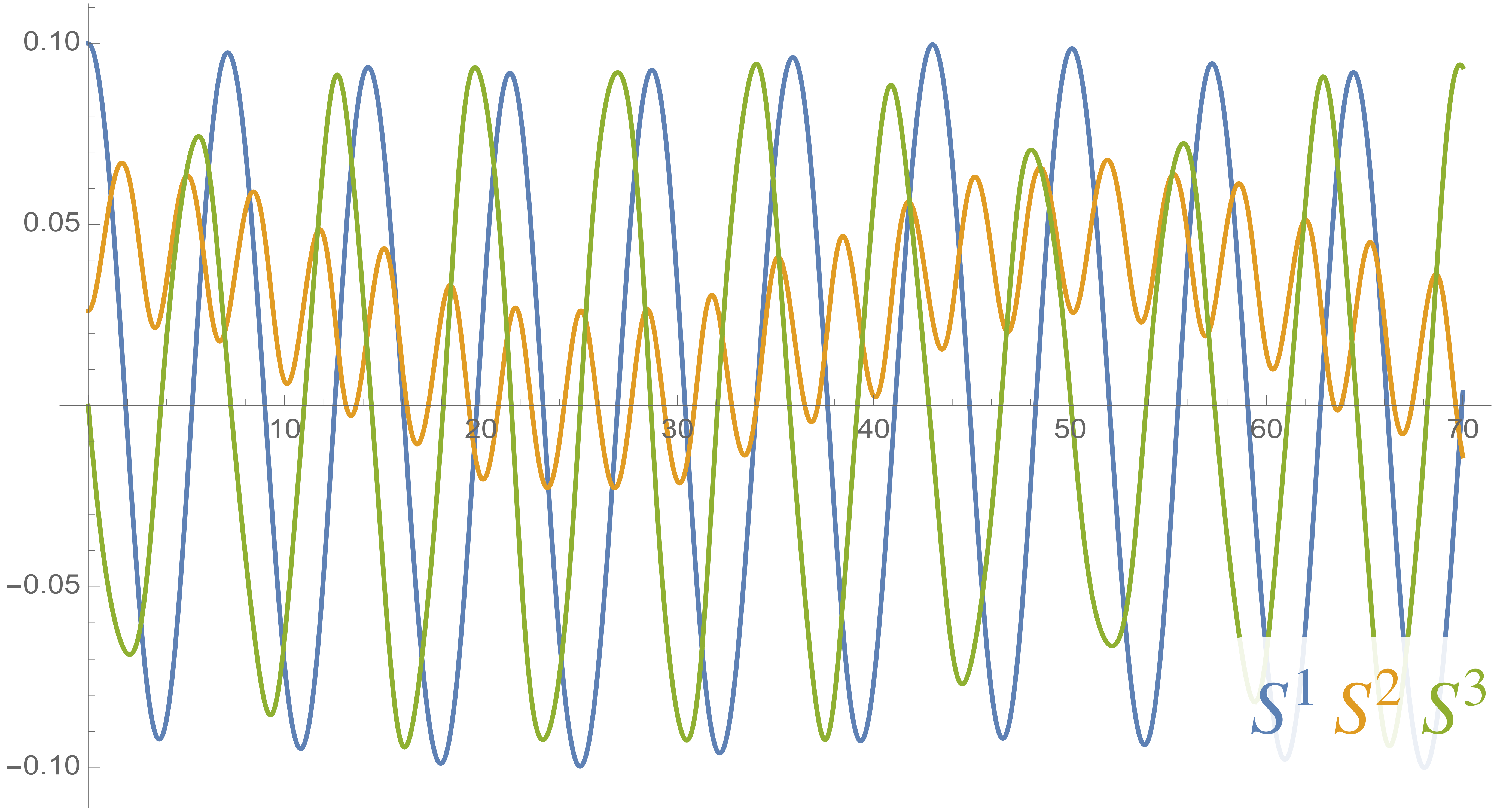
  and any orbital configuration (non-planar, eccentric, non-equatorial)
- Can do everything like geodesics: ISCO, separatrix, resonances, etc

Future *(currently in progress)*

- Whole framework generalises to Kerr (all key invariants still exist)
- Hamiltonian formulation generalises to quadratic-in-spin

Thank you !


20
10
-20
-10
10
20
-10
-20
$r(\psi)$


0.10
0.05
-0.05
-0.10
10
20
30
40
50
60
70
S^1
S^2
S^3

Backup

4-steps solution recipe *(e.g., bound orbit)*

1) Choose $\mathscr{K}$ and initial conditions

$$(t, r, \omega, \psi, s) = (\,0, r_p, \omega, 0, s(0)\,)$$

$$(\pi_t, \pi_r, \pi_\omega, \pi_\psi, \pi_s) = (\,-E, 0\,, J_z\,, J, \pi_s(0)\,)$$

s.t. bound orbit
(cf. bifurc. diagram)

periapsis

fixed mean plane

$\mathscr{K}$ ↗

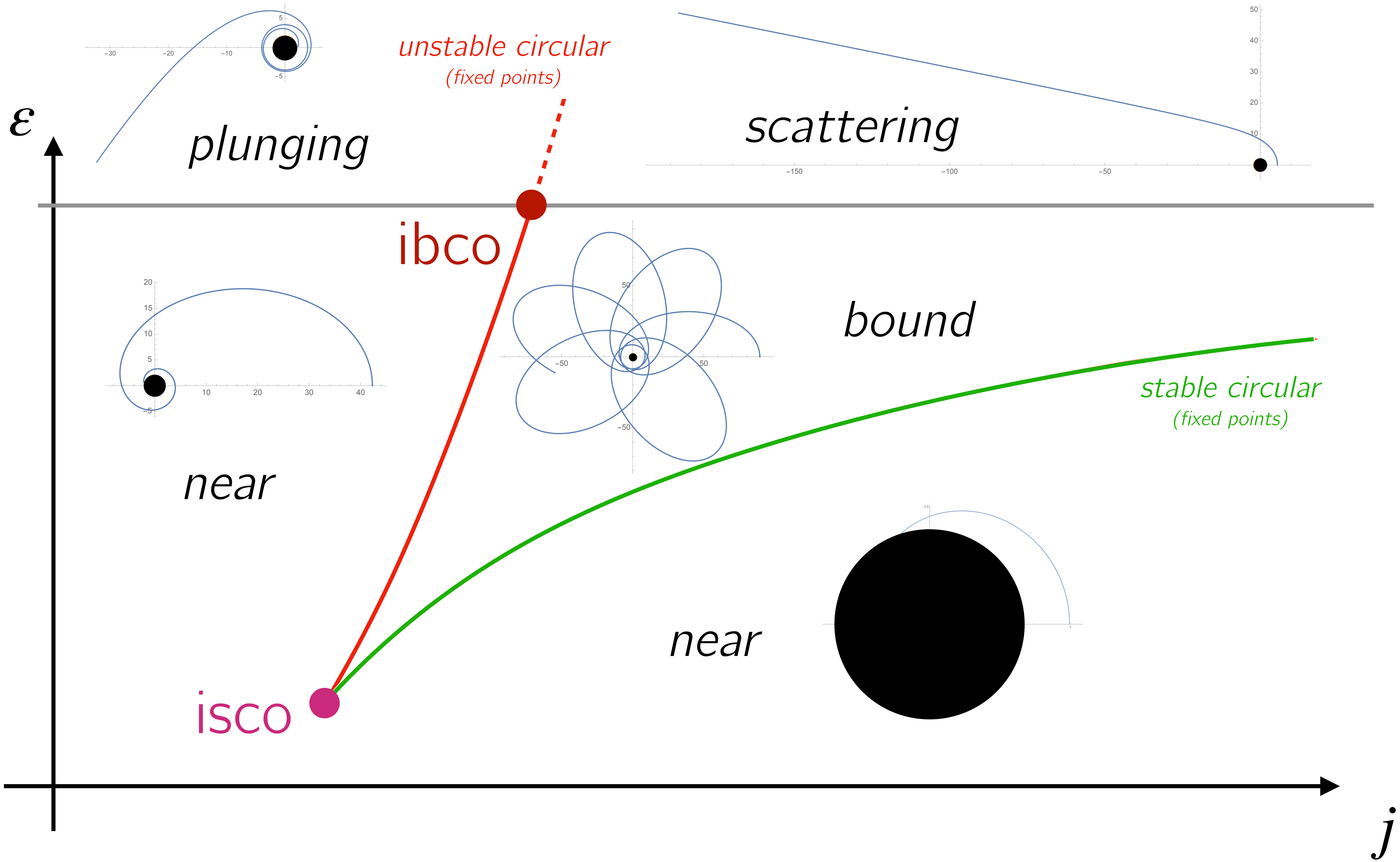
initial spin $\vec{S}$
configuration
(3 dofs)

2) Analytic solution for $r(\psi)$

3) Solve Hill equation for $s(\psi), \pi_s(\psi)$

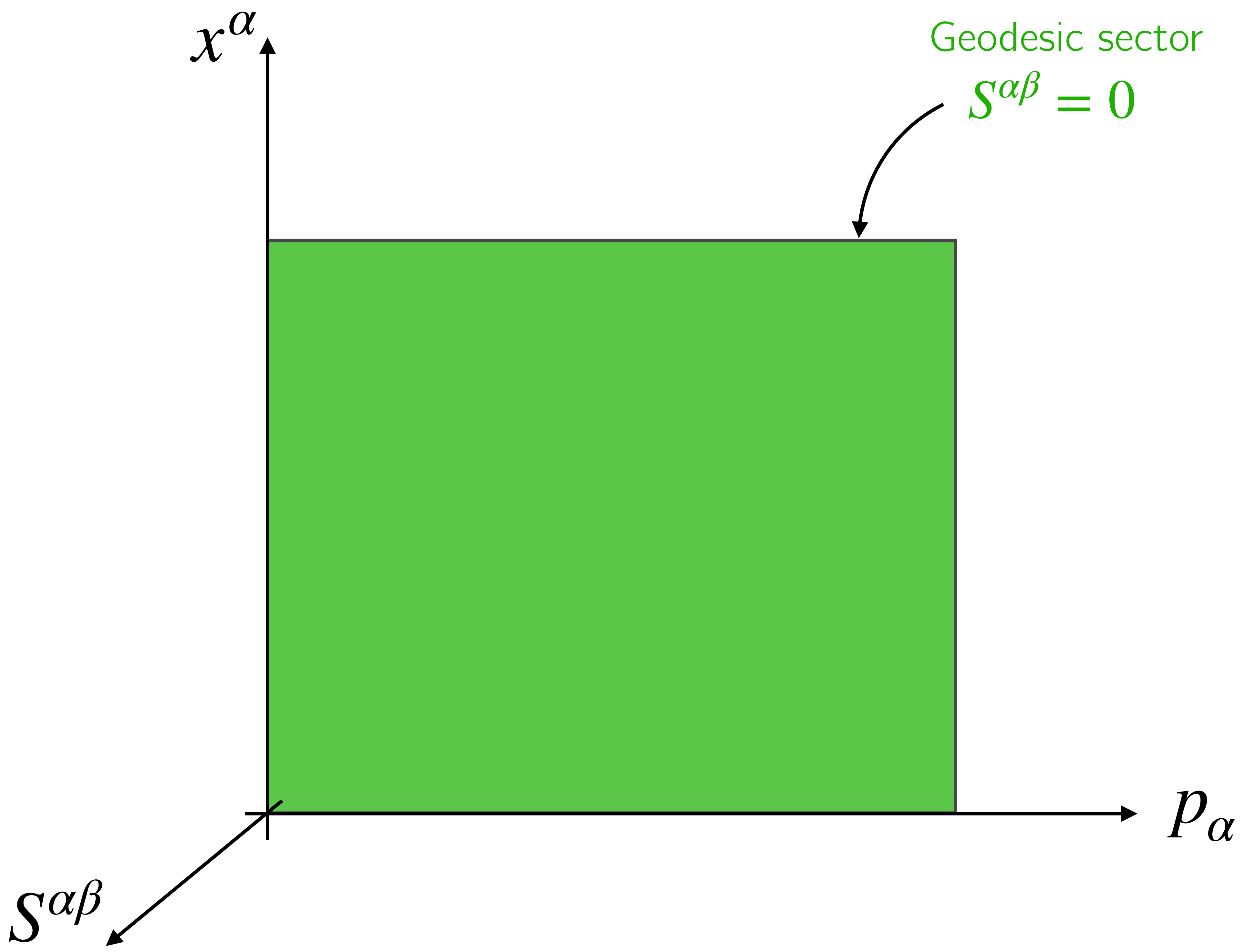
4) Algebraic relations for $(t, \theta, \varphi, S^\alpha)$ and $\psi(\tau)$

orbit config *spin config*	planar		non-planar	
	circ.	ecc.	circ.	eccentric
aligned	analytic	unphysical	analytic	unphysical
misaligned	unphysical	analytic	unphysical	semi-analytic

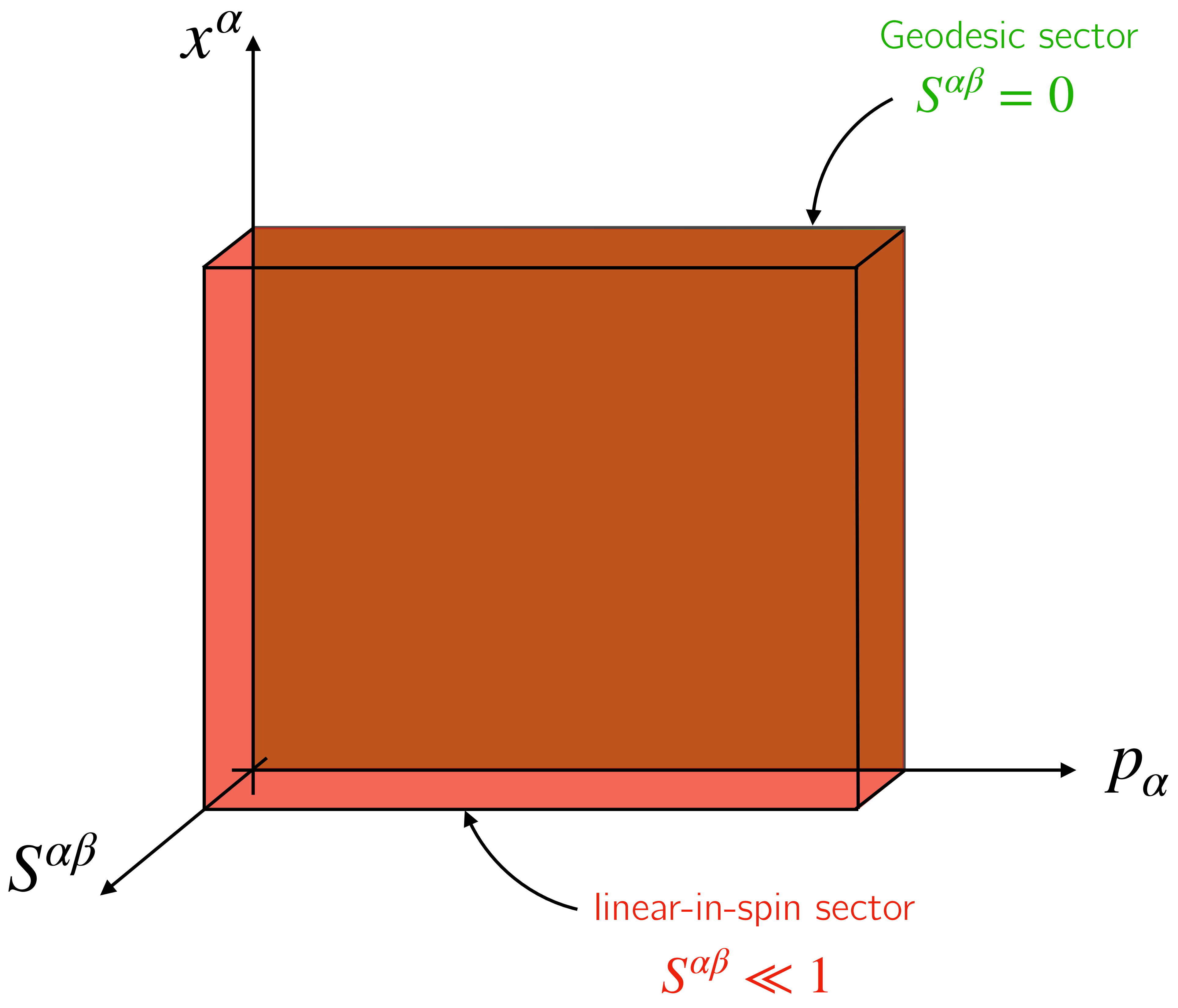


plunging
unstable circular
(fixed points)
scattering
ε
ibco
bound
stable circular
(fixed points)
near
near
isco
j

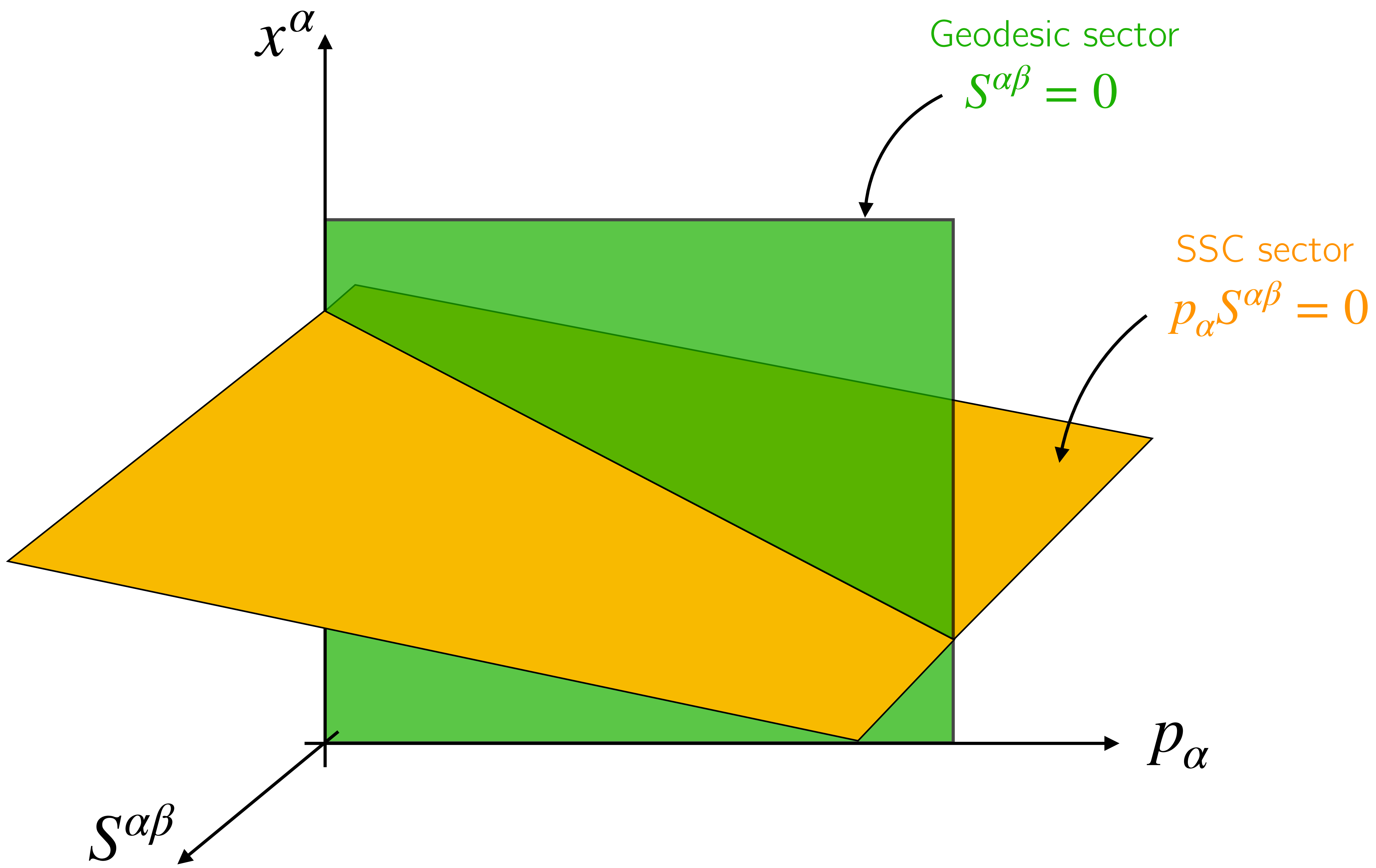
Dipolar particle phase space



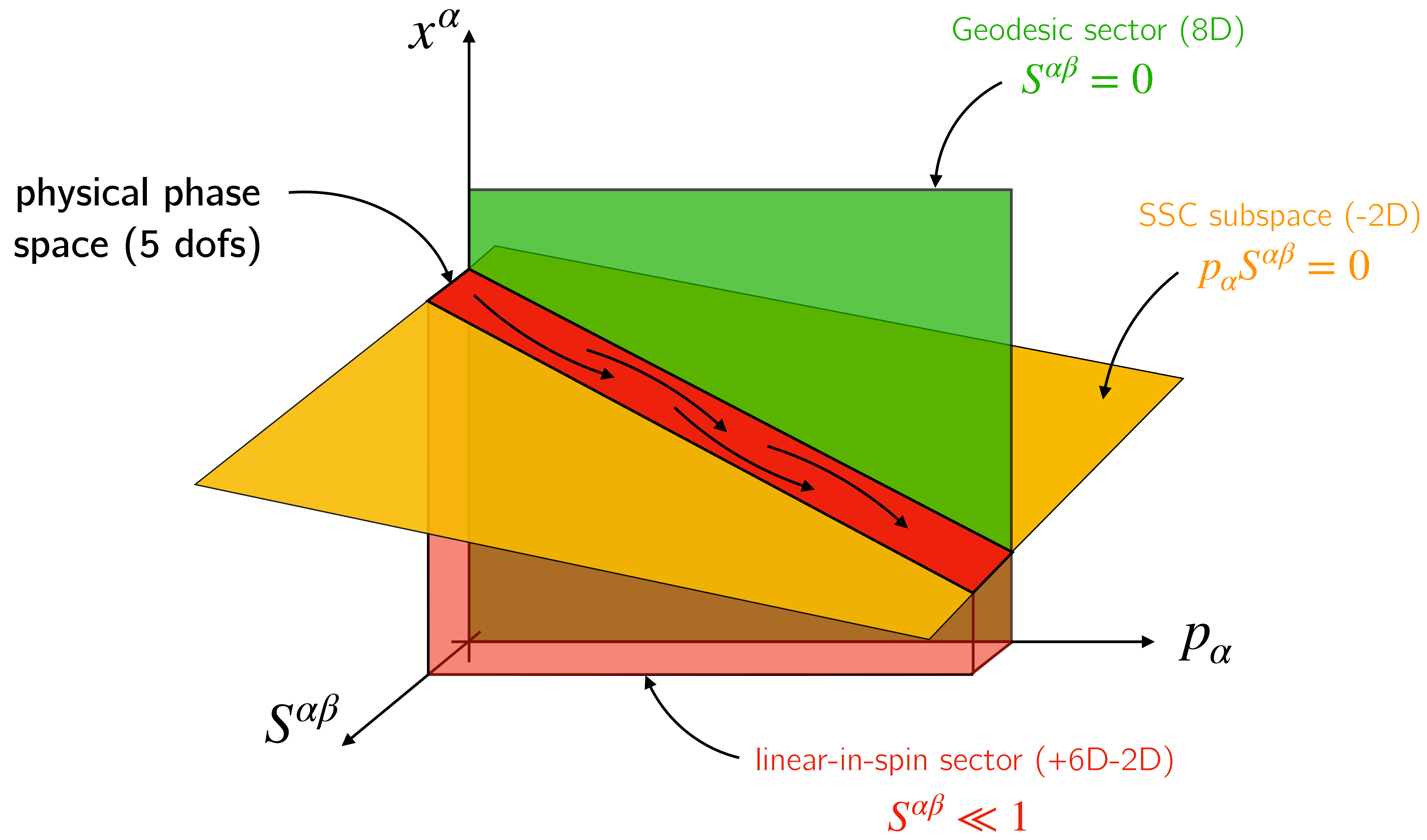
Dipolar particle phase space



Dipolar particle phase space



Dipolar particle phase space



4-steps solution recipe *(e.g., bound orbit)*

1) Choose initial conditions

$$(t, r, \omega, \psi) = (\,0, r_p, \omega, 0,)$$

$$(\pi_t, \pi_r, \pi_\omega, \pi_\psi) = (\,-E, 0\,, J_z\,, J\,)$$

fixed mean
plane

s.t. bound orbit
(cf. bifurc. diagram)

periapsis

2) Analytic solution for $r(\psi)$

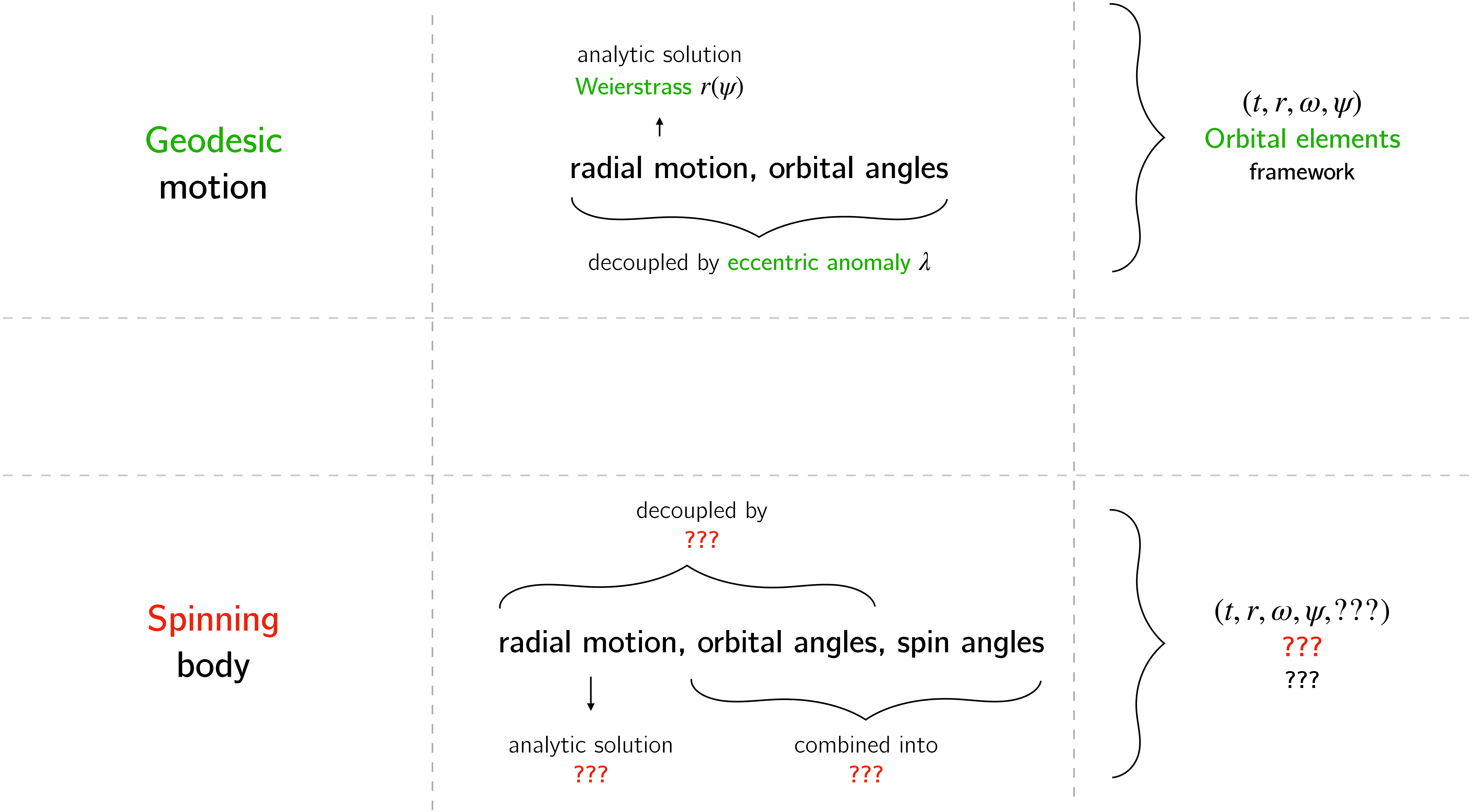
$$r(\psi) = \frac{p}{1 - e\,F(\psi)} \qquad F(\psi) = 1 + \frac{c_0}{\wp(\psi/2) + c_2}$$

3) Algebraic relations for (t, θ, φ) and $\psi(\tau)$

$$\cos\theta = -\sin\iota\cos\psi$$

$$\sin\theta\cos(\omega - \phi) = \sin\psi$$

$$\sin\theta\sin(\omega - \phi) = \cos\psi\cos\iota$$



Geodesic
motion

analytic solution
Weierstrass $r(\psi)$
↑
radial motion, orbital angles

decoupled by **eccentric anomaly** λ

(t, r, ω, ψ)
Orbital elements
framework

Spinning
body

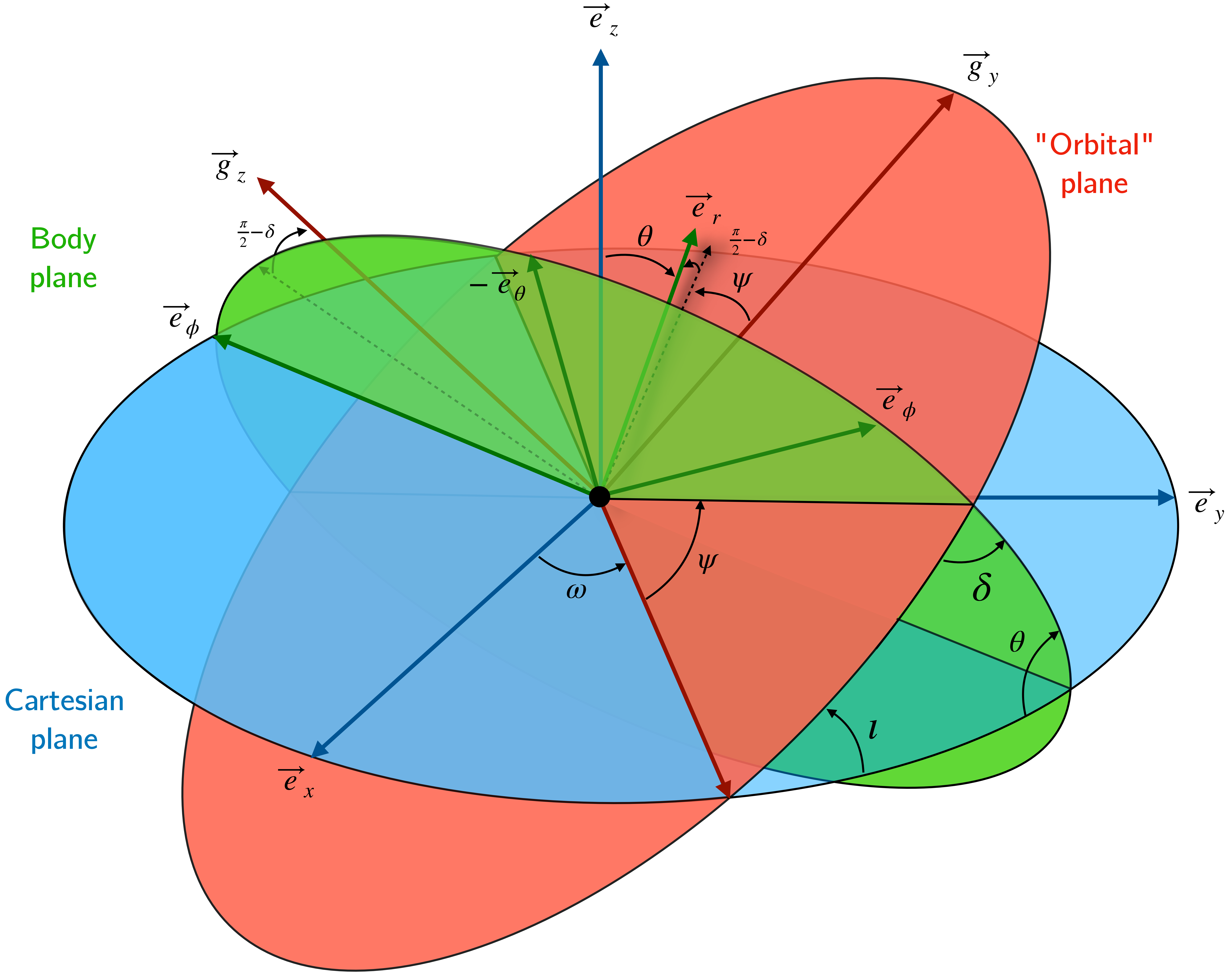
decoupled by
???

radial motion, orbital angles, spin angles

↓
analytic solution
???

combined into
???

$(t, r, \omega, \psi, ???)$
???
???


$\vec{e}_z$
$\vec{g}_y$
"Orbital" plane
$\vec{g}_z$
Body plane
$\frac{\pi}{2}-\delta$
$\vec{e}_r$
θ
$\frac{\pi}{2}-\delta$
$-\vec{e}_\theta$
ψ
$\vec{e}_\phi$
$\vec{e}_\phi$
$\vec{e}_y$
ω
ψ
δ
θ
ι
Cartesian plane
$\vec{e}_x$

An example in Schwarzschild coordinates

- New Hamiltonian $$H = -\frac{E^2}{2f} + \frac{f p_r^2}{2} + \frac{J^2}{2r^2}$$

1 dof Hamiltonian, parameterised by *E,J*

- Analytical solution

$$r(\psi) = \frac{p}{1 - e\,F(\psi)}$$

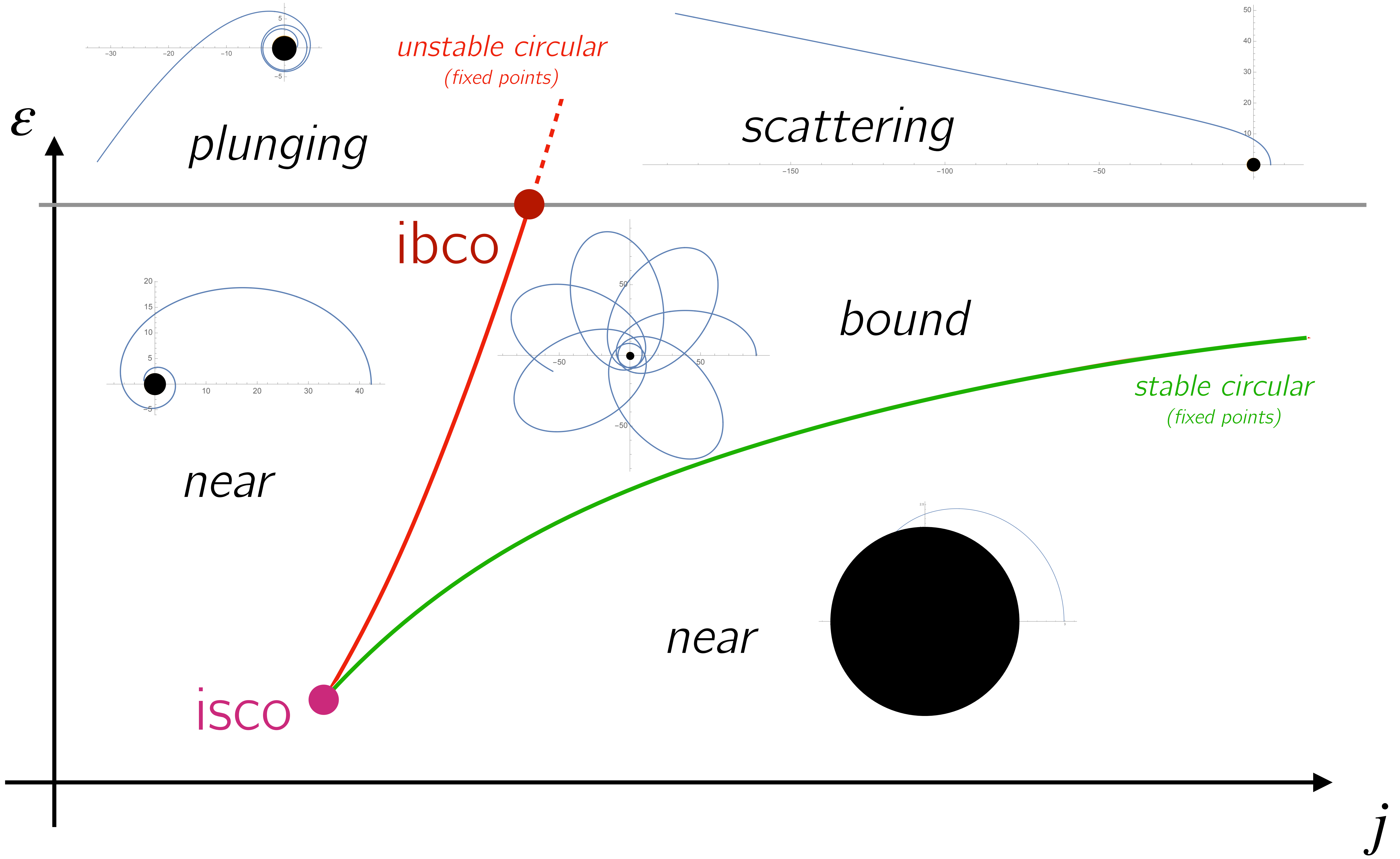
$$e = \frac{r_a - r_p}{r_a + r_p} \qquad p = \frac{2r_a r_p}{r_a + r_p}$$

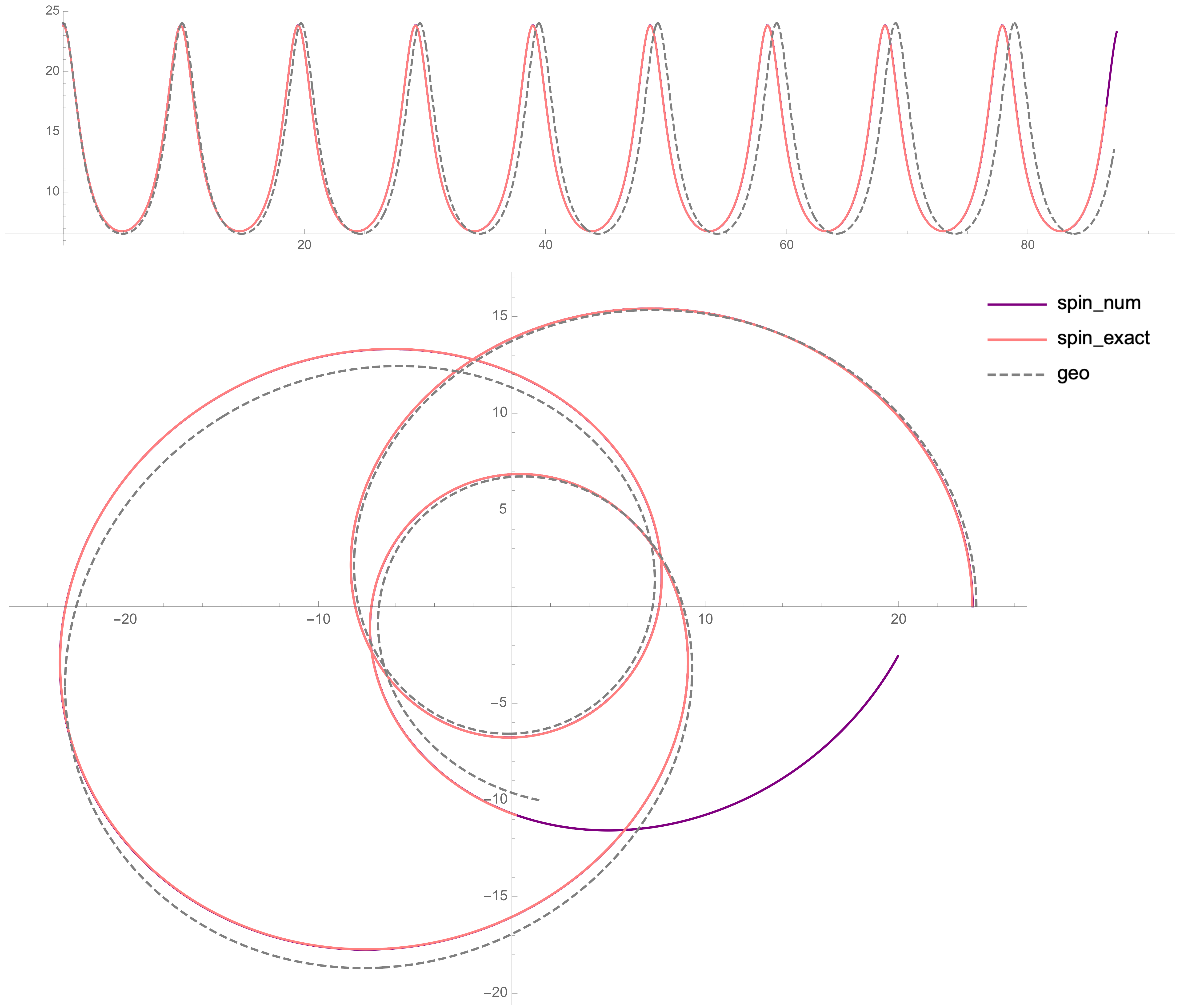
$$F(\psi) = 1 + \frac{2(2(1-f_a) - f_p)}{\wp(\psi/2) + f_a - 2/3}$$

Weierstrass elliptic function

Period:

1<F(ψ)<1




25
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80
spin_num
spin_exact
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15
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5
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−10
−15
−20
−20
−10
10
20

- Canonical transformation

$(t, p_t, r, p_r) \mapsto$ untouched

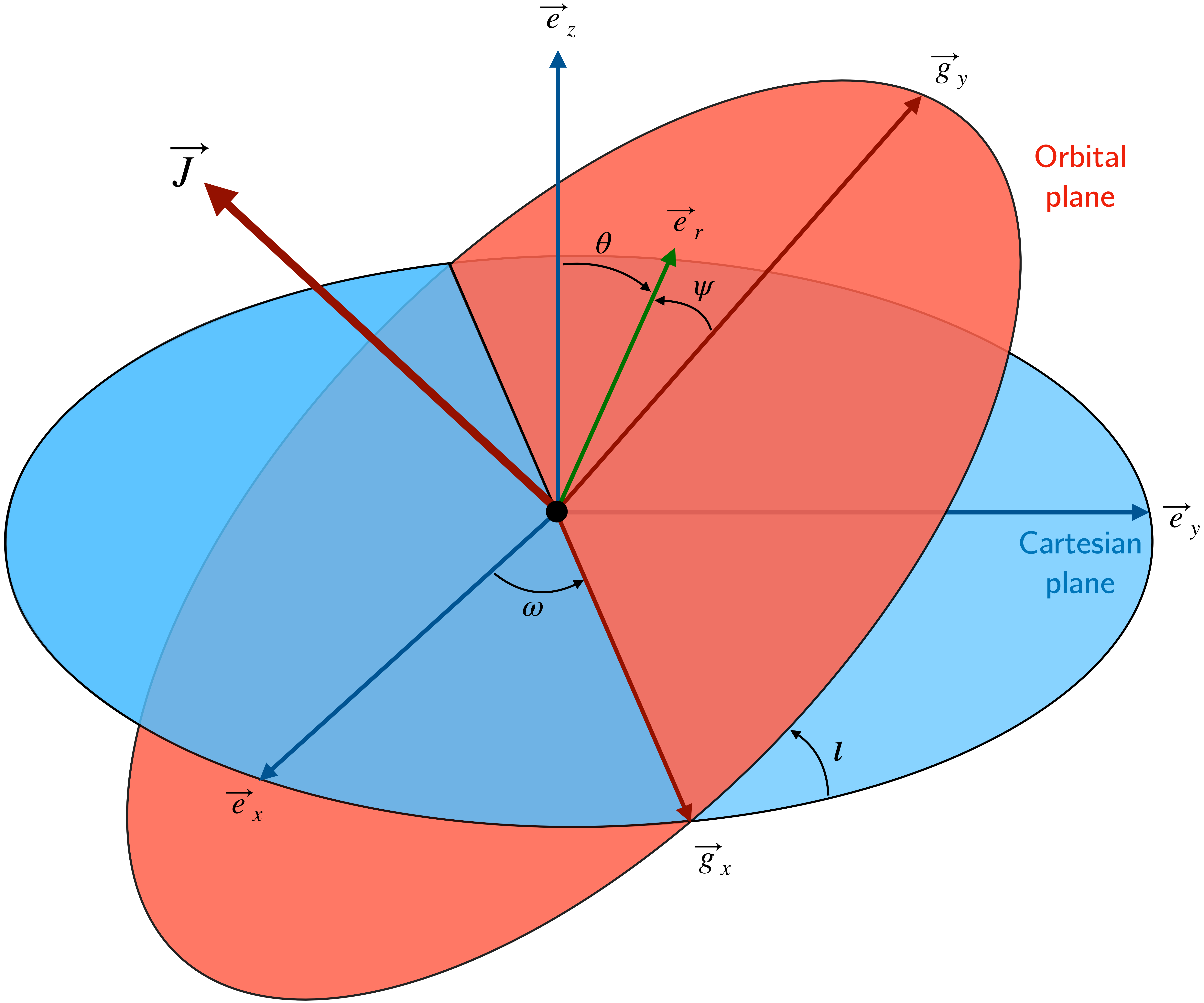
$$(\theta, \pi_\theta, \phi, \pi_\phi, \underbrace{\sigma, \pi_\sigma}_{\text{cano coord. for } \vec{S}}) \mapsto (\psi, \overbrace{J, \omega, J_z}^{\text{invariant plane}}, \underbrace{s, \pi_s}_{\substack{\text{combine}\\ \text{spin+orbital}\\ \text{symmetrically}}})$$

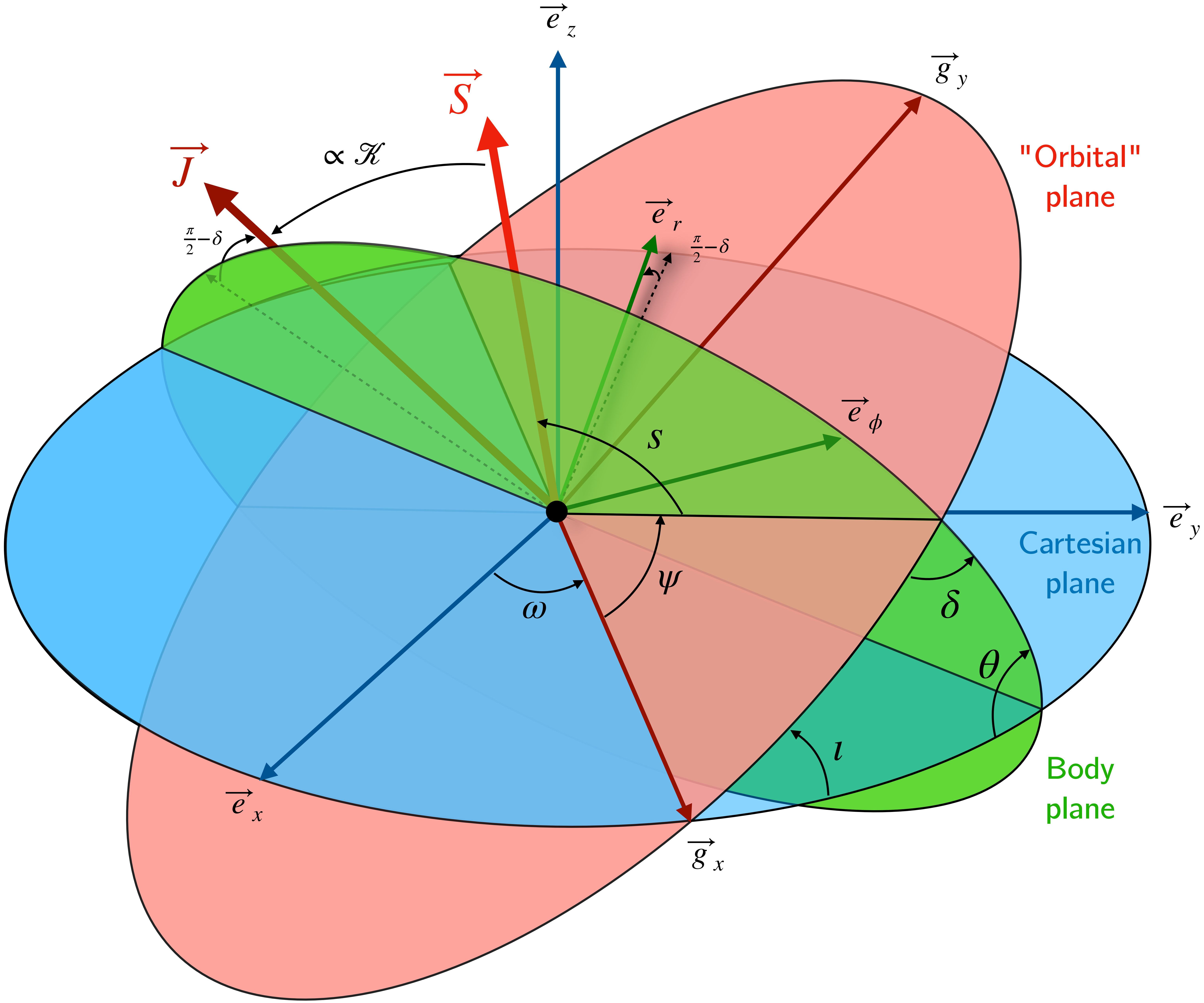
So-called "Andoyer" variables
1850's coordinates
for the *lunar problem* in
classical mechanics

- Canonical transformation

$(t, p_t, r, p_r) \mapsto$ untouched

$(\theta, p_\theta, \phi, p_\phi) \mapsto (\psi, J, \omega, J_z)$




$\vec{e}_z$
$\vec{S}$
$\vec{J}$
$\propto \mathscr{K}$
$\vec{g}_y$
"Orbital" plane
$\vec{e}_r$
$\frac{\pi}{2}-\delta$
$\frac{\pi}{2}-\delta$
$\vec{e}_\phi$
s
$\vec{e}_y$
Cartesian plane
ψ
ω
δ
θ
ι
Body plane
$\vec{e}_x$
$\vec{g}_x$

Completely generic orbits: eccentric, misaligned, non-planar

Decoupling between radial and rotational (spin+orbit) dynamics

Analytic solutions for radial sector + Hill equation for rotational

Physical interpretation, spin and orbital-plane precession

Radial classification is same as geodesics: near, plunge, bound, scattering

RADIAL SECTOR

- 1 dof Hamiltonian for radial sector

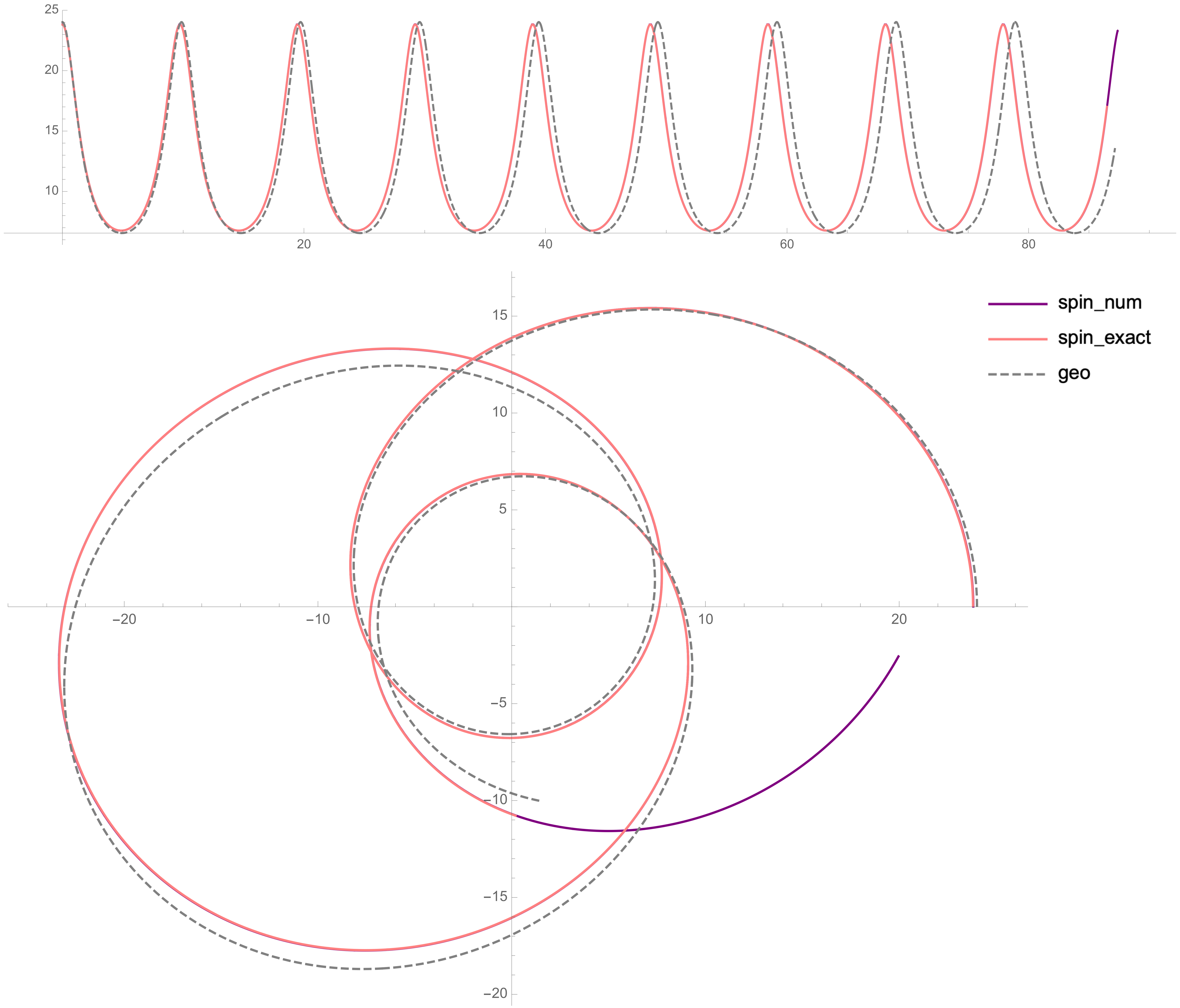
Killing invariants

$$H(r,\pi_r) = -\frac{E^2}{2f} + \frac{f\pi_r^2}{2} + \frac{L^2}{2r^2} - \frac{MEC_Y}{r^3 f}$$

- Analytical solution $r(\psi) = \dfrac{p}{1 - e\,F(\psi)}$

$$e = \frac{r_a - r_p}{r_a + r_p} \qquad p = \frac{2r_a r_p}{r_a + r_p}$$

$$F(\psi) = 1 + \frac{c_0}{\wp(c_1\psi) + c_2}$$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spin_num
spin_exact
geo
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−15
−20
−20
−10
10
20

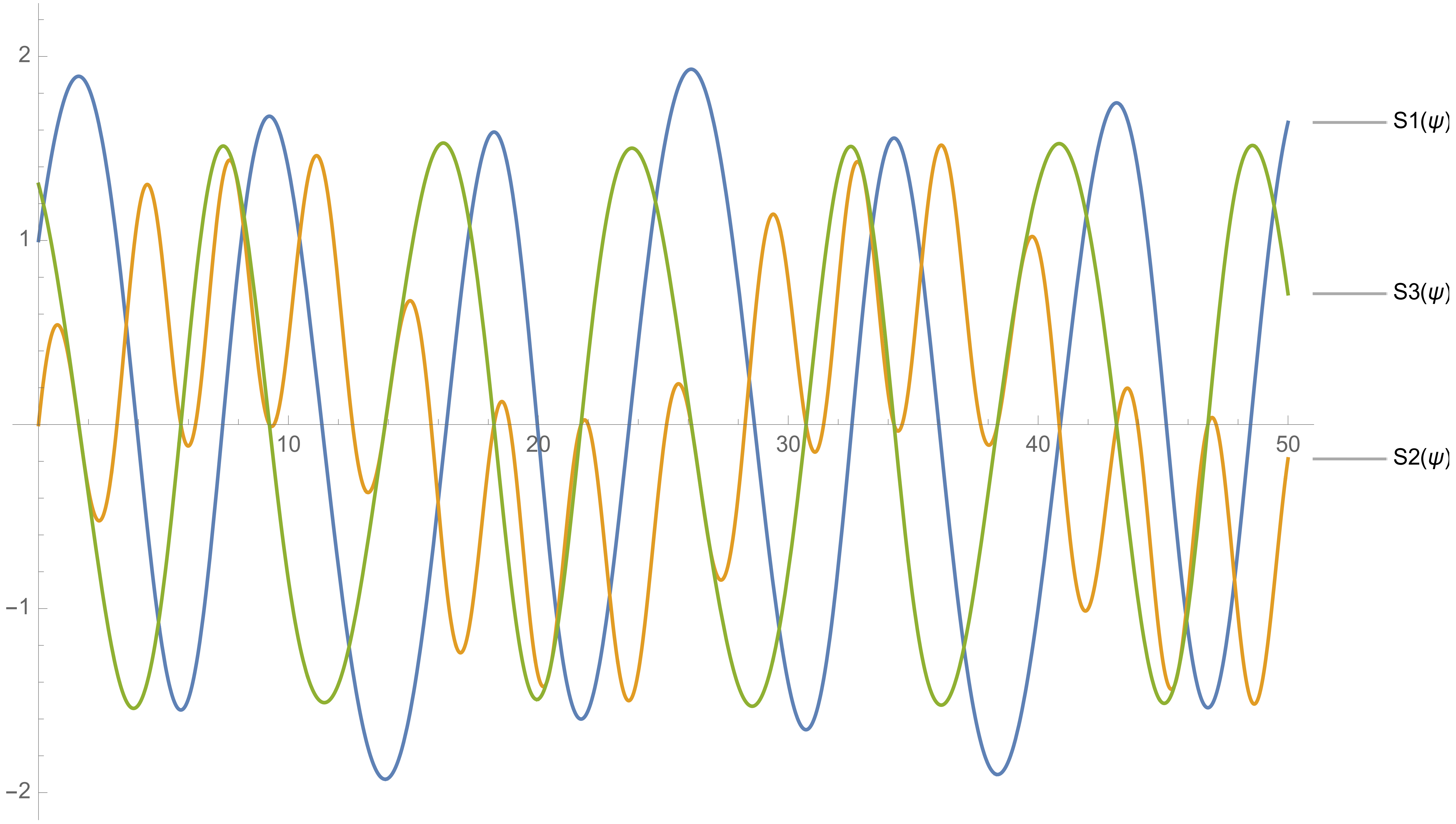
ROTATIONAL SECTOR

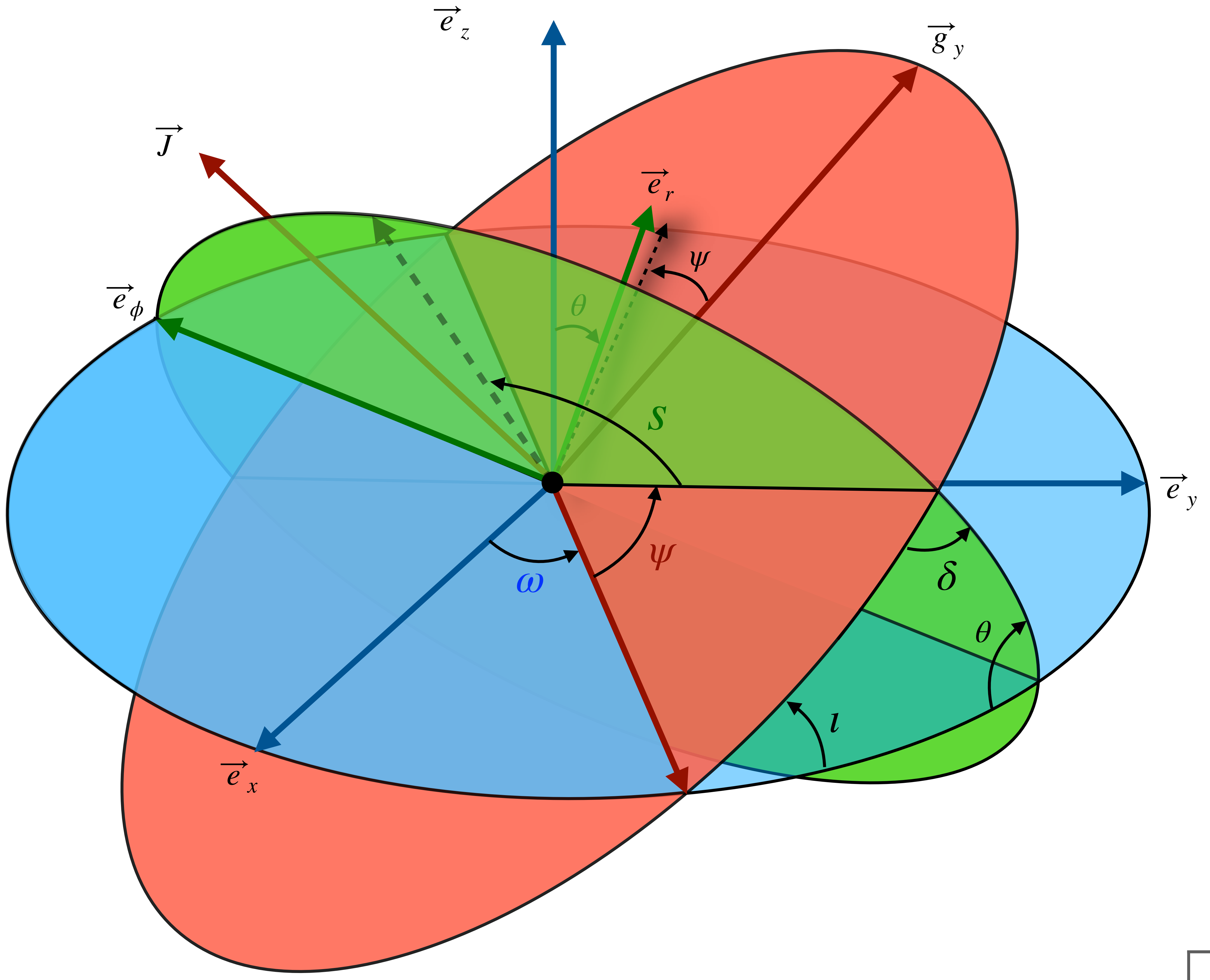
- Hill differential equation $Y = \pi_s$ and $Y = \tan s$

$$\frac{d^2Y(\psi)}{d\psi^2} + \left(1 - \frac{3M}{r(\psi)}\right) Y(\psi) = 0$$

$$Y_0(\psi + T_{r(\psi)}) = Y_0(\psi)$$

Aligned case:





1) Assume planar => e_r confined into some invariant plane
2) set **J'** orthogonal to that plane: its an invariant vector
3) the angle between **J** and **J'** is δ (or π/2-δ)
4) δ is constant, so π_s is const, so dot{π_s}=0
5) from EoM, dot{π_s}=0 => K tan(s)=0
6) either s=0,π (aligned, misaligned) or K=0 (perpendicular)

	Circular		Eccentric	
	Planar	Non-planar	Planar	Non-planar
(anti-)aligned	analytic	*impossible*	analytic	*impossible*
misaligned	*impossible*	analytic	*impossible*	semi-analytic

$$\cos \iota = \frac{J_z}{J}$$

$$\cos s = \frac{\pi_s}{J}$$