# Poles at infinity in on-shell diagrams

## NBI Joint Theory Seminar

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Based on [2212.06840] with J. Trnka and U. Öktem

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# Motivation

- Unitarity of the S-matrix has been an immensely important concept in the study of scattering amplitudes.
- For instance, factorization on poles $p_i \to 0$ leading to the soft bootstrap program.
- Unitarity of the S-matrix does not predict what happens with tree-level amplitudes (or loop integrands) on UV poles when the external momenta (or loop momenta) go to infinity
- Is there a notion of *unitarity at infinity*?
- On-shell diagrams are natural objects to consider (gauge invariance, factorization manifest)

# Motivation

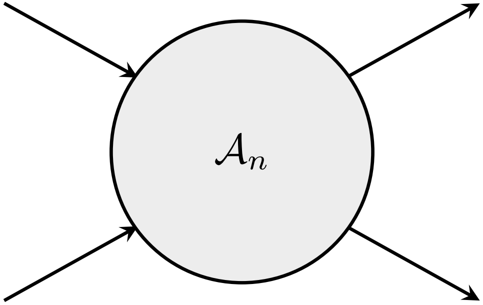
- We will study on-shell diagrams in mainly $\mathcal{N} = 3$ and show that there is a "factorization" property for diagrams with poles at infinity
- Important to note: the amplitudes themselves don't have poles at infinity, but the diagrams do.
- For $\mathcal{N} < 3$ these poles also start to show up in the amplitudes.

# Scattering amplitudes

## Motivation

From perspective of experimentalists:

- Amplitudes are highly non-local objects.
- They measure incoming and outgoing states at infinity

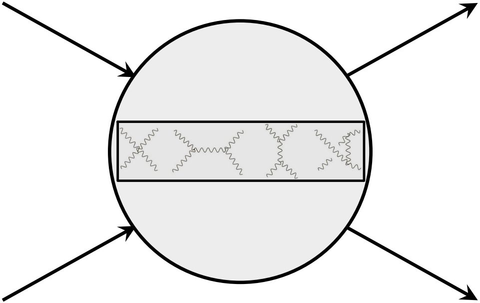


# Scattering amplitudes

## Motivation

From theorists perspective

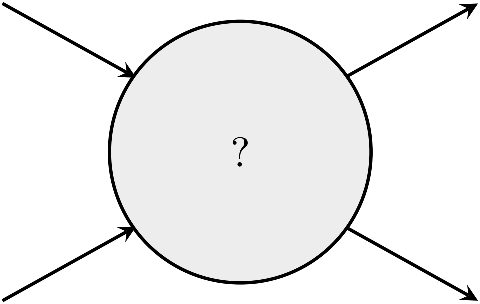
- Amplitudes are highly non-local objects.
- Traditionally described through very local processes, e.g. Feynman diagrams



# Scattering amplitudes

## Motivation

- Amplitudes are highly non-local objects.
- What else can we fill this blob with?



# Scattering amplitudes

## Color-ordering

We will focus on Yang-Mills where kinematic- and color information decompose in the amplitude.

$$\mathcal{A}_n^{\text{tree}} = \sum_{\sigma \in S_n/\mathbb{Z}_n} \mathrm{Tr}[T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}}] A_n^{\text{tree}}(\sigma(1), \ldots, \sigma(n)). \tag{1}$$

DDM Basis, from Jacobi identities

$$\mathcal{A}_n^{\text{tree}} = \sum_{\sigma \in S_{n-2}} f^{a_1 a_{\sigma(2)} b_1} f^{b_1 a_{\sigma(2)} b_2} \cdots f^{b_{n-3} a_{\sigma(n-2)} a_n} A_n(1, \sigma, n). \tag{2}$$

$\Rightarrow$ Only need to calculate color-ordered amplitudes.

# Scattering amplitudes
## sYM

- In the remaining we will work with super-symmetric Yang-Mills amplitudes
- Characterized by $\mathcal{N}$: number of super-symmetry generators $Q^A, Q^\dagger_A$.
- We will work mainly with $\mathcal{N} = 2, 3, 4$.

# Scattering amplitudes

## Spinor-Helicity

- We are going to focus on massless spin 1 particles
- All data needed about particles is helicities $\pm$ and momenta $p_i^\mu$.
- We then take

$$d = 4, \qquad p_i^\mu p_{i\mu} = 0, \qquad \sum_i p_i = 0 \tag{3}$$

- And define usual Pauli matrices

$$\begin{aligned} \sigma^\mu_{\alpha\dot\beta} &= (\mathbb{1}_{\alpha\dot\beta}, \sigma^1_{\alpha\dot\beta}, \sigma^2_{\alpha\dot\beta}, \sigma^3_{\alpha\dot\beta}) \\ (\bar\sigma^\mu)^{\dot\alpha\beta} &= (\mathbb{1}^{\dot\alpha\beta}, -(\sigma^1)^{\dot\alpha\beta}, -(\sigma^2)^{\dot\alpha\beta}, -(\sigma^3)^{\dot\alpha\beta}) \end{aligned} \tag{4}$$

# Scattering amplitudes

## Spinor-Helicity

- We can use this to define momentum bi-spinors from the four-momenta $p_\mu$

$$p_{\alpha\dot{\beta}} \equiv p_\mu \sigma^\mu_{\alpha\dot{\beta}}, \qquad\qquad p^{\dot{\alpha}\beta} \equiv p_\mu (\sigma^\mu)^{\dot{\alpha}\beta} \tag{5}$$

- Determinant of this is 0 from the massless condition
- $p_{\alpha\dot{\beta}}$ has rank 1 $\Rightarrow$ write as product of two 2-spinors $\lambda_\alpha$, $\tilde{\lambda}_{\dot{\beta}}$

# Scattering amplitudes

## Spinor-Helicity

- We can use the bra-ket notation to define spinor (braket) products

$$p_{\alpha\dot{\beta}} = \lambda_\alpha \tilde{\lambda}_{\dot{\beta}} \equiv |p\rangle_\alpha \, [p|_{\dot{\beta}} \tag{6}$$

as well as

$$\langle kp\rangle \equiv \langle k|^{\alpha} |p\rangle_\alpha, \text{ and } [kp] \equiv [k|_{\dot{\alpha}} |p]^{\dot{\alpha}} \tag{7}$$

# On-shell recursion

- Locality (i.e. point interactions) dictates: the only poles are propagators
$$\frac{1}{P^2}, \qquad \text{with } P = \sum_k p_k \tag{8}$$
- Unitarity of the S-matrix forces the amplitude to factorize on this pole,
$$A \xrightarrow[P^2=0]{} A_L \frac{1}{P^2} A_R \tag{9}$$
- This is true for IR poles, $P^2 \to 0$, no such structure is known for UV poles, $P^2 \to \infty$.

# On-shell recursion

## BCFW

- This is known as on-shell recursion, i.e. building higher point amplitude from lower point ones.
- An explicit example of this is BCFW (Britto, Cachazo, Feng, Witten) recursion.
- Perform complext shift

$$\begin{aligned} \lambda_i &\to \lambda_i - z\lambda_j \\ \widetilde{\lambda}_j &\to \widetilde{\lambda}_j + z\widetilde{\lambda}_i \end{aligned} \tag{10}$$

# On-shell recursion

## BCFW

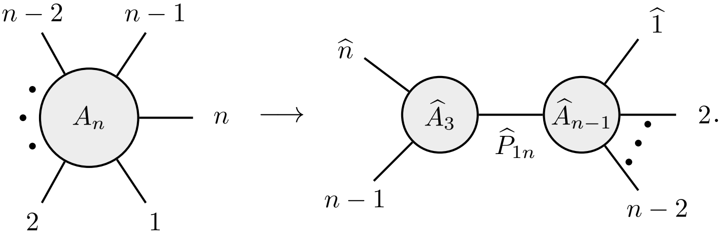
- Shift conserves momentum.
- Consider the holomorphic function $\frac{\widehat{A}_z}{z}$, i.e. the shifted amplitude.
- If amplitude behaves nicely for $z \to \infty$ we can use Cauchy's theorem to relate

$$A_n(z=0) = -\sum_k \mathrm{Res}_{z=z_k}\left[\frac{\widehat{A}_n(z)}{z}\right] \tag{11}$$

# On-shell recursion

## BCFW

- On each $z_I$ pole some propagator $\widehat{P}_I^2 = \left(\sum_{k\in I} \widehat{p}_k\right)^2$ goes on shell.
- On each pole the amplitude factorizes
- Example, $n$-point MHV amplitude, under a $1, n$ shift

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# On-shell recursion

## BCFW

- Recursively built amplitude from lower points.
- Keep recursing until everything is built from fundamental lowest point amplitudes.

# Unitarity

## 1-loop amplitude

- Unitarity allow us to write the amplitude as a linear combination of basis integrals with gauge-invariant on-shell prefactors

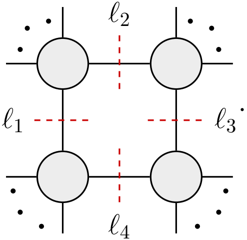
$$A_n^{\text{1-loop}} = \sum_k a_k \int \mathrm{d}I_k^{\square} + \overbrace{\sum_k b_k \int \mathrm{d}I_k^{\triangle} + \sum_k c_k \int \mathrm{d}I_k^{>\!\bigcirc\!<}}^{0 \text{ in } \mathcal{N}=4} \underbrace{+\mathcal{R}}_{0 \text{ in } \mathcal{N}=1,2}$$

- Coefficients determined using *Unitarity cuts*
- One-loop analogue of tree level unitarity

# Unitarity

## 1-loop amplitude

- When maximal number of propagators are cut, we have *maximal cuts*
- Cutting each propagator puts it on shell $\ell_i^2 = 0$
- For instance, we have the Quadruple cut:



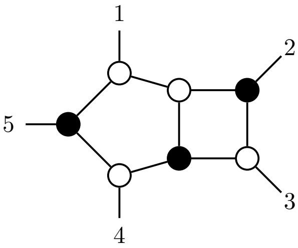

$$. \tag{13}$$

- For each helicity configuration the coefficient then is summed over solutions to cut conditions

$$a_k \sim \sum_{\text{sol}} A_1 \times A_2 \times A_3 \times A_4 \tag{14}$$

# On-Shell diagrams

- Iterate both types of cuts until all propagators are cut
- Remaining object is build entirely of fundamental amplitudes
- These are on-shell diagrams

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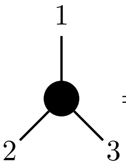
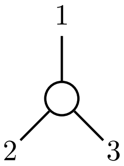
# On-shell diagrams

## Three-point amplitudes

The simplest amplitudes we can construct are three point. For $\mathcal{N} = 4$




$$= \frac{\delta^4(P)\delta^8(\mathcal{Q})}{\langle 12\rangle\langle 23\rangle\langle 31\rangle},$$




$$= \frac{\delta^4(P)\delta^4(\widetilde{\mathcal{Q}})}{[12][23][31]}$$

where,

$$P \equiv \lambda\cdot\widetilde{\lambda} = \lambda_1\widetilde{\lambda}_1 + \lambda_2\widetilde{\lambda}_2 + \lambda_3\widetilde{\lambda}_3, \quad \mathcal{Q} \equiv \lambda\cdot\widetilde{\eta} = \lambda_1\widetilde{\eta}_1 + \lambda_2\widetilde{\eta}_2 + \lambda_3\widetilde{\eta}_3,$$

$$\widetilde{\mathcal{Q}} \equiv [12]\widetilde{\eta}_3 + [23]\widetilde{\eta}_1 + [31]\widetilde{\eta}_2$$

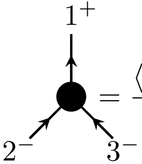
$$\Phi(\widetilde{\eta}) = g^+ + \widetilde{\eta}^I\ \widetilde{g}_I + \frac{1}{2!}\widetilde{\eta}^I\widetilde{\eta}^J\ \phi_{IJ}$$

$$+\frac{1}{3!}\epsilon_{IJKL}\widetilde{\eta}^I\widetilde{\eta}^J\widetilde{\eta}^K\ \widetilde{g}^L + \frac{1}{4!}\epsilon_{IJKL}\widetilde{\eta}^I\widetilde{\eta}^J\widetilde{\eta}^K\widetilde{\eta}^L\ g^-.$$

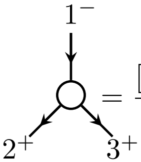
# On-shell diagrams

## Three-point amplitudes

The simplest amplitudes we can construct are three point. For any $\mathcal{N}$




$$= \frac{\langle 23\rangle^{4-\mathcal{N}}\delta^4(P)\delta^{2\mathcal{N}}(\mathcal{Q})}{\langle 12\rangle\langle 23\rangle\langle 31\rangle},$$




$$= \frac{[23]^{4-\mathcal{N}}\delta^4(P)\delta^{\mathcal{N}}(\widetilde{\mathcal{Q}})}{[12][23][31]}$$

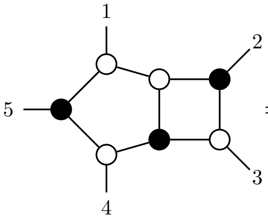
Since $0 = p_3^2 = (p_1 + p_2)^2 = 2p_1 \cdot p_2 = \langle 12\rangle\,[21]$, these obey constrained kinematics,

$$\tilde{\lambda}_1 \sim \tilde{\lambda}_2 \sim \tilde{\lambda}_3, \qquad \lambda_1 \sim \lambda_2 \sim \lambda_3 \tag{16}$$

# On-shell diagrams

## Three-point amplitudes

- On-shell diagrams are build by gluing these fundamental three-point vertices together.
- All vertices satisfy momentum conservation.
- Every propagator (internal line) is on-shell, $p^2 = 0$.

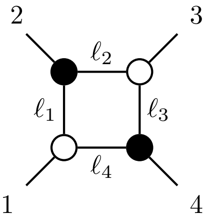



$$= \prod_k \int d^{\mathcal{N}} \widetilde{\eta}_k \int \frac{d^2\lambda_k \, d^2\widetilde{\lambda}_k}{\mathrm{GL}(1)} \left( \prod_j A_3^{(j)} \right) \tag{17}$$

# On-shell diagrams

## 4 point amplitude

- Simplest example




$$\ell_1 = \frac{\langle 23\rangle}{\langle 13\rangle}\lambda_1\widetilde{\lambda}_2, \qquad \ell_3 = \frac{\langle 14\rangle}{\langle 13\rangle}\lambda_3\widetilde{\lambda}_4,$$

$$\ell_2 = \frac{\langle 12\rangle}{\langle 13\rangle}\lambda_3\widetilde{\lambda}_2, \qquad \ell_4 = \frac{\langle 34\rangle}{\langle 13\rangle}\lambda_1\widetilde{\lambda}_4.$$

- Gluing is done by integrating over cut conditions

$$\Omega = \int d^4\widetilde{\eta}_1 \dots d^4\widetilde{\eta}_4 \int \frac{d^2\lambda_{\ell_1}\, d^2\widetilde{\lambda}_{\ell_1}}{\mathrm{GL}(1)} \dots \frac{d^2\lambda_{\ell_4}\, d^2\widetilde{\lambda}_{\ell_4}}{\mathrm{GL}(1)}$$
$$\times \left\{ A_3(1,\ell_1,\ell_4)A_3(2,\ell_1,\ell_2)A_3(3,\ell_2,\ell_3)A_3(4,\ell_3,\ell_4) \right\} \qquad (18)$$
$$= \frac{\delta^4(P)\delta^8(\mathcal{Q})}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

# On-shell diagrams

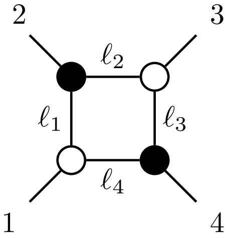
## 4 point amplitude

$$\Omega = \frac{\delta^4(P)\delta^8(\mathcal{Q})}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \tag{19}$$

- All poles in the amplitude correspond to sending one $\ell_i \to 0$.
- One pole is not present in the amplitude, $\langle 13\rangle$.
- This is the pole at infinite momentum $\ell_i \to \infty$, or the *UV pole*.

# On-shell diagrams

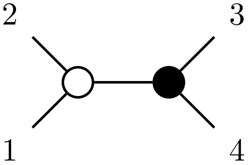
## 4 point amplitude




$$\ell_1 = \frac{\langle 23\rangle}{\langle 13\rangle}\lambda_1\widetilde{\lambda}_2, \qquad \ell_3 = \frac{\langle 14\rangle}{\langle 13\rangle}\lambda_3\widetilde{\lambda}_4,$$

$$\ell_2 = \frac{\langle 12\rangle}{\langle 13\rangle}\lambda_3\widetilde{\lambda}_2, \qquad \ell_4 = \frac{\langle 34\rangle}{\langle 13\rangle}\lambda_1\widetilde{\lambda}_4. \tag{20}$$

For example, sending $\langle 12\rangle = 0$ implies $\ell_2 = 0$ and we get

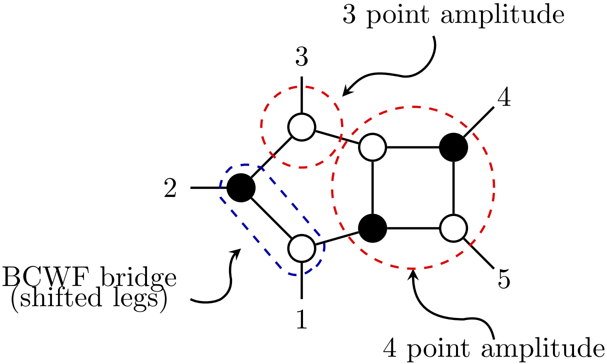



$$= \frac{\delta(\langle 12\rangle)\delta^4(P)\delta^8(\mathcal{Q})}{\langle 23\rangle\langle 34\rangle\langle 41\rangle}, \tag{21}$$

# On-shell diagrams

## Higher point amplitudes

- Higher point diagrams correspond to BCFW terms
- For MHV amplitudes we obtain 1 diagram at all points

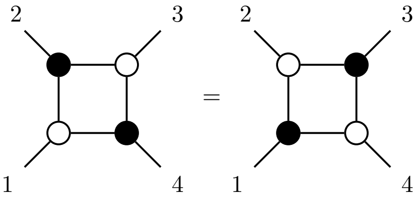


# On-shell diagrams

## Identity moves

- The following moves do not change the on shell function for the diagram – are *identity moves*

Square move:

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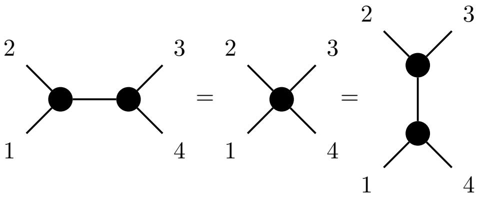
Merge-expand:

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# Dual Formulation

- Consider momentum conservation:

$$\delta^4(P) = \delta^4(\lambda \cdot \tilde{\lambda}) = \delta^4(\lambda_1 \tilde{\lambda}_1 + \ldots + \lambda_n \tilde{\lambda}_n) \tag{24}$$

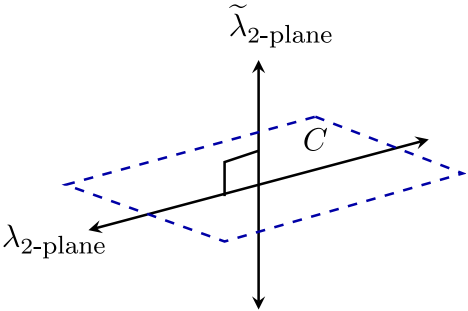
- Introduce a $k$-plane in $n$-dimensions represented by a $(k \times n)-$matrix (modded out by GL(k) since such row operations leave the $k$-plane invariant).
- This space is denoted by $G(k, n)$, the *Grassmannian*.
- A point in this space is represented by a $(k \times n)$ matrix, which we refer to as the $C$-matrix.
- Linearized momentum conservation condition

$$\delta(C \cdot Z) = \delta^{((n-k)\times 2)}(C^{\perp} \cdot \lambda)\delta^{(k\times 2)}(C \cdot \widetilde{\lambda})\delta^{(k\times \mathcal{N})}(C \cdot \widetilde{\eta}) \tag{25}$$

# Dual Formulation

## Interpretation

- Geometrically we can visualize this in "particle space"

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# Dual Formulation

## Interpretation

- What is the connection to on-shell diagrams?
- They parameterize $C$ in a certain way

# Dual Formulation

## Interpretation

- Introduce orientation for each diagram by assigning arrows to all edges where
  - Black vertices have two incoming and one outgoing arrow.
  - White vertices have one incoming and two outgoing arrows.
- Then assign edge variables to all edges, fixing in each vertex one variable to 1.
- The C matrix is then given by

$$C_{\alpha a} = \sum_{\Gamma_{\alpha \to a}} \prod_j \alpha_j \tag{27}$$

# Dual Formulation

## Interpretation

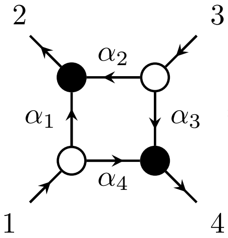
- The on-shell function associated with the an on-shell diagram in $\mathcal{N} = 4$ SYM theory is given by

$$\Omega = \int \prod_i \frac{d\alpha_i}{\alpha_i} \delta(C \cdot Z) \tag{28}$$

- where the $\delta$-functions let us determine the $\alpha$'s.

# Dual Formulation

As an example, take




$$\Rightarrow C = \begin{pmatrix} 1 & \alpha_1 & 0 & \alpha_4 \\ 0 & \alpha_2 & 1 & \alpha_3 \end{pmatrix}, C^{\perp} = \begin{pmatrix} -\alpha_1 & 1 & -\alpha_2 & 0 \\ -\alpha_4 & 0 & -\alpha_3 & 1 \end{pmatrix}.$$

which e.g. leads to

$$\delta^4(C^{\perp} \cdot \lambda) = \delta^2(-\alpha_1\lambda_1 + \lambda_2 - \alpha_3\lambda_3)\delta^2(-\alpha_4\lambda_1 - \alpha_2\lambda_3 + \lambda_4). \tag{29}$$

We can use this to solve for $\alpha$'s.

# Dual Formulation

Explicitly

$$\Omega = \frac{\delta^{8}\left(\mathcal{Q}\right)\delta^{4}\left(P\right)}{\alpha_1\alpha_2\alpha_3\alpha_4\langle 13\rangle^4}\delta\left(\alpha_2-\frac{\langle 12\rangle}{\langle 13\rangle}\right)\delta\left(\alpha_1-\frac{\langle 23\rangle}{\langle 13\rangle}\right) \times\delta\left(\alpha_3-\frac{\langle 14\rangle}{\langle 13\rangle}\right)\delta\left(\alpha_4-\frac{\langle 43\rangle}{\langle 13\rangle}\right) = \frac{\delta^{8}\left(\mathcal{Q}\right)\delta^{4}\left(P\right)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \tag{30}$$

# Dual Formulation

- So on-shell diagrams can be used to find the tree level amplitude, or the maximal cut of a loop integrand.

# Dual Formulation

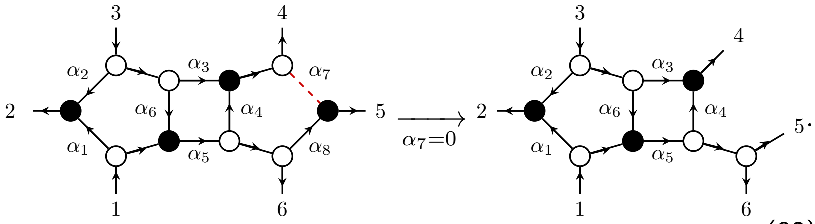
- With the basics introduced, we can focus on the pole structure.
- Importantly the UV structure will *not* correspond to that of the amplitudes themselves.
- But seeing how the on-shell diagrams behave can give a clue on what directions to look in for the amplitudes.

# Dual Formulation

## Interpretation

- Poles in $\Omega$ correspond to sending edges to $0$ or $\infty$
- For instance for 6-point MHV,

$$\Omega = \frac{\delta^8\left(\mathcal{Q}\right)\delta^4\left(P\right)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 61\rangle} \tag{31}$$

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here $\alpha_7 = \frac{\langle 56\rangle}{\langle 46\rangle}$

# Dual Formulation

## $\mathcal{N} \neq 4$

- This is simple enough. What about $\mathcal{N} \neq 4$?
- We have to modify

$$\Omega = \int \prod_i \frac{d\alpha_i}{\alpha_i} \delta(C \cdot Z) \mathcal{J}^{\mathcal{N}-4} \tag{33}$$

- The Jacobian $\mathcal{J}$ is given by

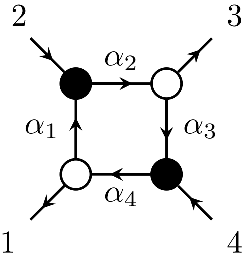
$$\mathcal{J} = 1 + \sum_i f_i + \sum_{\substack{\text{disjoint} \\ \text{pairs } i,j}} f_i f_j + \sum_{\substack{\text{disjoint} \\ \text{pairs } i,j,k}} f_i f_j f_k + \cdots \tag{34}$$

with $f_i$ is a clockwise-oriented product of edge-variables in closed cycles, and the sums are over disjoint collections of these closed cycles.

# Dual Formulation

## $\mathcal{N} \neq 4$

Example:




$$C = \begin{pmatrix} \alpha_2\alpha_3\alpha_4\delta & 1 & \alpha_2\delta & 0 \\ \alpha 4\delta & 0 & \alpha_1\alpha_2\alpha_3\delta & 1 \end{pmatrix} \quad (35)$$

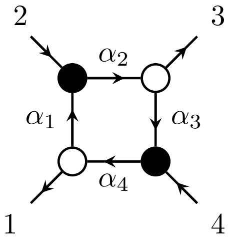
The $\delta$ that appears here corresponds to a geometric series, which can be written as:

$$\delta = \sum_{k=0}^{\infty} (\alpha_1\alpha_2\alpha_3\alpha_4)^k = \frac{1}{1 - \alpha_1\alpha_2\alpha_3\alpha_4} \quad (36)$$

# Dual Formulation

## $\mathcal{N} \neq 4$

Example:

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The Jacobian from the internal cycle is

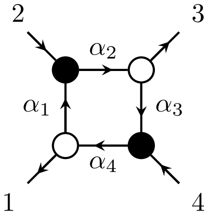
$$\mathcal{J} = 1 - \alpha_1\alpha_2\alpha_3\alpha_4 = \frac{\langle 13\rangle\langle 24\rangle}{\langle 12\rangle\langle 34\rangle} \tag{38}$$

The on-shell form is then given by

$$\Omega = \underbrace{\frac{\langle 24\rangle^{4-\mathcal{N}}\,\delta^4(P)\delta^{2\mathcal{N}}(Q)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}}_{\Omega_{\text{bare}}}\left(\frac{\langle 12\rangle\langle 34\rangle}{\langle 13\rangle\langle 24\rangle}\right)^{4-\mathcal{N}} \tag{39}$$

# Dual Formulation

## $\mathcal{N} \neq 4$




$$\alpha_1 = \frac{\langle 23\rangle}{\langle 13\rangle}, \quad \alpha_2 = \frac{\langle 13\rangle}{\langle 12\rangle}, \quad \alpha_3 = \frac{\langle 14\rangle}{\langle 13\rangle}, \quad \alpha_4 = \frac{\langle 13\rangle}{\langle 34\rangle}.$$

- Let us study this result

$$\Omega = \underbrace{\frac{\langle 24\rangle^{4-\mathcal{N}}\,\delta^4(P)\delta^{2\mathcal{N}}(Q)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}}_{\Omega_{\text{bare}}}\left(\frac{\langle 12\rangle\langle 34\rangle}{\langle 13\rangle\langle 24\rangle}\right)^{4-\mathcal{N}} \qquad (40)$$

- Deletes poles that stem from edges that are not removable
- Introduces pole at infinity $\langle 13\rangle$.

# Dual Formulation

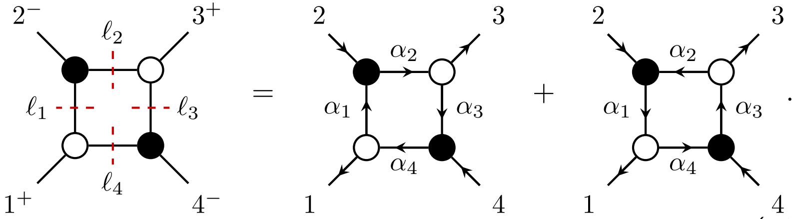
## $\mathcal{N} \neq 4$

- For now, focus on $\mathcal{N} = 3 \Rightarrow$ Simple Poles
- Important point: For $\mathcal{N}=4$ SYM theory each on-shell diagram directly corresponds to a cut of the loop integrand
- For $\mathcal{N}<4$ SYM one has to sum over all internal configurations for fixed external helicities

# Dual Formulation

## $\mathcal{N} \neq 4$

- For $\mathcal{N}<4$ SYM one has to sum over all internal configurations for fixed external helicities,
- Example: For legs $2, 4$ incoming there are two possible internal orientations,

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# Dual Formulation

## $\mathcal{N} \neq 4$

- For $\mathcal{N} = 3$ we get,

$$\mathrm{Cut}\ A_4(1^+2^-3^+4^-) = \frac{\langle 12\rangle\langle 23\rangle\, \delta^4(P)\delta^6(Q)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle\langle 13\rangle} + \frac{\langle 23\rangle\langle 14\rangle\, \delta^4(P)\delta^6(\mathcal{Q})}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle\langle 13\rangle}$$
$$= \frac{\langle 24\rangle\, \delta^4(P)\delta^6(\mathcal{Q})}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}.$$

So

$$\begin{aligned} \mathrm{Cut}\ \left[A^{\mathcal{N}=3}\right] &= \mathrm{Cut}\ \left[A^{\mathcal{N}=4}\right] \\ \text{ONS}\ \left[A^{\mathcal{N}=3}\right] &\neq \text{ONS}\ \left[A^{\mathcal{N}=4}\right] \end{aligned} \qquad (42)$$

where ONS: On-shell diagram

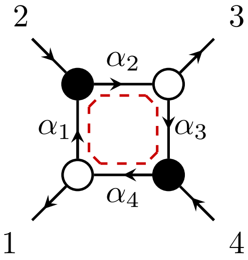
# Dual Formulation

## $\mathcal{N} \neq 4$

- For now, focus on on-shell diagrams in $\mathcal{N} = 3 \Rightarrow$ Simple Poles
- What happens if we sit on the pole at infinity?

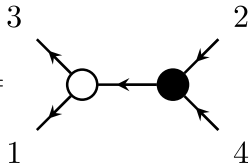
# Dual Formulation

## $\mathcal{N} \neq 4$




$$= \frac{\delta^4(P)\delta^6(Q)}{\langle 23\rangle\langle 14\rangle\langle 13\rangle} \tag{43}$$

$$\text{Res}(\Omega)_{\langle 13\rangle=0} = \frac{\delta^4(P)\delta^6(Q)\delta(\langle 13\rangle)}{\langle 23\rangle\langle 14\rangle} =$$




$$\equiv \Omega_{UV}$$

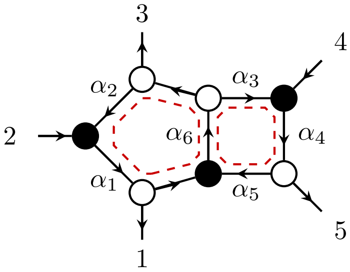
# Dual Formulation

## $\mathcal{N} \neq 4$

Video showing schematics:

# Dual Formulation

Other example




$$\alpha_1 = \frac{\langle 13\rangle}{\langle 23\rangle},\ \alpha_2 = \frac{\langle 12\rangle}{\langle 13\rangle},\ \alpha_3 = \frac{\langle 45\rangle}{\langle 35\rangle},$$
$$\alpha_4 = \frac{\langle 35\rangle}{\langle 34\rangle},\ \alpha_5 = \frac{\langle 13\rangle}{\langle 35\rangle},\ \alpha_6 = \frac{\langle 35\rangle}{\langle 15\rangle}. \tag{44}$$

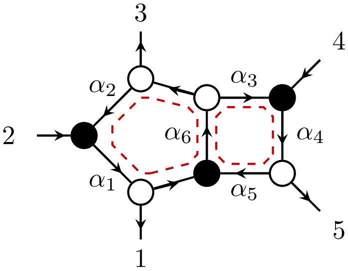
The on-shell function is then equal to

$$\Omega = \frac{\delta^4(P)\delta^6(Q)}{\langle 12\rangle\langle 45\rangle\langle 13\rangle\langle 35\rangle} \tag{45}$$

# Dual Formulation

## $\mathcal{N} \neq 4$

Blow up each loop individually



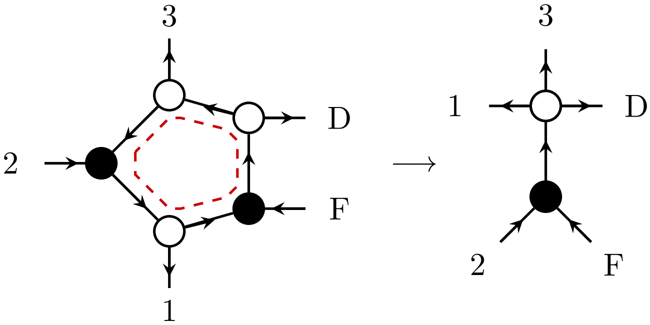

$$\Omega = \frac{\delta^4(P)\delta^6(Q)}{\langle 12\rangle\langle 45\rangle\langle 13\rangle\langle 35\rangle} \tag{46}$$

We know how the box blows up already

# Dual Formulation

## $\mathcal{N} \neq 4$

The pentagon on the other hand,

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where the 4-point vertex can be further expanded as a chain of 3-point vertices.

# Dual Formulation

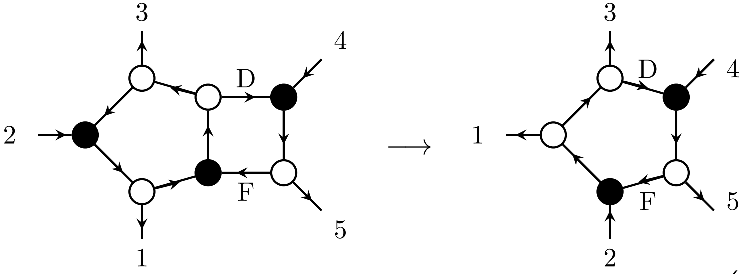
## $\mathcal{N} \neq 4$

Video showing schematics:

# Dual Formulation

## $\mathcal{N} \neq 4$

Gluing back with the right box we get a pentagon diagram,




(48)

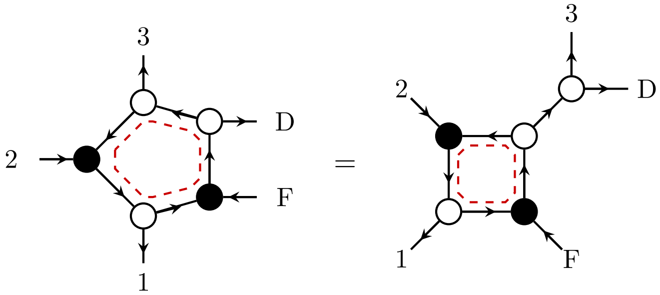
where

$$\Omega = \frac{\delta^4(P)\delta^6(Q)}{\langle 12\rangle\langle 45\rangle\langle 13\rangle\langle 35\rangle} \quad \longrightarrow \quad \Omega_{UV} = \frac{\delta^4(P)\delta^6(Q)\delta(\langle 13\rangle)}{\langle 12\rangle\langle 45\rangle\langle 35\rangle} \tag{49}$$

# Dual Formulation

## $\mathcal{N} \neq 4$

This should not be surprising since the pentagon was really just a box

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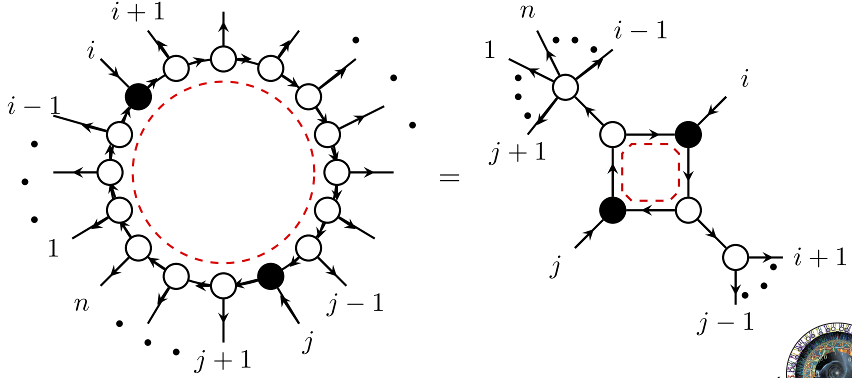
which we already knew how to do.

# Dual Formulation

## $\mathcal{N} \neq 4$

- Let us generalize this to higher points
- Sufficient in planar diagrams to treat $n$-gons
- These are secretly just boxes!

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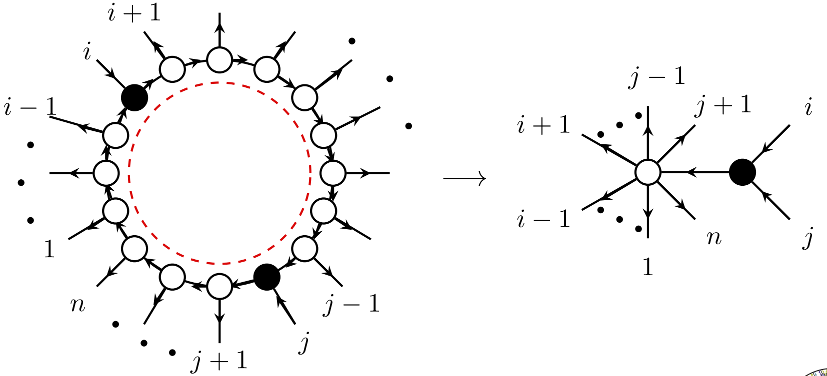
# Dual Formulation

## $\mathcal{N} \neq 4$

The residue is then simple to generalize

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# Dual Formulation

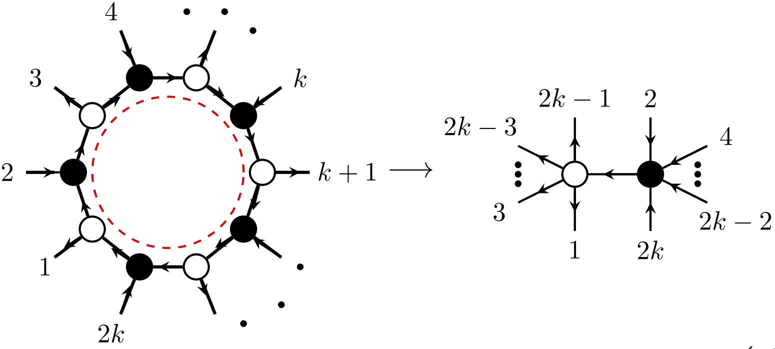
## $\mathcal{N} \neq 4$

- These all are MHV diagram with only two black vertices.
- For N$^{k}$MHV diagrams we have $k - 2$ black vertices.
- The expressions are a lot more complicated, but the result is similar
- Find result for n-gon then attach these to remaining diagram.

# Dual Formulation

## $\mathcal{N} \neq 4$

On UV pole on-shell diagrams behaves as

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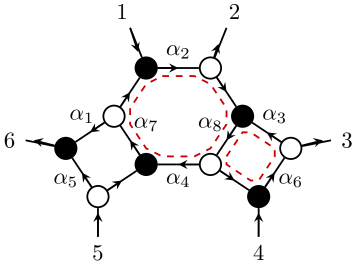
# Larger diagrams

## $\mathcal{N} \neq 4$

Take three-loop six-point NMHV leading singularity diagram



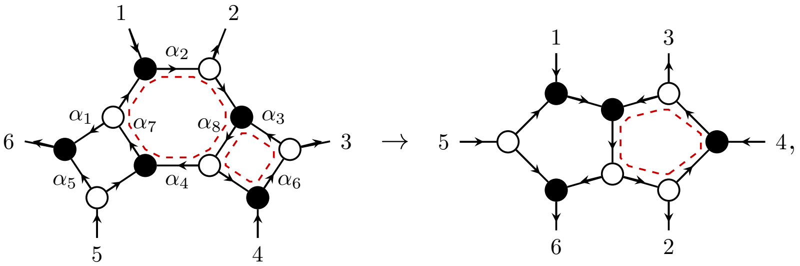

$$= \frac{\delta^6(\mathcal{Q})\delta^4(P)\delta([56]\widetilde{\eta}_1+[61]\widetilde{\eta}_5+[15]\widetilde{\eta}_6)}{\langle 34\rangle\langle 2|3+4|1]\langle 3|5+6|1]\langle 2|1+6|5][56]}$$

- Hexagon pole at $\langle 2|3+4|1] = 0$
- Box pole at $\langle 3|5+6|1]$

# Larger diagrams

## $\mathcal{N} \neq 4$

- Contracting the Hexagon into a tree and then attaching to the remaining diagram

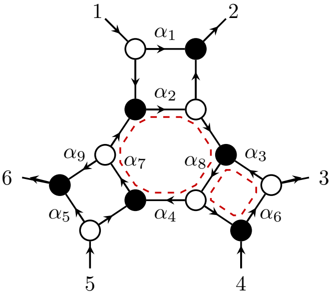



- Notice the change in orientation – this is a general feature since for larger diagrams this gives non planar diagram

# Larger diagrams

## $\mathcal{N} \neq 4$

- Let us consider a more intricate example – the top dimensional cell of $G_+(3,6)$
- This can be obtained by attaching a BCFW bridge to the previous diagram: $\widehat{\widetilde{\lambda}}_1 = \widetilde{\lambda}_1 + \alpha_1 \widetilde{\lambda}_2$ and $\widehat{\lambda}_2 = \lambda_2 - \alpha_1 \lambda_1$
- On shell conditions leave one parameter unfixed

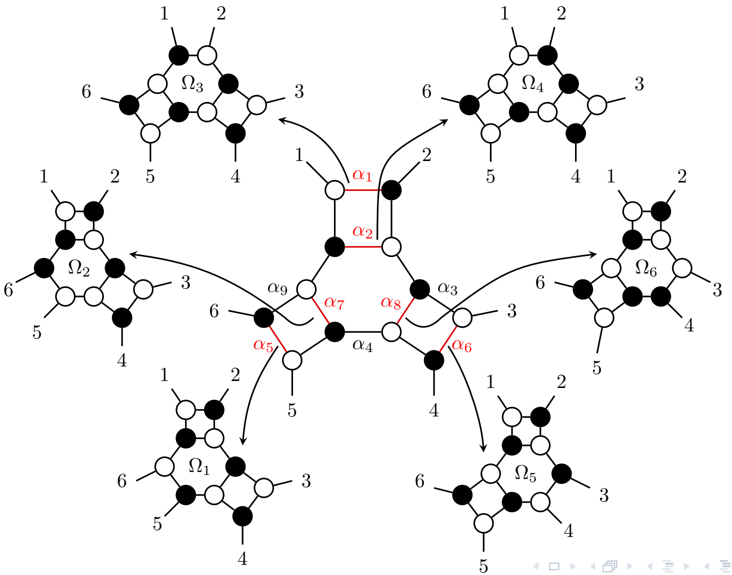



$$= \int \underbrace{\frac{d\alpha_1 \, \langle 4|\hat{1}+5|6]\delta(\Xi)}{\alpha_1 s_{\hat{1}56}\langle \hat{2}3\rangle\langle 34\rangle\langle \hat{2}|3+4|5]\langle 4|5+6|\hat{1}][\hat{1}6][56]}}_{\Omega^{\mathrm{bare}}} \times \underbrace{\frac{s_{\hat{1}56}\langle \hat{2}3\rangle\langle 4|5+6|\hat{1}][\hat{1}6]}{\langle \hat{2}|5+6|\hat{1}]\langle 3|5+6|\hat{1}]((\langle 4|\hat{1}+5|6])}}_{\mathcal{J}^{-1}} \qquad (54)$$

# GRT

## $\mathcal{N} \neq 4$

For $\mathcal{N} = 4$ the pole structure can be illustrated as follows

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# GRT

## $\mathcal{N} \neq 4$

- Where, through the Global Residue Theorem (GRT)

$$\sum_{i=1}^{6} \Omega_i = 0, \quad \text{with} \quad \Omega_1 = \frac{\delta^4(P)\delta^8(\mathcal{Q})\,\delta([34]\widetilde{\eta}_5+[45]\widetilde{\eta}_3+[53]\widetilde{\eta}_4)}{s_{345}[34][45]\langle 16\rangle\langle 12\rangle\langle 2|3+4|5]\langle 6|1+2|3]}$$

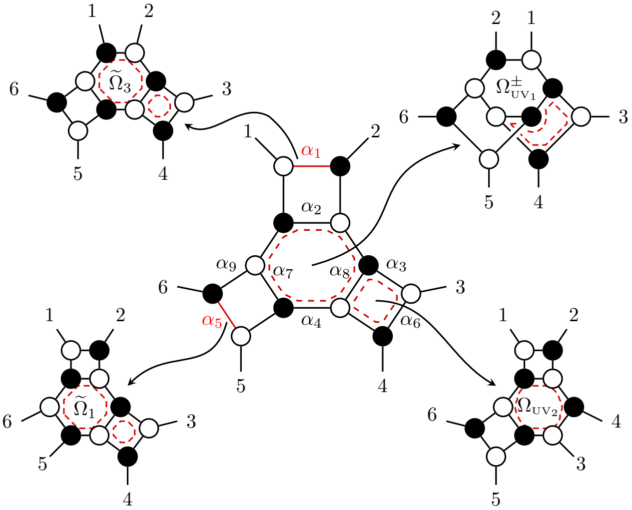
- This is directly linked to the six-point NMHV tree-level amplitude

$$\mathcal{A}_{6,3}^{\text{tree}} = \Omega_1 + \Omega_3 + \Omega_5 = -\Omega_2 - \Omega_4 - \Omega_6 \qquad (56)$$

# GRT

## $\mathcal{N} \neq 4$

For our $\mathcal{N} = 3$ example we have a new GRT, where we also explicitly see the non planar structure

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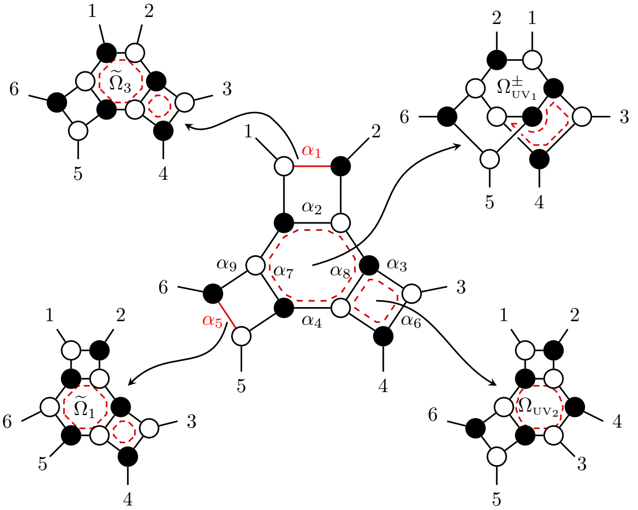
# GRT

## $\mathcal{N} \neq 4$

Let us highlight the Hexagon UV pole which gives the non-planar diagram:

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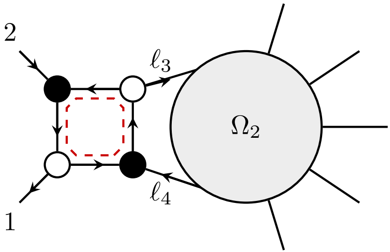
# Dual Formulation

## $\mathcal{N} \neq 4$

Video showing schematics:

More general problem

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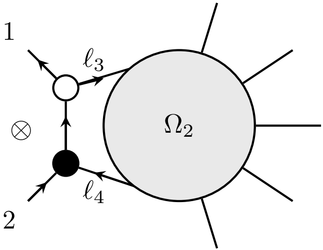
UV pole of this is obtained by acting on the collapsed diagram with a differential operator $\mathcal{O}$

# General $\mathcal{N}$

## $\mathcal{N} \neq 4$

$$\Omega_{\mathrm{UV}} = \mathcal{O}^{\mathcal{N}} \otimes \tag{60}$$




with

$$\mathcal{O}^{\mathcal{N}} = \frac{1}{(3-\mathcal{N})!} \left( \frac{\langle 2\ell_3\rangle[2\ell_3]}{\langle 12\rangle[1\ell_3]} \left\langle \lambda_2 \frac{d}{d\lambda_{\ell_3}} \right\rangle \right)^{3-\mathcal{N}} \tag{61}$$

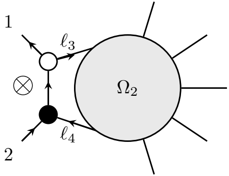
# General $\mathcal{N}$

## $\mathcal{N} \neq 4$

The procedure is as follows

- Calculate the bare on-shell function of the lower-loop on-shell diagram obtained by diagrammatic rules
  - Crucially this relies on leaving the integration over an unfixed leg $\lambda_{\ell_3}$ such that one can act with derivative.
  - Also includes an integration over the internal leg $\ell_4$ to eliminate the dependencies $\lambda_{\ell_3}$ from momentum conservation.
- Take the appropriate number of derivatives with respect to $\lambda_{\ell_3}$

$$\Omega_{\mathrm{UV}} = \mathcal{O}^{\mathcal{N}} \otimes \tag{62}$$



# General $\mathcal{N}$

## $\mathcal{N} \neq 4$

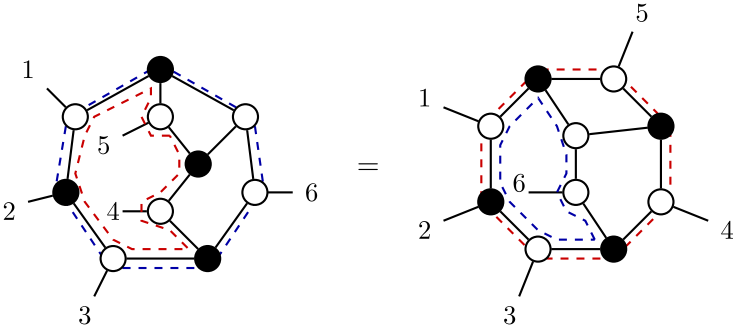
It is not surprising that higher order poles come with derivatives.

Finally, let us briefly study non-planar diagrams.

# Non planar

## $\mathcal{N} \neq 4$

Example

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# Summary and Outlook

# Summary and Outlook

To recap today's talk

- Amplitudes are defined by their factorization properties on their poles
- Using the dual formulation we introduced a procedure for taking the residue of an on-shell diagram on it's UV pole - this is not something that one expected apriori!
- This can point us to directions to look in for the UV structure of amplitudes

# Summary and Outlook

Outlook

- How does this procedure for the UV poles relate to the actual amplitudes and loop integrands (i.e. not just for the on-shell diagram)?
- Results have implications in search for positive geometries that could capture amplitudes in theories with non-trivial UV physics.
- Non-planar results are needed for instance for $\mathcal{N} = 8$ SuGRA.

Thank you for your attention!