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Henrik Jeldtoft Jensen
Department of Mathematics & Centre for Complexity Science
Imperial College London

The following exercises are meant as suggestions for discussions that can illuminate important aspects of the complexity science approach to systems consisting of many interrelated components.

Exercise 1 – Consequences of criticality
Discuss in what way the response and/or control of a system supporting critical behaviour differ from a system with short range correlations and Gaussian distributed event sizes.

Exercise 2 – Power law probability distribution
Consider a probability distribution for event sizes s given by

$$P(s) = \begin{cases} As^{-\tau} & \text{if } s \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

where A is a normalisation factor.

(a) Explain in what sense power laws are scale invariant.

(b) Derive the normalisation constant

(c) Compute the moments of the distribution and derive criteria on τ for the existence of the a-th moment.

Exercise 3 – Hierarchies
Make a summary of the hierarchies of processes, nested within each other, you can identify when you think of systems around us such as climate, economics, a human being, etc.

Exercise 4 – "Things" versus processes
Discuss emergence from the viewpoint that reality actually consists of processes and not of "things"

The following references are of relevance to this discussion:

- A.N. Withehead, Process and Reality (Gifford Lectures), Macmillan, USA, 1979.
- V. Lowe, Understanding WHitehead. The Johns Hopkins Press, 1962.
- D.W. Sherburne, A Key to Whitehead's Process and Reality. University of Chicago Press, 1981.
- D. J. Chalmers. Strong and weak emergence. In P. Clayton and P. Davies (eds), The Re-mergence of Emergence. Oxford University Press, 2006.
- S. Gibb, R. F. Hendry and T. Lancaster (eds). The Routledge Handbook of Emergence. Routledge, 2019.

Exercise 5 – Emergence of structure and intermittency
List examples from geophysics, climate, biology, sociology and neuroscience/psychology which involves the emergence of some kind of

(1) Robust many component structures.

(2) Intermittency in time.

Exercise 6 – Intermittency, transitions and tipping points
Make a list of as many types of intermittency and transitions as possible in as many different complex systems as you can think of. For each transition try to suggest ways to check if the transition involves a diverging length scale and some type of scale invariance.

Exercise 7 – Fluctuations as early warning signal
Tipping points and fluctuations.

(a) List examples of abrupt changes, or tipping points, which may be precede by an increased in the fluctuations of some observable parameter.

(b) Now list examples of abrupt changes, or tipping points, which are not precede by an increased in the fluctuations of some (known) observable parameter.

Exercise 8 – Branching process and power laws
Consider a branching process with the following branching probabilities.

$$\begin{aligned} p_0 &= q, \\ p_1 &= 0, \\ p_2 &= p, \\ p_k &= 0 \ \ \forall k \geq 2. \end{aligned} \tag{1}$$

Let $P(T)$ denote the probability that extinction occurs at generation T. And let $P(S)$ denote the probability that the total number of nodes generated before extinction is equal to S

- For which value of p is this process critical.
- By direct simulation, consider the limit $\mu \to 1^-$ and estimate the value of the exponent of the power law $P(T)$ is approaching. At criticality, the theoretical value for the exponent corresponding to $P(S)$ is $a = -3/2$ and the exponent corresponding to $P(T)$ is $b = -2$.

Exercise 9 – Entropic driven dynamics
We consider a particle moving on the lattice structure depicted in Fig. 1. We assume it moves to each of its nearest neighbour sites with equal probability and that it moves in a way such that within each box it is equally like to be found at any of the L_i^2 squares, where L_i is the linear size of box i, i.e. $L_i = 2^i$ with $i = 1, 2, 3, ...$

(a) Assume for a moment that the passage between boxes is blocked. What is the entropy of the particle when placed in box i.

Imagine we have an ensemble of systems like the one in Fig. 1. We can then think of what on average is happening in this set of equivalent systems. It is natural to estimate the average time between recurrent visits to a given site in box i as given by the inverse of the probability to visit a site.

(b) Assume the particle is in box i. What is the probability that the particle is located next to an escape site, see Fig. 1.

(c) What is the probability per time that a particle in box i moves to box $i + 1$.

(d) What is the probability per time that a particle in box $i + 1$ moves to box i.

Now imagine one particle is in box i and one in box $i + 1$.

(e) Show that the net probability (right move minus left move) for a particle being move from box i to $i + 1$ is given by $\frac{1}{3} \cdot \frac{1}{2^{i+1}}$

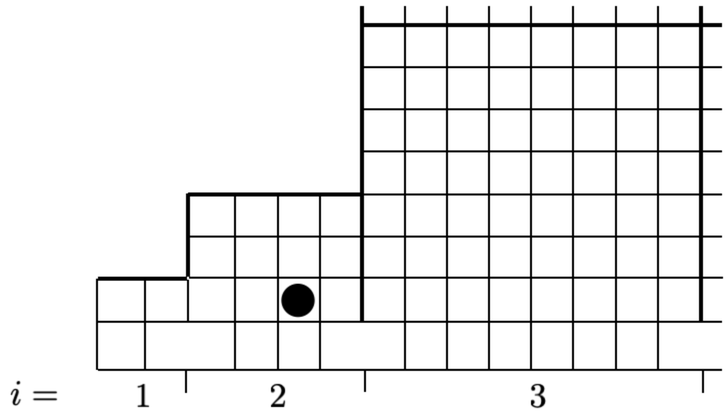

Figure 1: Particle moving in boxes. Imagine the figure extends to the right with ever more boxes $i = 1, 2, 3, 4, ...$ added. The particle is able to move between neighbour boxes though the hole at the bottom right corner of box i.

We see the particles will move towards higher entropy. Considerations like these were used in R. Arthur and A. Nicholson, *Selection Principle for Gaia* (J. Theo. Biol. **533**, 110940, 2022) to argue about Gaia, evolution and "entropic pressure".

Exercise 10 – Early warning signal

In stead of the increase in fluctuations one may in principle use the overlap between $\delta\mathbf{n}(t)$ and the normalised eigenvector $\mathbf{e}_{\max}$ corresponding to the most positive eigenvalue $\lambda_{\max}$. We define a warning signal, $Q(t)$, from the growth predicted during the succeeding Δt time units by the expansion of $\delta\mathbf{n}(t)$ component along the most unstable direction

$$Q(t) = |e^{\lambda_{\max}\Delta t}\delta\mathbf{n}(t) \cdot \mathbf{e}_{\max}|. \tag{2}$$

As a first step, let us look at the the temporal evolution of two populations $x(t)$ and $y(t)$. The x population evolves according to the following Ornstein-Uhlenbeck process with a time independent $\eta > 0$ and $\bar{X} > 0$ given by

$$x(t+1) = [x(t) - \eta(x(t) - \bar{X})] \exp(-\alpha y(t)) + \sqrt{x(t)}\chi(t) \tag{3}$$

and the restoring term exponentially damped, $\alpha > 0$, by coupling to a growth process controlling the y population according to

$$y(t+1) = \begin{cases} y(t) + \nu x(t) & \text{with prob. } p \\ y(t)(1+\beta) + \sqrt{y(t)}\chi(t) & \text{with prob. } 1-p, \end{cases} \tag{4}$$

where we assume the two constants ν and β to be positive. We can loosely think of $x(t)$ and $y(t)$ as being proportional to the size of the populations of two types X and Y. Type

Y may become populated as a result of a mutation from population X, which occurs with probability p

Simulate the process in Eqns. (3) and (4). Construct $Q(t)$ in Eq. (2) for the motion in the vicinity of the fixed point at $(x, y) = (\bar{X}, 0)$. Starting from different initial points on the x-axis near $x = X$, study the fluctuations $|\boldsymbol{\delta}(t)| = |(x(t) - \bar{X}, y(t) - 0)|$ away from the fixed point at $(\bar{X}, 0)$ and the temporal behaviour of $Q(t)$. Compare the behaviour of $|\boldsymbol{\delta}(t)|$ and $Q(t)$ in terms of early warning signals.

You can for example consider the following parameters: $\eta = 10^{-1}$, $\alpha = \beta = \nu = 10^{-2}$, $\bar{X} = 1$, $p = 10^{-4}$.

Text books – There very are many relevant and good text books. Ask me for other suggestions. Here are three I know particularly well. Click here for link to books.