Loop Amplitudes in QCD, Geometry and Ansätze

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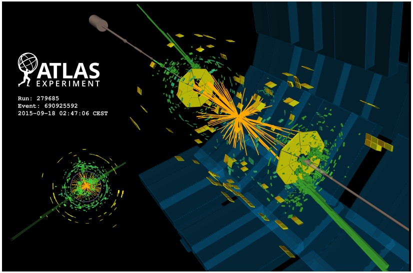
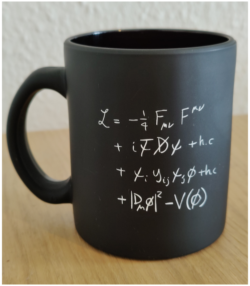
Marie Skłodowska-Curie Actions

CERN

The Future of Precision at the LHC



$\longleftrightarrow$



- LHC experiments will increase in statistics over next 2 decades.
- Experimental precision will reach 1% level for many observables.
- Precise theoretical predictions needed to match experimental error.

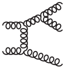
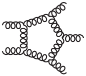
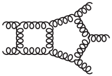
Precise Perturbative Predictions

- Perturbation theory is our major tool for making predictions.

- As a proton machine, the LHC is dominated by QCD.

$$\sigma[\alpha_S] \sim \sigma_{\text{LO}} + \alpha_S \delta\sigma_{\text{NLO}} + \alpha_S^2 \delta\sigma_{\text{NNLO}} + \mathcal{O}(\alpha_S^3).$$

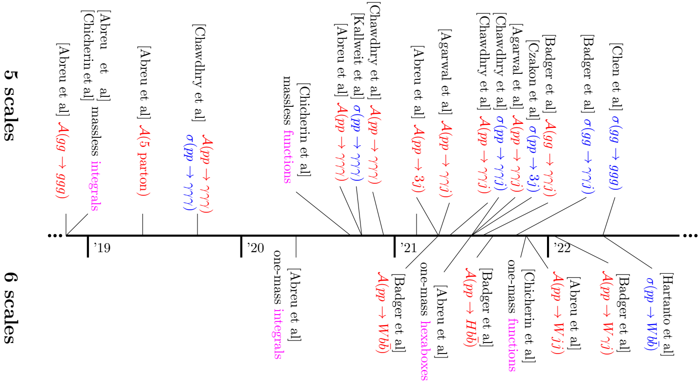
- At LHC energies, $\alpha_S \sim 0.1$, $\Rightarrow$ we require NNLO QCD.
  (+ parton-shower + resummation).

$A_5 = \Big[$  $+ \cdots \Big] + \alpha_S \Big[$  $+ \cdots \Big] + \alpha_S^2 \Big[$  $+ \cdots \Big] + \mathcal{O}(\alpha_s^3).$

$\Rightarrow$ Need two-loop scattering amplitudes.

Large Progress in Multi-Scale NNLO Calculations

Integrals, Amplitudes,
Cross Sections.



Progress in $2 \to 3$ calculations sparked by appropriate techniques.

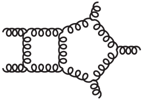
Outline

- Part 1: Outline and Status of Modern Techniques.
- Part 2: A Geometric Approach to Amplitude Construction.

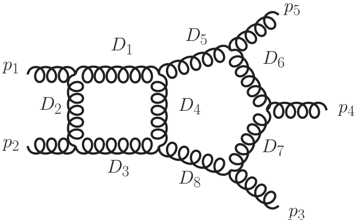
Part 1: Modern Techniques For QCD Amplitudes

Anatomy of a Loop Amplitude

- Textbook: A loop amplitude is the sum of Feynman diagrams.

$$A^{(2)}(p_1, \ldots, p_5) =$$  $$+\, \mathcal{O}(10000) \text{ diagrams.}$$

- Each (gauge-dependent) diagram leads to Feynman integrals:



$$= \int [d^D \vec{\ell}] \frac{N(\ell_1, \ell_2)}{D_1 D_2 D_3 D_4 D_5 D_6 D_7 D_8}.$$

- QCD numerators N are polynomials of $\vec{\ell}$ with 1000s of terms.
- Function of many variables with complex branch cut structure.

The Modern Perspective and Challenges

Aim: efficient, stable evaluation of amplitude over phase space.

- Too many integrals $\Rightarrow$ write $A^{(2)}$ in terms of master integrals:

$$A^{(2)}(p_1, \ldots, p_5) = \sum_k \underbrace{\mathcal{C}_k(p_1, \ldots, p_5)}_{\text{rational functions}} \underbrace{\mathcal{I}_k(s_{12}, s_{23}, s_{34}, s_{45}, s_{51})}_{\text{master integrals}}.$$

- Coefficients $\mathcal{C}_k$ are process specific, integrals $\mathcal{I}_k$ are universal.

Challenges

- $\mathcal{I}_k$: analytic complexity $\Rightarrow$ efficient, stable evaluation difficult.
- $\mathcal{C}_k$: intermediate expression swell, poorly understood structure.

Master Integrals

$$\sum_k \mathcal{C}_k(p_1, \ldots, p_5)\, \mathcal{I}_k(s_{12}, s_{23}, s_{34}, s_{45}, s_{51})$$

Differential Equations and Master Integrals

- Integrals are special functions depending on many variables.

$$\mathcal{I}_k \sim \int [d^D \vec{\ell}] \frac{N(\ell_1, \ell_2)}{D_1 D_2 D_3 D_4 D_5 D_6 D_7 D_8} \qquad D_i = \Big(\sum_j a_{ij} \ell_j + \sum_j b_{ij} p_j \Big)^2 - m_i^2.$$

- Analytic structure is manifested by differential equations.

$$\mathrm{d}\mathcal{I}_k = \mathbf{M}_{kl}(\epsilon, \vec{s})\mathcal{I}_l.$$

[Gehrmann, Remiddi '01]

- For (pure) polylogarithmic integrals, **M** simplifies: [Henn '13]

$$\mathbf{M}_{kl}(\epsilon, \vec{s}) = \underbrace{\epsilon}_{\text{regulator}} \sum_\alpha \underbrace{M^\alpha_{k,l}}_{\text{rational numbers}} \mathrm{d}\log\big(\underbrace{W_\alpha[\vec{s}]}_{\text{algebraic functions}} \big).$$

- Powerful DE construction/initial value approaches exist.

[Abreu, BP, Zeng '18] [Abreu, Ita, Moriello, BP, Tschernow, Zeng '20]

Numerically Solving the Differential Equation

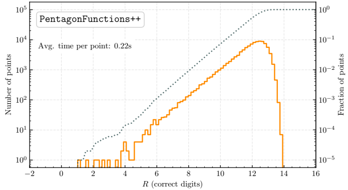
Aim: numerically evaluate $\mathcal{I}_j$ as ϵ series, $\mathcal{I}_j(\vec{p},\epsilon) = \sum_{k=-4}^{0} d_{j,k}\epsilon^k h_k + \mathcal{O}(\epsilon)$.

Pentagon functions: [Gehrmann, Henn, lo Presti '18]

- Dedicated iterated integral code:

$$h_k \sim \int_0^1 \mathrm{d}\log(W_n[t_n]) \cdots \int_0^{t_2} \mathrm{d}\log(W_1[t_1]).$$

- Very efficient for five-point.



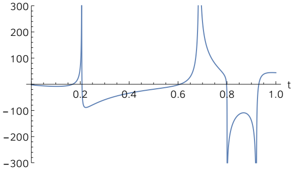
[Chicherin, Sotnikov '20]

Series expansions: [Moriello '19]

- Patch together h_k from power series:

$$h_k \sim \sum_{j_1,j_2} (t-t_0)^{j_1/2} \log(t-t_0)^{j_2}.$$

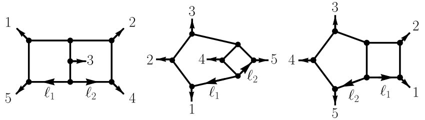
- Implementations: DiffExp/AMFlow. [Hidding '20] [Liu, Ma '22]
- Promising for elliptic five-point.



[Abreu, Ita, Moriello, BP, Tschernow, Zeng '20]

Master Integrals – State of the Art (Two-Loop Five-Point)

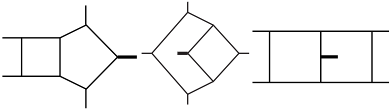
- Five-point Massless



[Papadopoulos, Tommasini, Wever '15]
[Gehrmann, Henn, lo Presti '18]
[Abreu, Page, Zeng '18]
[Chicherin, Gehrmann, Henn, lo Presti, Mitev, Wasser '18]

[Abreu, Dixon, Herrmann, Page, Zeng '18]
[Chicherin, Gehrmann, Henn, Wasser, Zhang, Zoia '18]
[Gehrmann, Henn, lo Presti '18]
[Chicherin, Sotnikov '20]

- Five-point One-Mass



[Abreu, Ita, Moriello, Page Tschernow, Zeng '20]
[Abreu, Ita, Page, Tschernow '21]
[Papadopoulos, Tommasini, Wever 15]

[Canko Papadopoulos, Syrrakos 20]
[Kardos, Papadopoulos, Smirnov, Syrrakos, Wever 19]
[Chicherin, Sotnikov, Zoia '21]

[Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia. to appear]

Integral Coefficients

$$\sum_k \mathcal{C}_k(p_1, \ldots, p_5)\, \mathcal{I}_k(s_{12}, s_{23}, s_{34}, s_{45}, s_{51})$$

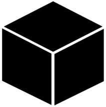
The Analytic Reconstruction Approach

- Analytic coefficients built from numerical samples via Ansatz.

$$\mathcal{C}_k(p_1, \ldots, p_n) = \sum_{j=1}^{N_k} c_{jk} \mathfrak{a}_{jk}(p_1, \ldots, p_n), \qquad c_{jk} \in \mathbb{Q}.$$

- Numerical evaluations provide constraints on unknown c_{jk}.

$$(p_1^{(0)}, \ldots, p_n^{(0)}) \longrightarrow$$



$$\longrightarrow \mathcal{C}_1(p_1^{(0)}, \ldots, p_n^{(0)}).$$

- Made practical by finite field methods (working modulo p).
[Schabinger, von Manteuffel '14; Peraro '16]

- Sidesteps complex algebra – only intermediate numerics!

Ansätze and [Known] Structure of QCD Coefficients

- Re-express $\mathcal{C}_a$ as rational function of Mandelstam variables.

$$\mathcal{C}_a(\varepsilon_1, p_1, \ldots, \varepsilon_n, p_n) = \underbrace{\phi(\varepsilon_1, p_1, \ldots, \varepsilon_n, p_n)}_{\text{phase weight}} \frac{\mathcal{N}_a\left[s_{ij}, \epsilon(p_i, p_j, p_k, p_l)\right]}{\prod_\alpha W_\alpha(s_{ij})^{q_{a\alpha}}}.$$

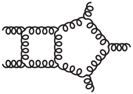
- W_α are symbol letters, from differential equation. [Abreu, Dormans, Febres Cordero, Ita, BP '18]

- $\mathcal{N}$ is polynomial in Lorentz invariants, e.g. (5-point massless):

$$\mathcal{N} = \underbrace{\mathcal{N}_+(s_{12}, s_{23}, s_{34}, s_{45}, s_{51})}_{\text{polynomial}} + \epsilon(p_1, p_2, p_3, p_4) \underbrace{\mathcal{N}_-(s_{12}, s_{23}, s_{34}, s_{45}, s_{51})}_{\text{polynomial}}.$$

Gathering Numerical Data: Numerical Unitarity

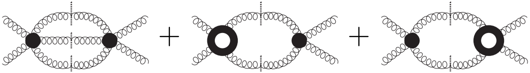
Aim: Numerically reduce Feynman diagram sum to masters.



$$+ \mathcal{O}(10000) \text{ diagrams} = \int [\mathrm{d}^D \vec{\ell}]\, \mathcal{A}(\vec{\ell}) = \sum_j \mathcal{C}_j \mathcal{I}_j.$$

- Unitarity lets us build integrand from product of trees.

$$\mathrm{Disc}\,[A] =$$



- Non-uniqueness of integrand captured by surface terms σ.

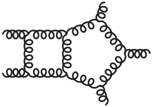
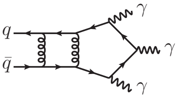
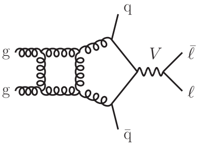
$$\mathcal{A}(\vec{\ell}) \sim \mathcal{A}(\vec{\ell}) + \sigma(\vec{\ell}) \qquad \text{where} \qquad \int [\mathrm{d}^D \vec{\ell}]\, \sigma(\vec{\ell}) = 0.$$

- Surface terms built with algebraic geometry approaches.

[Gluza et al '11; Ita '15; Larsen et al '15, Abreu, Febres Cordero, Ita, Jaquier, BP, Zeng '17]

Complexity of Analytic Reconstruction

- Currently $T_{\text{reconstruct}} = \mathcal{O}(\text{laptop})$, $T_{\text{evaluate}} = \mathcal{O}(\text{cluster})$.
- Number of evaluations for recent QCD helicity amplitudes:

	Three-jet	Three-photon	W + two-jets*
Process			
# Samples	$\sim 10^5$	$\sim 10^5$	$\sim 10^6$.

*After simplification via [Badger et al '20]

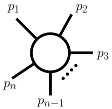
Lesson: Need to better understand rational functions in QCD.

Part 2: Structure of $\mathcal{C}_k$ from Algebraic Geometry

(based on [arXiv:2203.04269] w/ Giuseppe De Laurentis)

Spinor Helicity Methods

- Amplitudes conventionally represented in momentum space.



$$\rightarrow \qquad \sum_{i=1}^{n} p_i^\mu = 0, \qquad p_i^2 = m_i^2.$$

- Yang-Mills amplitudes compactly expressed in spinor variables.

$$p_i^2 = 0 \qquad \Rightarrow \qquad p_{i\mu}\sigma^{\mu\alpha\dot{\alpha}} = \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}}.$$

- p_i is automatically on-shell, momentum conservation becomes

$$\sum_{i=1}^{n} \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}} = 0.$$

- Invariant quantities represented in terms of spinor brackets.

$$\langle ij \rangle = \lambda_i^\alpha \lambda_{j\alpha}, \qquad [ij] = \tilde{\lambda}_{i\dot{\alpha}} \tilde{\lambda}_j^{\dot{\alpha}}.$$

Constraints from Singular Limits

- Consider six-point one-quark line amplitude $\mathcal{A}^{\text{tree}}_{q^+g^+g^+\overline{q}^-g^-g^-}$.

$$\mathcal{A} = \frac{\mathcal{N}^*}{\langle 12\rangle\langle 23\rangle\langle 34\rangle[45][56][61]s_{345}}.$$

*$\mathcal{N}$ is a degree 6 polynomial in spinor brackets.

- Doubly singular behaviour can be used to constrain $\mathcal{A}$.

[De Laurentis, Maître '19]

- Consider setting $\langle 12\rangle$, $\langle 23\rangle$ are small:

$$\lambda_2^\alpha \sim \epsilon \quad \Rightarrow \quad \mathcal{A} \sim \epsilon^{-2}$$
$$\langle 12\rangle \sim \langle 23\rangle \sim \langle 13\rangle \sim \epsilon \quad \Rightarrow \quad \mathcal{A} \sim \epsilon^{-1}.$$

How do we systematically build an Ansatz with this behavior?

Challenges to Overcome

Aim Build Ansatz $\{\mathfrak{a}_1, \dots, \mathfrak{a}_N\}$ matching singular behaviour.

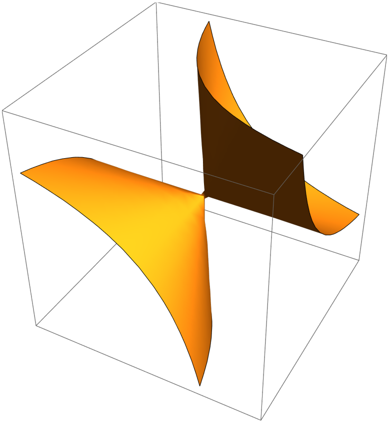
$$\mathcal{C}_i(\lambda, \tilde{\lambda}) = \frac{\mathcal{N}_i(\lambda, \tilde{\lambda})}{\prod_{j=0}^{n-1} \mathcal{D}_j(\lambda, \tilde{\lambda})^{q_{ij}}} = \sum_{j=1}^{N} c_{ij} \mathfrak{a}_j(\lambda, \tilde{\lambda}), \qquad c_{ij} \in \mathbb{Q}.$$

Strategy: Find polynomial space enjoying $\mathcal{N}$'s vanishing behavior.

Difficulties:

- How do we handle linear dependencies in spinor space?
- How to systematically study (define) all singular surfaces?
- How to algorithmically turn vanishing constraints into Ansatz?

Algebraic Geometry Warmup



Algebraic Geometry and Spinors

- Workhorse: ring of polynomials in spinor components – S_n.

$$S_n = \mathbb{C}[\lambda_1^\alpha, \ldots, \lambda_n^\alpha, \tilde{\lambda}_1^{\dot{\alpha}}, \ldots, \tilde{\lambda}_n^{\dot{\alpha}}].$$

- "Ideal" is set of combinations of polynomials $q_i \in S_n$.

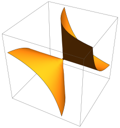
$$\underbrace{\langle q_1, \ldots, q_k \rangle}_{\text{generators}}{}_{S_n} = \left\{ \sum_{i=1}^{k} a_i q_i, \;\; a_i \in S_n \right\}.$$

- "Variety" associated to ideal is set of points satisfying $q_i = 0$.

$$V\Big(\langle q_1, \ldots, q_k \rangle_{S_n}\Big) = \Big\{ (\lambda, \tilde{\lambda}) \in \mathbb{C}^{4n} : q_i(\lambda, \tilde{\lambda}) = 0 \text{ for } 1 \le i \le k \Big\}.$$

Momentum Conservation and the Quotient Ring

- Momentum conservation ideal J_{Λ_n}. Associated variety $V(J_{\Lambda_n})$.

$$J_{\Lambda_n} = \left\langle \sum_{i=1}^{n} \lambda_{i\alpha}\tilde{\lambda}_{i\dot{\alpha}} \right\rangle_{S_n}, \qquad V(J_{\Lambda_n}) \sim$$  .

- Polynomials differing by J_{Λ_n} element are physically equivalent.

$$p \sim q \iff p - q \in J_{\Lambda_n}.$$

- Simple example at four points:

$$\lambda_1^\alpha\langle 12\rangle = -\lambda_3^\alpha\langle 32\rangle - \lambda_4^\alpha\langle 42\rangle \quad \Leftrightarrow \quad \lambda_1^\alpha\langle 12\rangle + \lambda_3^\alpha\langle 32\rangle + \lambda_4^\alpha\langle 42\rangle \in J_{\Lambda_4}.$$

- Polynomial spinor functions on $V(J_{\Lambda_n})$ live in quotient ring*:

$$R_n = S_n / J_{\Lambda_n}.$$

*Systematically organized by Gröbner basis techniques.

Algebra and Geometry of Singular Varieties

Singular Varieties

- Denominator of coefficients $\mathcal{C}_i$ factorizes.

$$\mathcal{C}_i(\lambda, \tilde{\lambda}) = \frac{\mathcal{N}_i(\lambda, \tilde{\lambda})}{\prod_{j=0}^{n-1} \mathcal{D}_j(\lambda, \tilde{\lambda})^{q_{ij}}}.$$

- Example $\mathcal{D}_j$, five point two-loop massless amplitudes:

$$\vec{\mathcal{D}}^{(\text{5-point})} = \{\langle 12\rangle, \langle 13\rangle, \ldots, [12], [13], \ldots, [1|2+3|1\rangle, \ldots\}$$

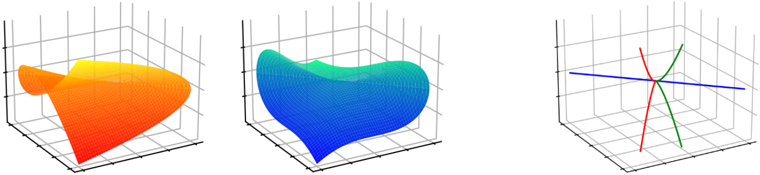
- We will consider varieties where two denominators are singular:

$$\mathcal{D}_i = \mathcal{D}_j = 0.$$

Must control branching structure and vanishing functions.

Branching of Surfaces Defined by Polynomials

- When we intersect surfaces, we may have multiple branches.



$xy^2 + y^3 - z^2 = 0 \qquad x^3 + y^3 - z^2 = 0, \qquad xy^2 + y^3 - z^2 = x^3 + y^3 - z^2 = 0.$

- Our double denominator zero surface has two branches:

$$\langle 12 \rangle = \langle 23 \rangle = 0 \quad \Leftrightarrow \quad \langle 12 \rangle = \langle 23 \rangle = \langle 13 \rangle = 0 \quad \text{or} \quad \lambda_2^\alpha = 0.$$

- Compute branchings with primary decomposition techniques.

[De Laurentis, BP '22], see also [Zhang '12].

Polynomials that Vanish on a Variety

- Polynomials vanishing on all points of variety U form ideal

$$I(U) = \Big\{ q \in S_n \quad \text{where} \quad q(x) = 0 \quad \text{for all } x \in U \Big\}.$$

- Consider if $\mathcal{N}_i$ vanishes to order k on U,

$$\mathcal{N}_i(x^{(\epsilon)}) = \mathcal{O}(\epsilon^k), \quad \text{where} \quad |x - x^{(\epsilon)}| \leq \epsilon \quad \text{and} \quad x \in U.$$

- **Zariski-Nagata**: $\mathcal{N}_i$ belongs to k^{th} "symbolic power" $I(U)^{\langle k \rangle}$.

$$I(U)^k = I(U)^{\langle k \rangle} \cap J_{e_1} \cap \ldots \cap J_{e_n}, \qquad \text{s.t.} \qquad V(J_{e_i}) \subset U.$$

- Ideal power, J^k computed by multiplying generators, e.g.

$$J^2 \; = \; \langle q_1, q_2, q_3 \rangle^2 \; = \; \langle q_1^2, q_2^2, q_3^2, q_1 q_2, q_2 q_3, q_1 q_3 \rangle.$$

From Geometry to an Ansatz

Ansatz Construction Algorithm, Sketched

1. Construct branches of varieties where two $\mathcal{D}_k$ vanish.

$$\mathcal{D}_i = \mathcal{D}_j = 0 \qquad \longrightarrow \qquad \mathcal{V} = \{U_1, U_2, \ldots\}.$$

2. Sample near surface to determine degree of divergence.

$$U: \quad \mathcal{D}_i \sim \mathcal{D}_j \sim \epsilon \qquad \Rightarrow \qquad \mathcal{C}_k \sim \frac{1}{\epsilon^{\kappa_U}}.$$

3. Ansatz is basis of intersection of associated ideals of vanishing polynomials. Constructed using Gröbner basis techniques.

$$\mathcal{N}_k \in \bigcap_{U \in \mathcal{V}} I(U)^{\langle \kappa_U \rangle}.$$

Studying a Six-Point Tree

$$\mathcal{A} = \frac{\mathcal{N}}{\langle 12\rangle\langle 23\rangle\langle 34\rangle[45][56][61]s_{345}}.$$

- Probe 108 varieties where pairs of $\langle ij\rangle, [ij], s_{ijk}$ are small.

$$\text{e.g.} \qquad [12] \sim [13] \sim [23] \sim \mathcal{O}(\epsilon) \qquad \Rightarrow \qquad \mathcal{A} \sim \mathcal{O}(\epsilon^2).$$

- $\mathcal{N}$ vanishes non-trivially on 28 varieties $\Rightarrow$ ideal memberships:

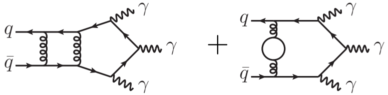
$$\mathcal{N} \quad \in \quad \Big\langle [12], [13], [23] \Big\rangle^2 \cap \Big\langle \langle 12\rangle, \langle 34\rangle \Big\rangle \cap \Big\langle \langle 12\rangle, [16] \Big\rangle \cap (25 \text{ more}).$$

- Imposing mass dimension constraints gives one term Ansatz:

$$\mathcal{N} = c_0\Big(\langle 12\rangle[21]\langle 45\rangle[54]\langle 4|2+3|1]\rangle + [16]\langle 6|1+2|3]\langle 34\rangle s_{123}\Big), \quad c_0 \in \mathbb{Q}.$$

Proof-of-Concept Remainders for $q\bar{q} \to \gamma\gamma\gamma$

$\mathcal{A} =$



$+ \cdots .$

(Analytic input from [Abreu, Page, Pascual, Sotnikov '20]).

- Codimension two surface study requires 317 $\mathbb{Q}_p$ samples.
- Fitting Ansatz now requires at most 566 $\mathbb{F}_p$ samples.

Amplitude	$R^{(2,0)}_{-++}$	$R^{(2,N_f)}_{-++}$	$R^{(2,0)}_{+++}$	$R^{(2,N_f)}_{+++}$
Old Ansatz Dim	41301	2821	7905	1045
New Ansatz Dim	566	20	18	6

Summary

- Large progress in $2 \to 3$ scattering, including phenomenology!
- Algebraic geometry provides insight into rational coefficients.
- For $q\overline{q} \to \gamma\gamma\gamma$ now only $\mathcal{O}(1000)$ numerical samples required.

From Ideals To Ansätze

- $\mathcal{N}$ is a polynomial of dimension d and little group weight $\vec{\phi}$:

$$\mathcal{M}_{d,\phi} = \left\{ a \in \mathcal{R}_n : [a] = d, \{a\}_j = \phi_j \right\}.$$

- Construct independent spinor monomials $m_{\alpha,\beta}$ via Groebner.

- Numerator in each ideal. Intersect with $\mathcal{M}_{d,\vec{\phi}}$ (by Groebner).

$$\mathcal{M}_{d,\vec{\phi}} \cap \left[\bigcap_{U \in \mathcal{V}^{(2)}} I(U)^{\langle k_U \rangle} \right] = \bigcap_{U \in \mathcal{V}^{(2)}} \left[\mathcal{M}_{d,\vec{\phi}} \cap I(U)^{\langle k_U \rangle} \right].$$

- Basis of intersection is Ansatz incorporating all constraints.

Massless 5-Point 2-Loop Singular Varieties (i)

- $\mathcal{V}^{(2)}$. Built from 10 parity/permutation generators:

V_i	1	2	3	4	5	6	7	8	9	10
# Perms	20	10	2	30	10	60	120	15	30	20

$$I(V_1) = \langle \langle 12 \rangle, \langle 13 \rangle, \langle 23 \rangle, [45] \rangle_{R_5} \, ,$$

$$I(V_2) = \langle \lambda_1^\alpha \rangle_{R_5} \, ,$$

$$I(V_3) = \langle \langle 12 \rangle, \langle 23 \rangle, \langle 34 \rangle, \langle 45 \rangle, \langle 15 \rangle, \langle 13 \rangle, \langle 14 \rangle, \langle 24 \rangle, \langle 25 \rangle, \langle 35 \rangle \rangle_{R_5} \, ,$$

$$I(V_4) = \langle \langle 12 \rangle, \langle 34 \rangle, \lambda_1^\alpha [15] + \lambda_2^\alpha [25] \rangle_{R_5} \, ,$$

$$I(V_5) = \langle \langle 12 \rangle, [12] \rangle_{R_5} \, ,$$

Massless 5-Point 2-Loop Singular Varieties (ii)

$$I(V_6) = \langle \langle 12 \rangle, [13] \rangle_{R_5} \,,$$

$$I(V_7) = \langle \langle 12 \rangle, \langle 3|1+4|3], \lambda_1^\alpha [14][35] + \lambda_2^\alpha [25][34] \rangle_{R_5} \,,$$

$$I(V_8) = \langle \langle 1|2+3|1], \langle 1|2+4|1], [1|2|4|1] - [1|2|3|1] + [1|3|4|1], \lambda \leftrightarrow \tilde{\lambda}, \\ \langle 45 \rangle \langle 12 \rangle \langle 13 \rangle - \langle 15 \rangle \langle 23 \rangle \langle 14 \rangle, \lambda \leftrightarrow \tilde{\lambda} \rangle_{R_5} \,,$$

$$I(V_9) = \langle \langle 1|2+3|1], \langle 2|1+3|2], \\ \tilde{\lambda}_1^{\dot\alpha} \langle 12 \rangle \langle 13 \rangle - \tilde{\lambda}_2^{\dot\alpha} \langle 12 \rangle \langle 23 \rangle - \tilde{\lambda}_3^{\dot\alpha} \langle 13 \rangle \langle 23 \rangle, \lambda \leftrightarrow \tilde{\lambda} \rangle_{R_5} \,,$$

$$I(V_{10}) = \langle \langle 1|2+3|1], \langle 2|1+4|2], \\ \lambda_1^\alpha [13][14][25] + \lambda_2^\alpha [12][24][35] + \lambda_3^\alpha [13][24][35], \lambda \leftrightarrow \tilde{\lambda} \rangle_{R_5} \,.$$

Irreducible Decomposition Examples

- The once singular variety at four point is reducible

$$V\left(\langle\,\langle 12\rangle\,\rangle_{R_4}\right) = V\left(\langle\langle 12\rangle, [34]\rangle_{R_4}\right) \cup V\left(\langle\langle 12\rangle, \langle 13\rangle, \langle 14\rangle, \langle 23\rangle, \langle 24\rangle, \langle 34\rangle\rangle_{R_4}\right).$$

- Many doubly singular varieties are reducible at five point, e.g.

$$V\left(\langle\langle 12\rangle, \langle 34\rangle\rangle_{R_5}\right) = V\left(\langle\langle 12\rangle, \langle 34\rangle, \lambda_1^\alpha[15] + \lambda_2^\alpha[25]\rangle_{R_5}\right) \cup V_{\text{angle}},$$

$$V\left(\langle\langle 12\rangle, \langle 13\rangle\rangle_{R_5}\right) = V\left(\langle\langle 12\rangle, \langle 13\rangle, \langle 23\rangle, [45]\rangle_{R_5}\right) \cup V\left(\langle\lambda_1^\alpha\rangle_{R_5}\right) \cup V_{\text{angle}},$$

where

$$V_{\text{angle}} = V\left(\langle\langle 12\rangle, \langle 23\rangle, \langle 34\rangle, \langle 45\rangle, \langle 15\rangle, \langle 13\rangle, \langle 14\rangle, \langle 24\rangle, \langle 25\rangle, \langle 35\rangle\rangle_{R_5}\right).$$

Lorentz Invariance

- Coefficients are Lorentz invariant functions of spinor brackets.

$$\mathcal{C}(\lambda, \tilde{\lambda}) = \mathcal{C}(\langle\rangle, []).$$

- Relevant ring is Lorentz invariant subring of S_n.

$$\mathcal{S}_n = \mathbb{F}\Big[\langle 12\rangle, \ldots, \langle (n-1)n\rangle, [12], \ldots [(n-1)n]\Big].$$

- Variables are brackets, now have "Schouten identities".

$$\mathcal{J}_{\Lambda_n} = \left\langle \sum_{j=1}^{n} \langle ij\rangle[jk], \langle ij\rangle\langle kl\rangle - \langle ik\rangle\langle jl\rangle - \langle il\rangle\langle kj\rangle, \langle\rangle \leftrightarrow [] \right\rangle.$$

- Physical spinor bracket functions also form a quotient ring.

$$\mathcal{R}_n = \mathcal{S}_n / \mathcal{J}_{\Lambda_n}.$$

Bases of Spinor Space and Polynomial Reduction

- Numerators are $\mathbb{Q}-$linear combinations of spinor monomials.

$$m_\alpha = \prod_i v_i^{\alpha_i} \qquad \text{where} \quad \vec{v} = \{\langle 12 \rangle, \langle 23 \rangle, \ldots [12], [23], \ldots\}.$$

- Little group/mass dimension constraints pick finite set of α.

- Polynomial reduction writes p in terms of generators g_i.

$$p = \Delta_{\{g_1, \ldots, g_k\}}(p) + \sum_{i=1}^{k} c_i g_i.$$

- Polynomial in ideal if and only if Groebner remainder is 0.

$$\Delta_{\mathcal{G}(J)}(p) = 0 \qquad \Leftrightarrow \qquad p \in J.$$

- Monomials irreducible by $\mathcal{G}(\mathcal{J}_{\Lambda_n})$ form basis. Related [Zhang '12]

$$\text{basis} = \{m_\alpha \text{ such that } \Delta_{\mathcal{G}(\mathcal{J}_{\Lambda_n})}(m_\alpha) = m_\alpha\}.$$

How To Perform Numerical Investigations?

- Need to find phase-space points $(\lambda^\epsilon, \tilde{\lambda}^\epsilon)$ where $\mathcal{D}_i$ are small.

$$\mathcal{D}_i(\lambda^\epsilon, \tilde{\lambda}^\epsilon) \quad \sim \quad \mathcal{D}_j(\lambda^\epsilon, \tilde{\lambda}^\epsilon) \quad \sim \quad \epsilon.$$

- Conflict with modern techniques: no small elements in $\mathbb{F}_p$.

$$|0|_{\mathbb{F}_p} = 0, \qquad \text{and} \qquad a \neq 0 \;\Rightarrow\; |a|_{\mathbb{F}_p} = 1.$$

- Approaching with complex numbers plagued by instabilities.

Enter the p-adic numbers – a middle ground between $\mathbb{F}_p$ and $\mathbb{C}$.

Introduction to the p-adic Numbers

- The p-adic numbers roughly correspond to Laurent series in p.

$$x = \sum_{i=\nu}^{\infty} a_i p^i = a_\nu p^\nu + a_{\nu+1} p^{\nu+1} + \cdots, \qquad \left(\begin{smallmatrix} a_i \in [0, p-1], \\ a_\nu \neq 0. \end{smallmatrix} \right).$$

- The p-adic numbers form a field. $x, y \in \mathbb{Q}_p \Rightarrow$

$$x+y \in \mathbb{Q}_p, \quad -x \in \mathbb{Q}_p, \quad x \times y \in \mathbb{Q}_p, \quad \frac{1}{x} \in \mathbb{Q}_p \text{ (if } x \neq 0).$$

- The p-adic absolute value allows for small numbers ($p \sim \epsilon$).

$$|x|_p = p^{-\nu}, \quad \Rightarrow \quad |p|_p < |1|_p.$$

Computing with p-adic Numbers

- For computing purposes* we truncate to finite order.

$$x = p^{\nu(x)}\Big(\underbrace{\tilde{x}}_{\text{mantissa}} + \mathcal{O}(p^k)\Big).$$

*Try [https://github.com/GDeLaurentis/pyadic] to investigate yourselves.

- Truncation reduces to finite field case for $\nu = 0, k = 1$.

- Arithmetic $(+ - /*)$ is essentially performed modulo p^k, e.g.

$$x \times y = p^{\nu(x)+\nu(y)}\left(\tilde{x}\tilde{y} + \mathcal{O}(p^k)\right).$$

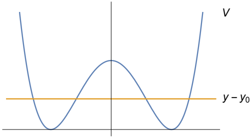
- Mantissa inverse computed with extended euclidean algorithm.

P-adic (Integer) Points Near an Irreducible Variety

- Want to find $(\lambda^{(\epsilon)}, \tilde{\lambda}^{(\epsilon)})$ "close" to $U = V(\langle q_1, \ldots, q_m \rangle_{R_n})$:

$$q_i\left(\lambda^{(\epsilon)}, \tilde{\lambda}^{(\epsilon)}\right) = pc_i + \mathcal{O}(p^k), \qquad \sum_{i=1}^{n} \lambda^{(\epsilon)}_{i\alpha} \tilde{\lambda}^{(\epsilon)}_{i\dot{\alpha}} = 0 + \mathcal{O}(p^k).$$

- First, find finite field $x \in U$ by intersecting with random plane.

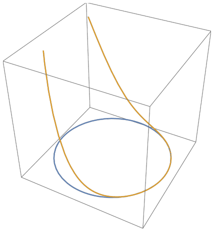


- Arbitrarily extend $\mathbb{F}_p$ point $(\lambda, \tilde{\lambda})$ to k digits. Trivially near U.
- To satisfy momentum conservation, perturb by $(p\delta, p\tilde{\delta})$.

$$(\lambda^{(\epsilon)}, \tilde{\lambda}^{(\epsilon)}) = (\lambda + p\delta, \tilde{\lambda} + p\tilde{\delta}).$$

Examples of Symbolic Powers

- A function vanishing to fourth order at a point on the circle:

$$\langle x - 1 \rangle^{\langle 4 \rangle}_{\mathbb{F}[x,y]/\langle x^2+y^2-1\rangle} \sim$$  .

- Often the symbolic power coincides with standard power, e.g.

$$\langle \langle 12 \rangle, [12] \rangle^{\langle 2 \rangle}_{R_5} = \langle \langle 12 \rangle, [12] \rangle^{2}_{R_5} = \langle \langle 12 \rangle^2, \langle 12 \rangle [12], [12]^2 \rangle_{R_5}.$$

- Symbolic/standard power may not coincide. E.g. in $\mathbb{F}[x, y, z]$

$$\langle xy, xz, yz \rangle^{\langle 2 \rangle} = \langle x^2y^2, x^2z^2, y^2z^2, xyz \rangle \neq \langle xy, xz, yz \rangle^2$$