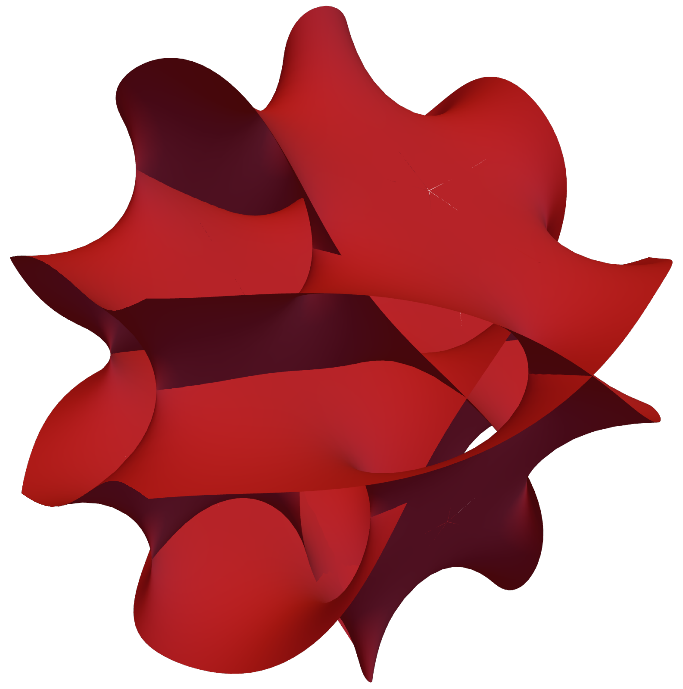
Geometry in Feynman Integrals

Of Calabi—Yaus, Curves, and more…



Sebastian Pögel, University of Mainz
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`2211.04292` (PRL 130 (2023) 10, 101601)
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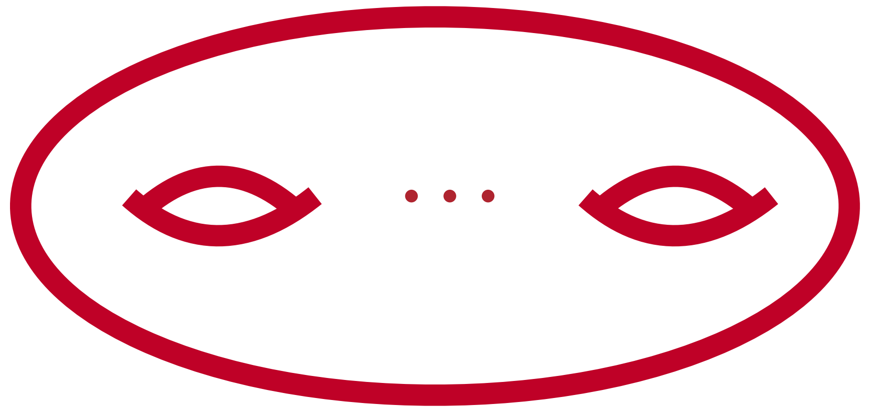
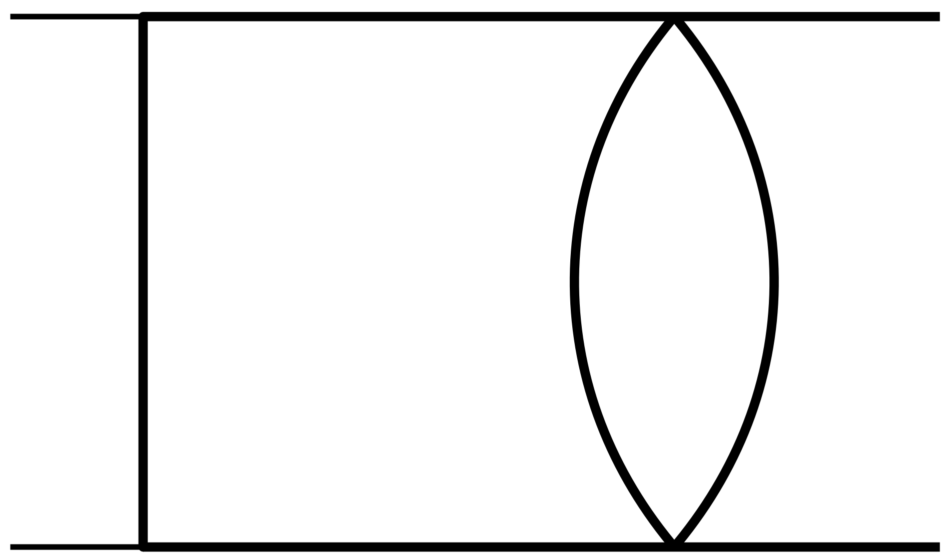
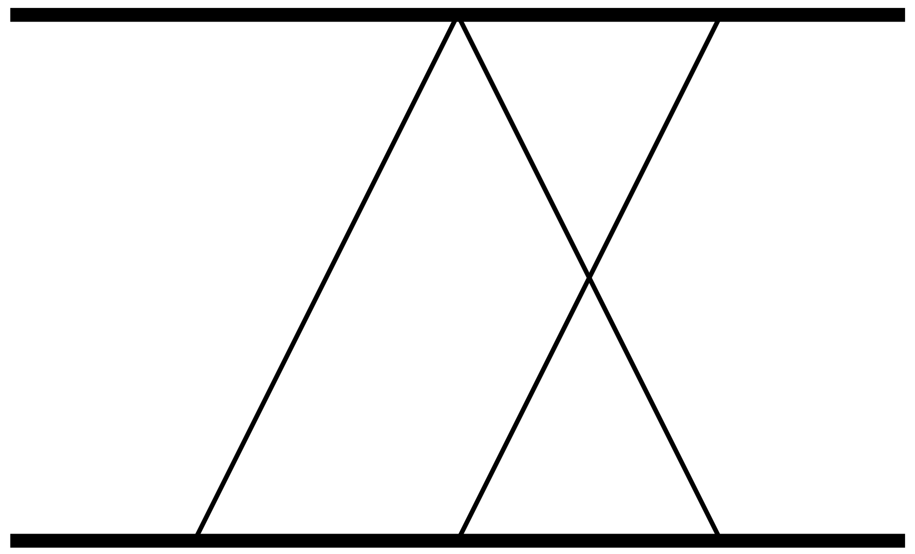


Feynman Integrals

QCD



Gravity



Theory independent building blocks capturing most loop-level information

Boil em,
mash em,
stick em in an amplitude

Integrals associated to geometries
Determines suitable function space

Sphere → **MPLs**

Elliptic curve → **Elliptic Integrals, modular forms, EMPLs**

. . .

What is there beyond elliptics?

Feynman Integral Evaluation
A How To

1. List all integrals appearing in your problem
2. Use identities (integration-by-parts, symmetries, etc.) to obtain basis integrals $\vec{I}$ (“Master Integrals”)
3. Write down a differential equation $\mathrm{d}\vec{I} = A\vec{I}$
4. Solve differential equation (???)
5. Obtain expressions for Laurent series in ε of $\vec{I}$

Trick: Choose Master Integrals such that

$$dI = \varepsilon A I$$ [Henn '13]

Find basis and variables, such that

- A independent of ε
- A consisting of functions we "understand well"

Analytic understanding
and/or
fast numerical evaluation

Given boundary value I_0

Can then trivially evaluated at any order in ε: $I = \mathbb{P}\exp\left(\varepsilon \int A\right) I_0$

Geometry associated to integral determines space of forms in A

Fantastic Geometries

and where to find them

How do we identify geometry of integrals?

Graph Polynomial

$$I \sim \int \prod \mathrm{d}\alpha_i \alpha_i^{\nu_i - 1} \frac{U^{\nu-(l+1)D/2}}{F^{\nu-lD/2}}$$

U/F define projective variety

Extracting geometry is hard!

Maximal Cuts

$$\mathrm{MaxCut}(I) \sim \int \frac{\prod \mathrm{d}\ell_j^D}{\prod_i D_i} /.\{D_i \to \delta(D_i)\}$$

Homogeneous solution to differential equation of full integral

Skeletonized version of integral

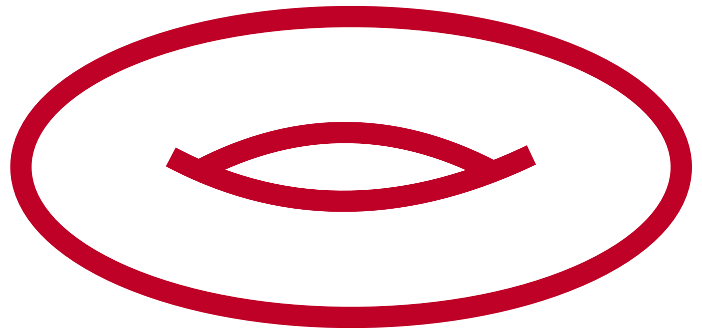
Much simpler to extract geometry

How do we generalize from the elliptic integrals?

Elliptic Feynman integrals are phenomenological state of the art

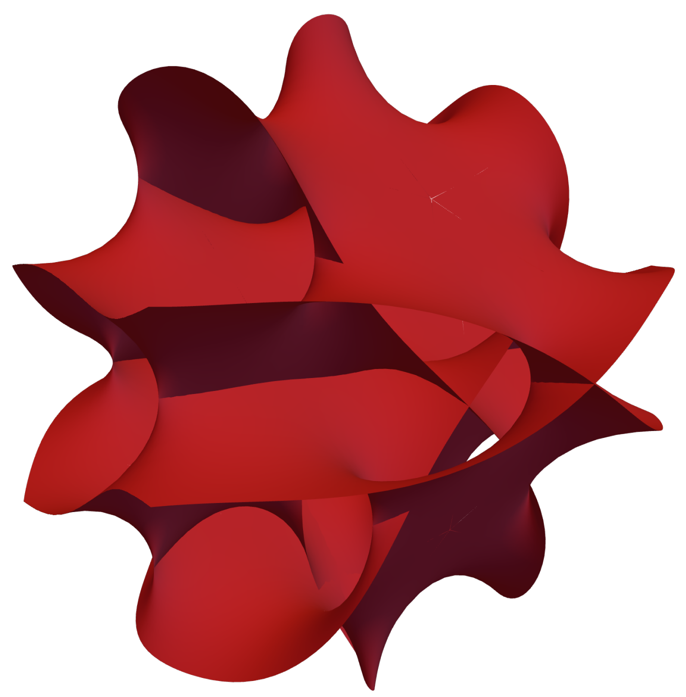
What else is there?

Elliptic Curve



Dimension
1

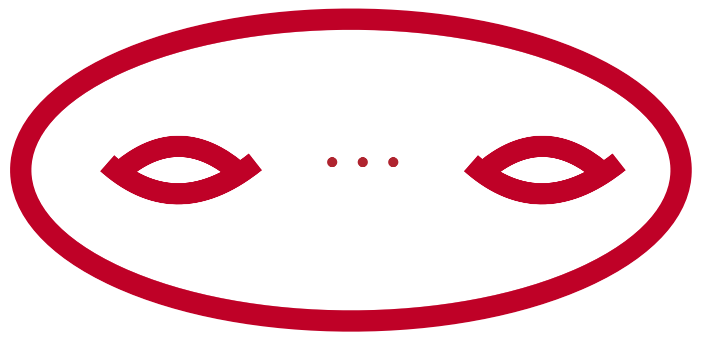
Genus
1



Higher dimensional varieties

e.g. Calabi—Yaus

Curves of higher genus



e.g. Hyperelliptic curves

Calabi—Yaus

Calabi—Yaus in Feynman Integrals

Compute maximal cut
and takes as many residues
as possible

$$\mathrm{MaxCut}\, I \sim \int \frac{\mathrm{d}\alpha_1 \ldots \mathrm{d}\alpha_n}{\sqrt{P(\alpha_1, \ldots, \alpha_n)}}$$

Calabi—Yau n-fold

Hypersurface in weighted projective space

[Bourjaily, McLeod, Vergu, Volk, von Hippel, Wilhelm, '20]

$$[1 : \alpha_1 : \ldots : \alpha_n : y] \in \mathbb{WP}^{1,1,\ldots,1,(n+1)}$$

$$y^2 = P(\alpha_1, \ldots, \alpha_n) \quad \text{with} \quad \deg P = 2(n+1)$$

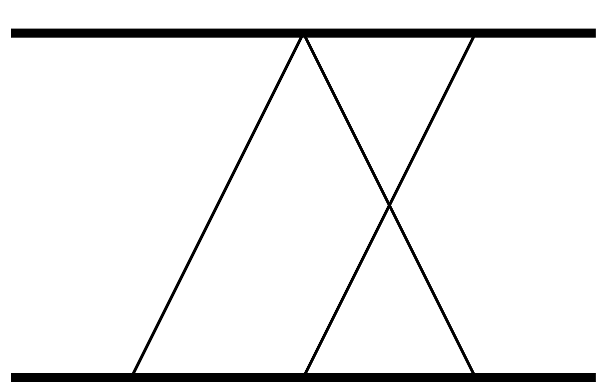
Codimension 1 = Dimension n

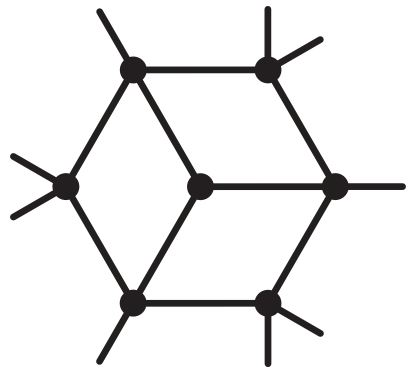
$\mathrm{MaxCut}(I)$ **is a so-called period of the Calabi—Yau**

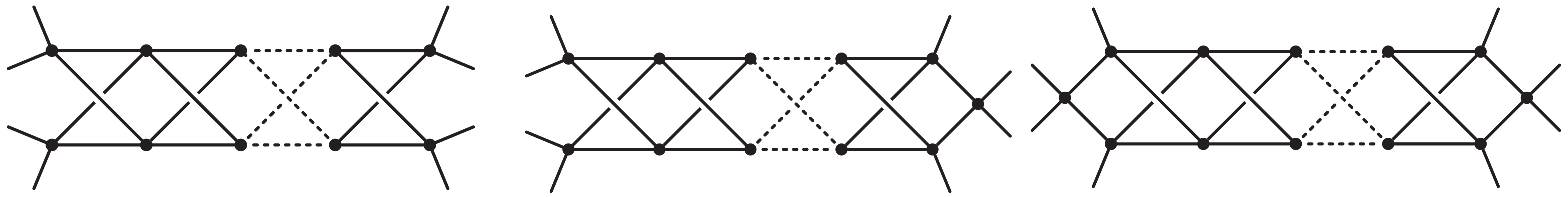
Calabi–Yaus: “A (bounded) bestiary”

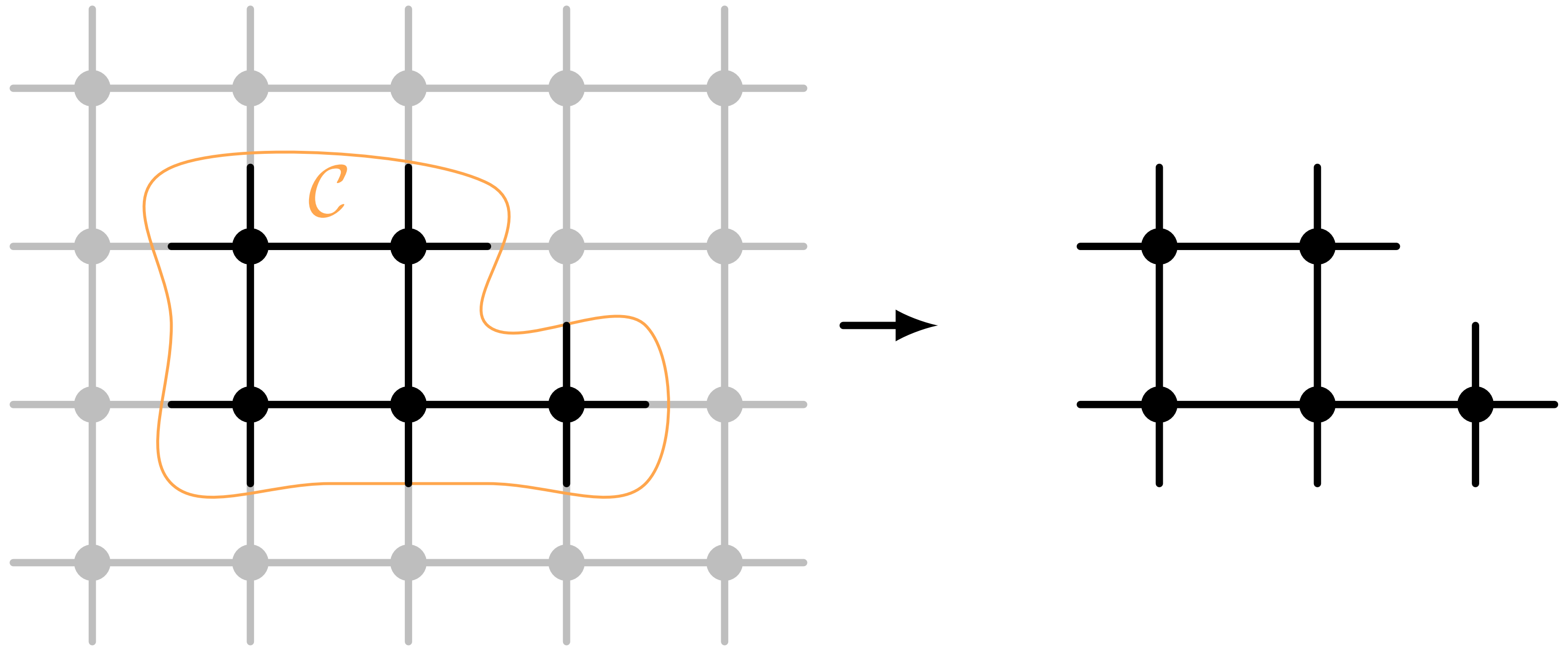
[Bourjaily, McLeod, von Hippel, Wilhelm, ’19]

[Duhr, Klemm, Loebbert, Nega, Porkert, Tancredi, ’22, ’23]





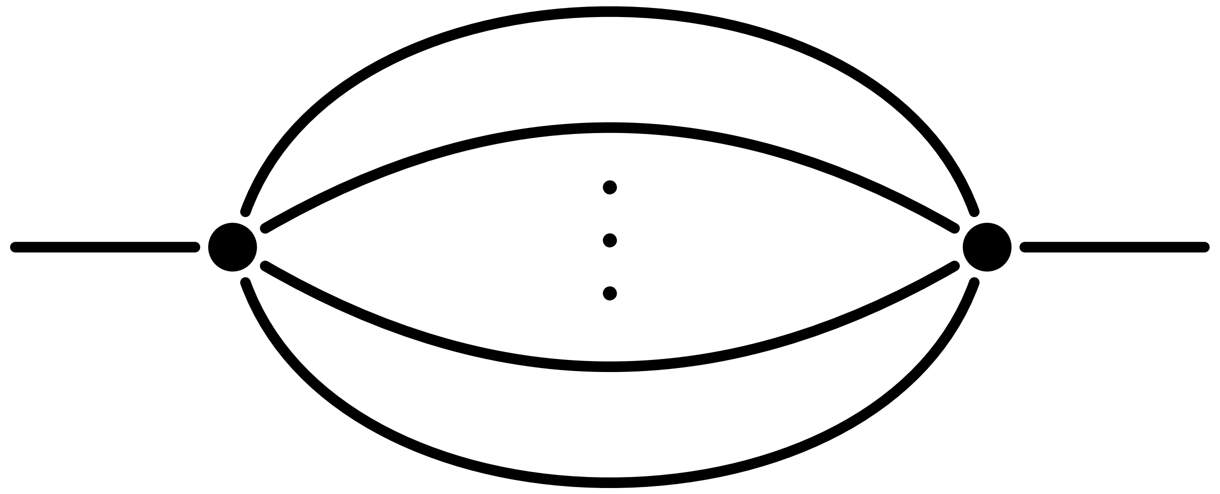




and more...

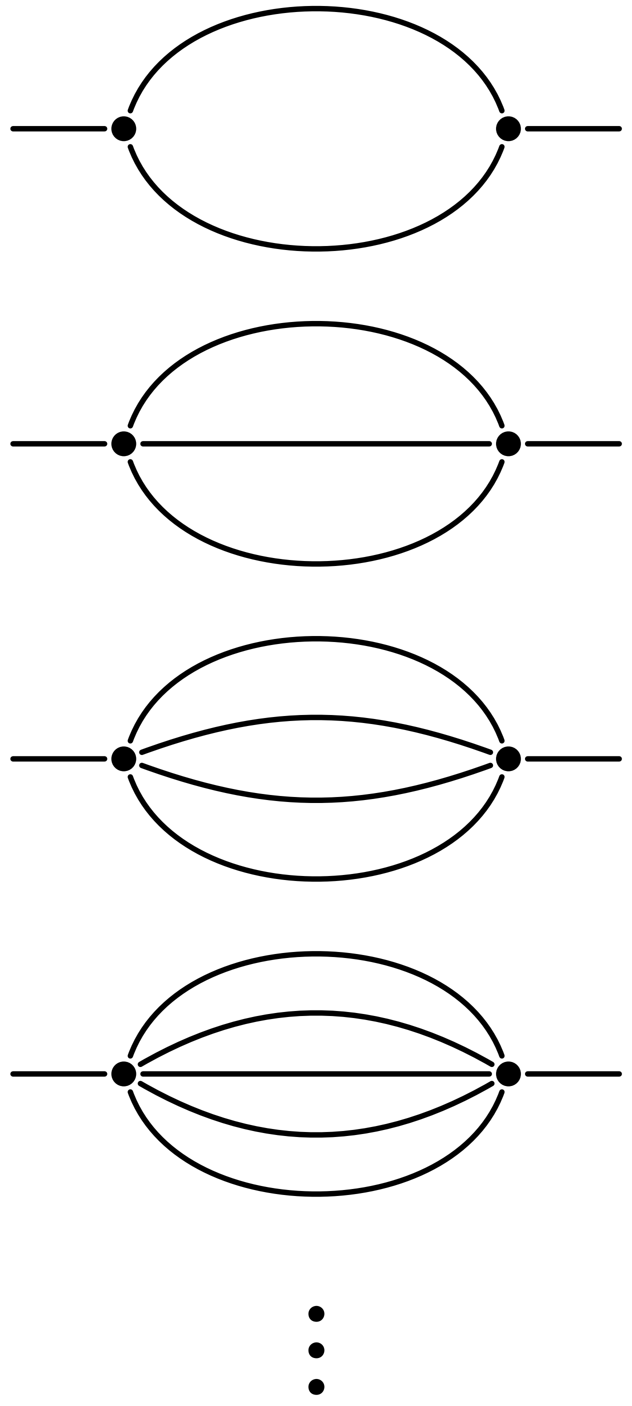
Simplest Example:
Banana Integrals

in $D = 2 - 2\varepsilon$



Bananas: A Calabi–Yau Prototype

Calabi–Yau…



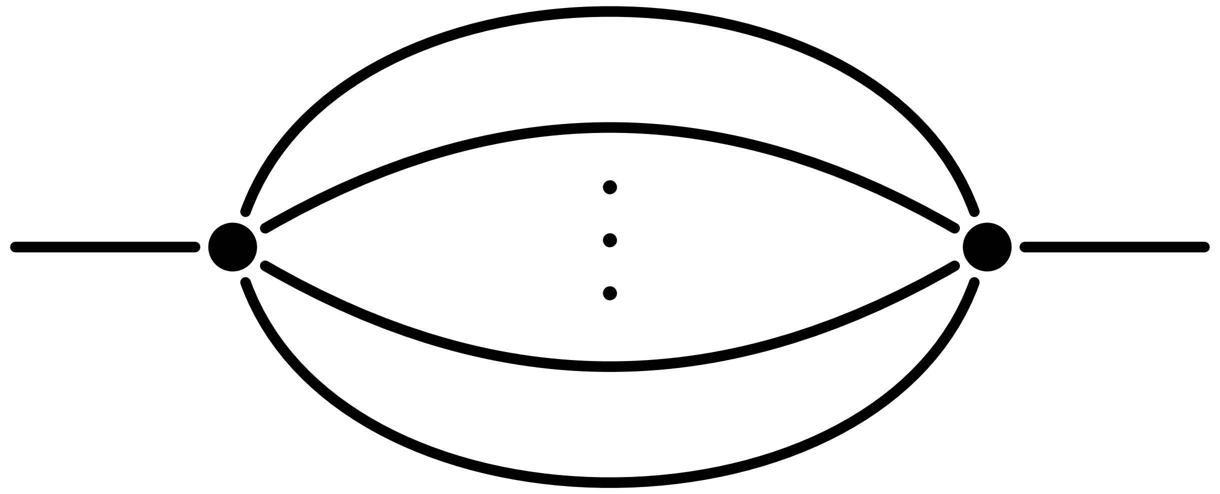
0-fold

1-fold = elliptic curve
"Sunrise"

2-fold = K3 surface

3-fold

⋮



ℓ-loop Banana integral

$$\hat{=}$$

$(\ell - 1)$-fold Calabi–Yau manifold

ℓ-loop banana program [Bönisch, Duhr, Klemm, Nega, Safari; Kreimer; Forum, von Hippel]

Simplification: Equal-mass → single scale

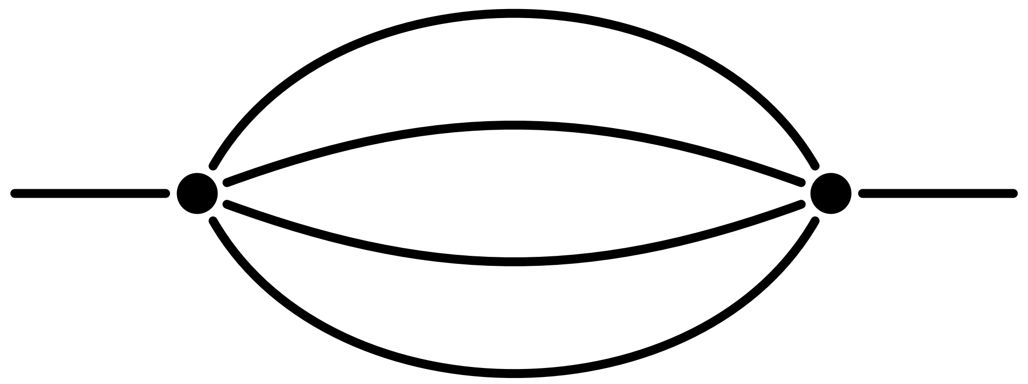
Kinematic variable

$$x = \frac{p^2}{m^2} \qquad y = -\frac{m^2}{p^2}$$

“Trivial” Calabi–Yaus

***Essentially* elliptic**

Three-loop Banana



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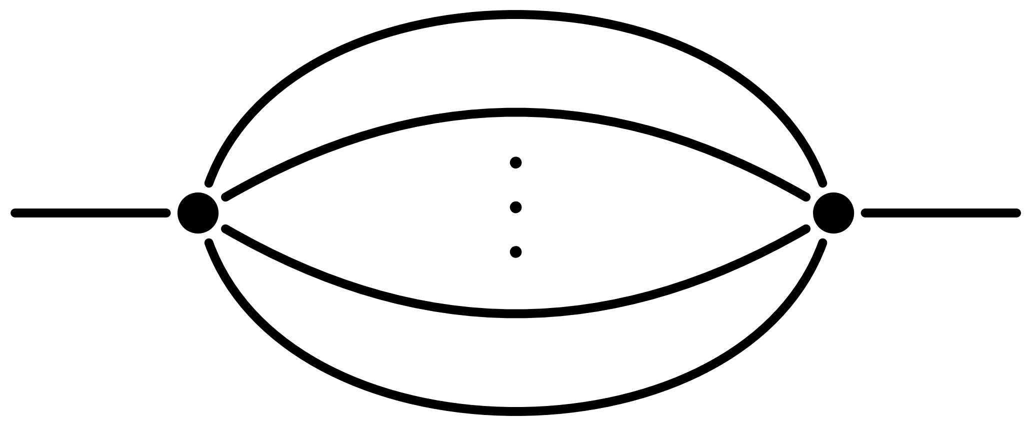
“Non-trivial” Calabi–Yaus

Non-elliptic

(≥Four)-loop Banana



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The Three-Loop Banana Integral

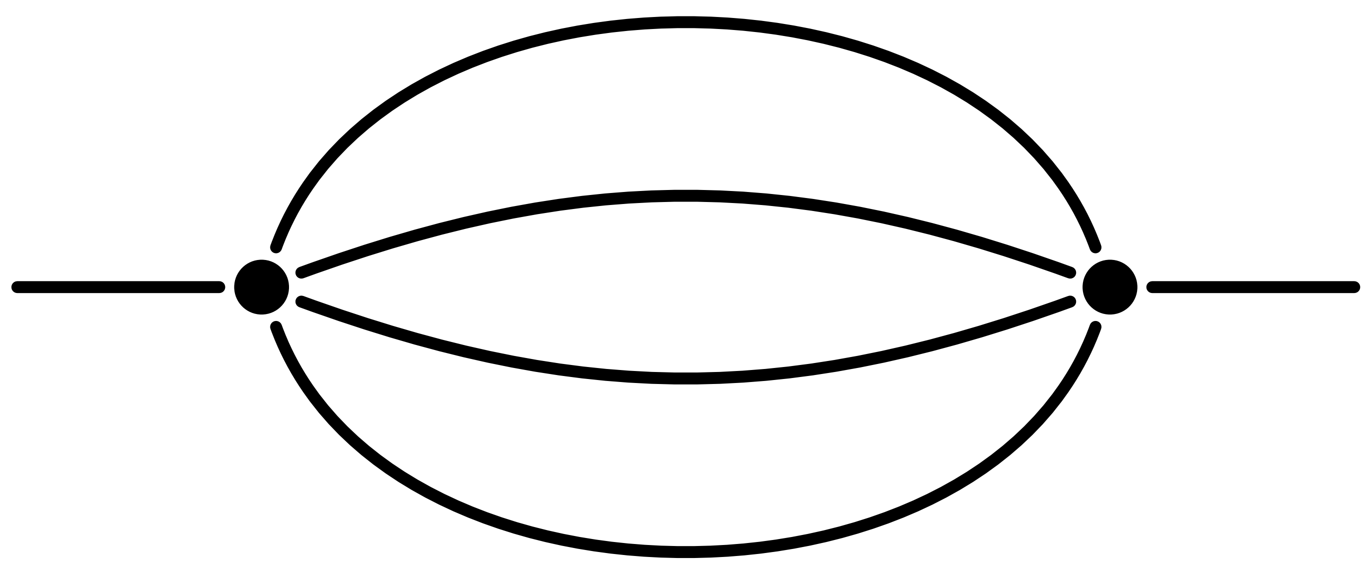
Simplest example of Feynman integral beyond elliptic:

Calabi–Yau 2-fold

Equal-mass case: closely connected to sunrise integral

Extensively studied in the past:

- Leading term in ε [Bloch, Kerr, Vanhove, 14']
- ε-factorized form [Primo, Tancredi,17']
- Master integrals in $d=2$ in terms of eMPLs $\tilde{\Gamma}$ [Broedel, Duhr, Dulat, Marzucca, Penante, 19']
- DEQ with meromorphic modular forms [Broedel, Duhr, Matthes, 21']
- ℓ-loop banana program [Bönisch, Duhr, Klemm, Nega, Safari; Kreimer; Forum, von Hippel]



Singularities:

$$x = \frac{p^2}{m^2} = 0,4,16,\infty$$

Picard-Fuchs Differential Operator

Annihilates $\mathrm{MaxCut}(I)$ (periods of Calabi–Yau)

Defines geometry

3-loop banana in $D = 2$:

$$\mathcal{L}_3^{(0)} = \frac{d^3}{dx^3} + \left[\frac{3}{x} + \frac{3}{2(x-4)} + \frac{3}{2(x-16)}\right]\frac{d^2}{dx^2} + \frac{7x^2-68x+64}{x^2(x-4)(x-16)}\frac{d}{dx} + \frac{1}{x^2(x-16)}.$$

with solutions $\mathcal{L}_3^{(0)}\,\omega_i = 0$ where $\omega_i = \mathrm{MaxCut}(I_{1111})|_{\gamma_i}$ on three independent contours γ_i

$\mathcal{L}_3^{(0)}$ is a symmetric square

[Verrill, 96'; Joyce, 72']

There exists an operator

$$\mathcal{L}_2^{(0)} = \frac{d^2}{dx^2} + \left[\frac{1}{x} + \frac{1}{2(x-4)} + \frac{1}{2(x-16)}\right]\frac{d}{dx} + \frac{(x-8)}{4x(x-4)(x-16)}$$

Sunrise in disguise

with solutions ψ_1, ψ_2, $\mathcal{L}_2^{(0)}\,\psi_i = 0$ such that

$$\omega_i \in \langle \psi_1^2,\, \psi_1\psi_2,\, \psi_2^2 \rangle$$

ε-Factorization: Sunrise

Make the ansatz

$$\begin{aligned} I_1 &= \varepsilon^2 I_{110}, \\ I_2 &= \varepsilon^2 \frac{\pi}{\psi_1} I_{111}, \\ I_3 &= \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_2 + F_{32} I_2, \end{aligned}$$

$$\tau = \frac{\psi_2}{\psi_1}$$

Periods of elliptic curve

$$\mathrm{d}I = \varepsilon \begin{pmatrix} 0 & 0 & 0 \\ 0 & \eta_2 & 1 \\ \eta_3 & \eta_4 & \eta_2 \end{pmatrix} I$$

A organised by modular weight

η_k : Modular forms of $\Gamma_1(6)$ of weight k, independent of ε

✓ A independent of ε

✓ A consists of $\underbrace{\text{modular forms}}_{\text{“well understood”}}$

ε-Factorization: Three-loop Ansatz

Make the ansatz

$$I_1 = \varepsilon^3 I_{1110},$$
$$I_2 = \varepsilon^3 \frac{1}{\omega} I_{1111},$$
$$I_3 = \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_2 + F_{32} I_2,$$
$$I_4 = \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_3 + F_{42} I_2 + F_{43} I_3.$$

No assumptions for ω and τ required

$$\downarrow$$

$$dI = \tilde{A} I$$

Requiring $\tilde{A} = \varepsilon A$ $\rightarrow$ constraints on ω, J, F_{32}, F_{42}, F_{43}

J: $\frac{\mathrm{d}x}{\mathrm{d}\tau}$

Eliminate Non-ε-Factorized Pieces

Already ε-factorized

$$dI = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \tilde{A}_{2,2} & 1 & 0 \\ 0 & \tilde{A}_{3,2} & \tilde{A}_{3,3} & 1 \\ \tilde{A}_{4,1} & \tilde{A}_{4,2} & \tilde{A}_{4,3} & \tilde{A}_{4,4} \end{pmatrix} I$$

$\tilde{A}_{4,k}$ contains term ε^{-4+k} through ε

Five variables, six constraints

$\omega, J, F_{32}, F_{42}, F_{43}$

$\rightarrow$ **One non-trivial constraint!**

Satisfied for $\omega = (x\psi_1^{\text{sun}})^2 \quad \tau = \dfrac{\psi_2^{\text{sun}}}{\psi_1^{\text{sun}}}$

Periods of elliptic curve

$$\frac{\mathrm{d}I}{\mathrm{d}\tau} = (2\pi i)\varepsilon \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -f_{2,a} - f_{2,b} & 1 & 0 \\ 0 & f_{4,b} & -f_{2,a} + 2f_{2,b} & 1 \\ f_{4,a} & f_6 & f_{4,b} & -f_{2,a} - f_{2,b} \end{pmatrix} I$$

Constraints allow symmetry

Alphabet: $\mathcal{A} = \{1, f_{2,a}, f_{2,b}, f_{4,a}, f_{4,b}, f_6\}$.

Function space of Alphabet

Meromorphic modular forms + Special function F_2

Iterated integral of meromorphic modular form of weight 6

$F_2 = I(1, g_6; \tau) \quad g_6 = \frac{x\,(x-8)\,(x+8)^3}{864\,(4-x)^{\frac{3}{2}}\,(16-x)^{\frac{3}{2}}} \left(\frac{\psi_1}{\pi}\right)^6$

$$I_2 = \varepsilon^3 \left(\frac{4}{3}\zeta_3 + I(1,1,f_{4,a};\tau)\right) + \mathcal{O}(\epsilon^4)$$

Obtained expressions for all masters up to ε^6

Numerics via q-expansion

“Trivial” Calabi–Yau Summary

ε**-factorized form:** Ansatz, then solve constraints algorithmically

Symmetric square: Three-loop banana integral related to elliptic curve

Function space: Meromorphic modular forms, plus iterated integrals thereof (F_2)

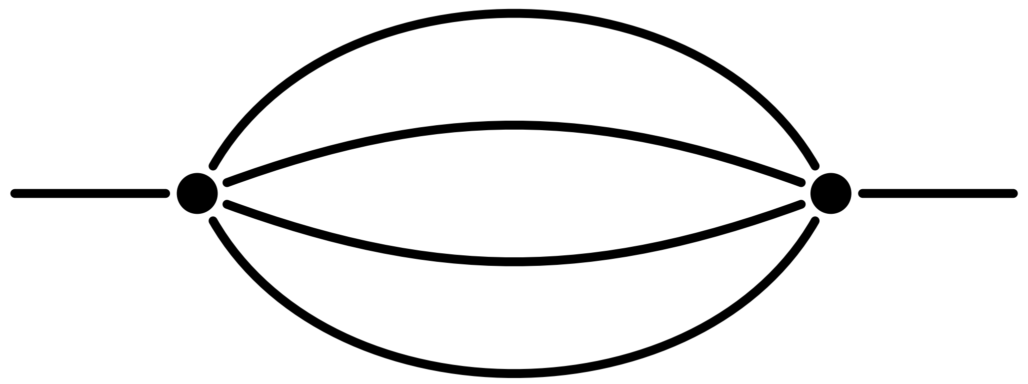
Expectation: This generalizes beyond the banana!

Single scale +
2-fold Calabi–Yau integral +
degree 3 Picard–Fuchs operator
$\longrightarrow$
“trivial” Calabi–Yau
i.e. symmetric power
of elliptic curve
$\longrightarrow$
ε-factorised DEQ from Ansatz
Function space
essentially elliptic

“Trivial” Calabi–Yaus

***Essentially* elliptic**

Three-loop Banana



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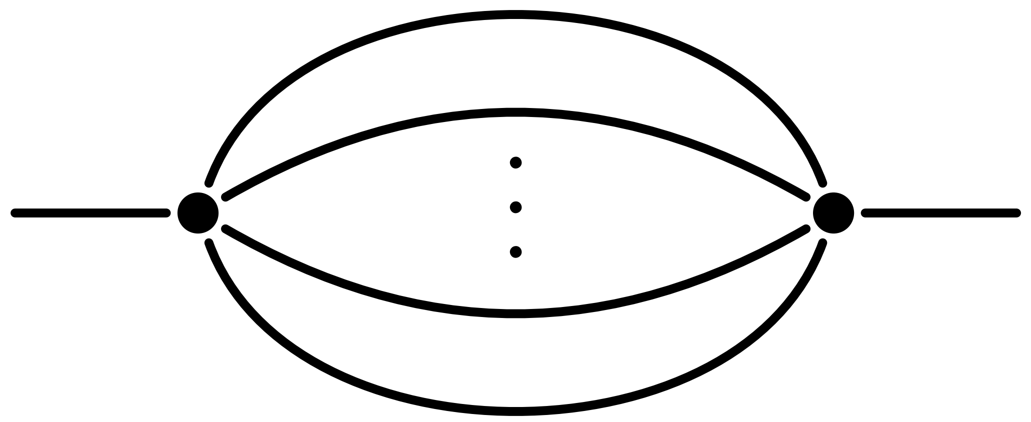
“Non-trivial” Calabi–Yaus

Non-elliptic

(≥Four)-loop Banana



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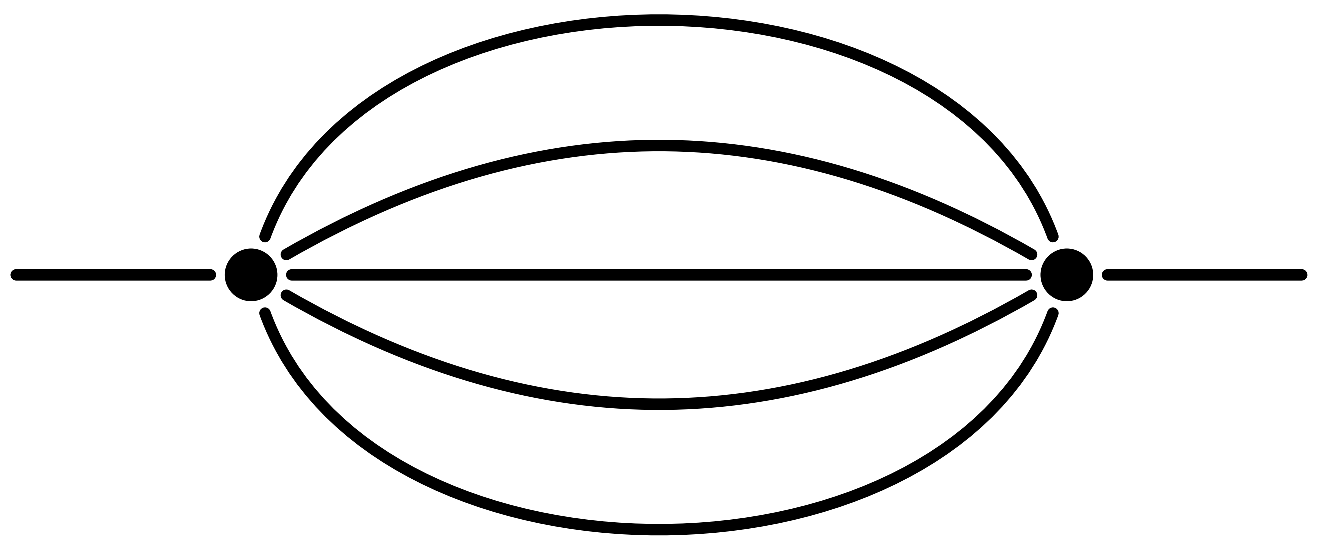
The Four-Loop Banana Integral

First banana integral with “non-trivial” Calabi–Yau:

Not related to elliptic curves



Integral already studied in the past

ℓ-loop banana program [Bönisch, Duhr, Klemm, Nega, Safari; Kreimer; Forum, von Hippel]

Singularities:

$y = -\frac{m^2}{p^2} = 0, -1, -\frac{1}{9}, -\frac{1}{25}, \infty$

Algebraic Variety from graph polynomial
Hypersurface in $\mathbb{CP}^4$ with

$$1/y = (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)\left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} + \frac{1}{\alpha_4} + \frac{1}{\alpha_5}\right)$$

Calabi–Yau very well known

Studied in [Hulek, Verrill, 05’; …]
Known as `AESZ34` [Almkvist, van Enckevort, van Straten, Zudilin]

ε-Factorization: Four-loop Ansatz

Guess the pattern?

$$
\begin{aligned}
I_1 &= \varepsilon^4 I_{11110}, \\
I_2 &= \varepsilon^4 \frac{1}{\omega} I_{11111}, \\
I_3 &= \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_2 + F_{32} I_2, \\
I_4 &= \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_3 + F_{42} I_2 + F_{43} I_3 \\
I_5 &= \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_4 + F_{52} I_2 + F_{53} I_3 + F_{54} I_4.
\end{aligned}
$$

?

$dI = \varepsilon A I$ leads to inconsistent constraints!
$\rightarrow$ No solution!

ε-Factorization: Four-loop Ansatz (fixed)

Modify ansatz!

$$I_1 = \varepsilon^4 I_{11110},$$

$$I_2 = \varepsilon^4 \frac{1}{\omega} I_{11111},$$

$$I_3 = \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_2 + F_{32} I_2,$$

$$I_4 = \frac{1}{\varepsilon}\boxed{\frac{1}{K_1}}\frac{\mathrm{d}}{\mathrm{d}\tau} I_3 + F_{42} I_2 + F_{43} I_3$$

$$I_5 = \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_4 + F_{52} I_2 + F_{53} I_3 + F_{54} I_4.$$

ℓ-loop Banana Integrals define special Calabi–Yau manifolds

Picard–Fuchs operators are **Calabi–Yau operators**

[Almkvist, van Enckevort, van Straten, Zudilin, 05']
[M. Bogner, 13']

K_1 is a Y-invariant of Calabi–Yau operator

Start appearing at 3-fold

$dI = \varepsilon A I$ leads to consistent constraints!

No prior knowledge of K_1 required!
Fixed by constraints (up to rescaling)

What is the function space of “non-trivial” Calabi–Yaus to solve constraints?

Currently unknown

But for fast numerics, imitate elliptics:

q-expansion

Expansion point

$y = -m^2/p^2 = 0$ **(MUM-point)**

Frobenius basis:

$\omega_1, \omega_2, \omega_3, \omega_4$

Expansion coordinate:

$q = \exp(2\pi i \tau)$, $\tau = \omega_2/\omega_1$

Canonical variables for Calabi–Yau operators

Generalization of
τ (ratio of periods)
q (nome)
from elliptic case $\ell = 2$

Four-Loop solutions

$q(y) = \exp\left(2\pi i \omega_2/\omega_1\right)$	$y - 8y^2 + 92y^3 - 1288y^4 + 20398y^5 + \mathcal{O}(y^6)$
$\omega = \omega_1$	$q + 3q^2 + q^3 + 23q^4 - 101q^5 + \mathcal{O}(q^6)$
$K_1 = \mathrm{d}^2/\mathrm{d}\tau^2(\omega_3/\omega_1)$	$1 - q + 17q^2 - 253q^3 + 3345q^4 - 43751q^5 + \mathcal{O}(q^6)$
J	$q + 16q^2 + 108q^3 + 672q^4 + 2570q^5 + \mathcal{O}(q^6)$
F_{32}	$c_{32} + 8q - 32q^2 + 512q^3 - 5872q^4 + 70008q^5 + \mathcal{O}(q^6)$
F_{42}	$c_{42} + 8q - 240q^2 + 4816q^3 - 90448q^4 + 1444008q^5$ $+ c_{32}(-9q + 176q^2 - 2956q^3 + 44568q^4 - 611106q^5)$ $+ c_{32}^2(q - 16q^2 + 220q^3 - 2600q^4 + 30018q^5)$ $+ \mathcal{O}(q^6)$
$\vdots$	

Predictable from just Picard–Fuchs operator (rows $q(y)$ to J)

Need to solve constraints (rows F_{32}, F_{42}, …)

Remaining freedom c_{32}, c_{42}, etc.
$\rightarrow$ can impose symmetry on A

Fast numerical evaluation
(Within convergence radius)

Five-, Six-, All-Loop Ansatz

$$I_1 = \varepsilon^\ell I_{1\ldots10},$$
$$I_2 = \varepsilon^\ell \frac{1}{\omega} I_{1\ldots1},$$
$$I_3 = \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_2 + F_{32} I_2,$$
$$I_4 = \frac{1}{\varepsilon}\frac{1}{K_1}\frac{\mathrm{d}}{\mathrm{d}\tau} I_3 + F_{42} I_2 + F_{43} I_3$$
$$I_5 = \frac{1}{\varepsilon}\frac{1}{K_2}\frac{\mathrm{d}}{\mathrm{d}\tau} I_4 + F_{52} I_2 + F_{53} I_3 + F_{54} I_4$$
$$\vdots$$
$$I_{\ell-1} = \frac{1}{\varepsilon}\frac{1}{K_2}\frac{\mathrm{d}}{\mathrm{d}\tau} I_{\ell-2} + \sum_{i=2}^{\ell-2} F_{\ell-1,i} I_i$$
$$I_{\ell} = \frac{1}{\varepsilon}\frac{1}{K_1}\frac{\mathrm{d}}{\mathrm{d}\tau} I_{\ell-1} + \sum_{i=2}^{\ell-1} F_{\ell,i} I_i$$
$$I_{\ell+1} = \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_{\ell} + \sum_{i=2}^{\ell} F_{\ell+1,i} I_i$$

Checked up to seven loops

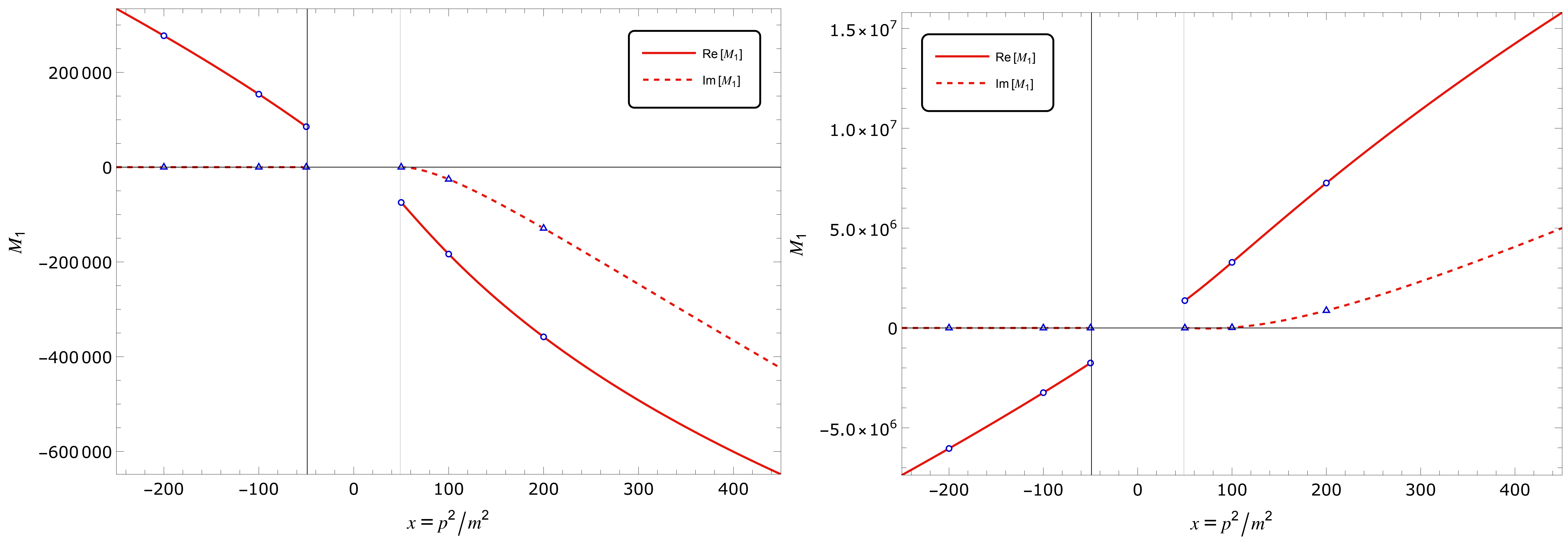
Ansatz with K_i being Y-invariants leads to consistent constraints

Checked up to six loops

Analytic expressions for Masters in terms of iterated integrals

$$I_2 = \left[I(1,K_1,K_2,K_1,1,A_{71};\tau) + \text{boundary}\right]\varepsilon^7 + \mathcal{O}(\varepsilon^8) \quad \text{etc.}$$

Numeric evaluation using q-expansion: agrees with `SecDec`



"Non-Trivial" Calabi–Yau Summary

ε-factorized form: Ansatz with information from Calabi–Yau operators
$\rightarrow$ Solve constraints algorithmically

Function space: currently unknown

Numerics: can obtain fast converging q-expansion

Expectation: Generalizes to other Calabi–Yau integrals

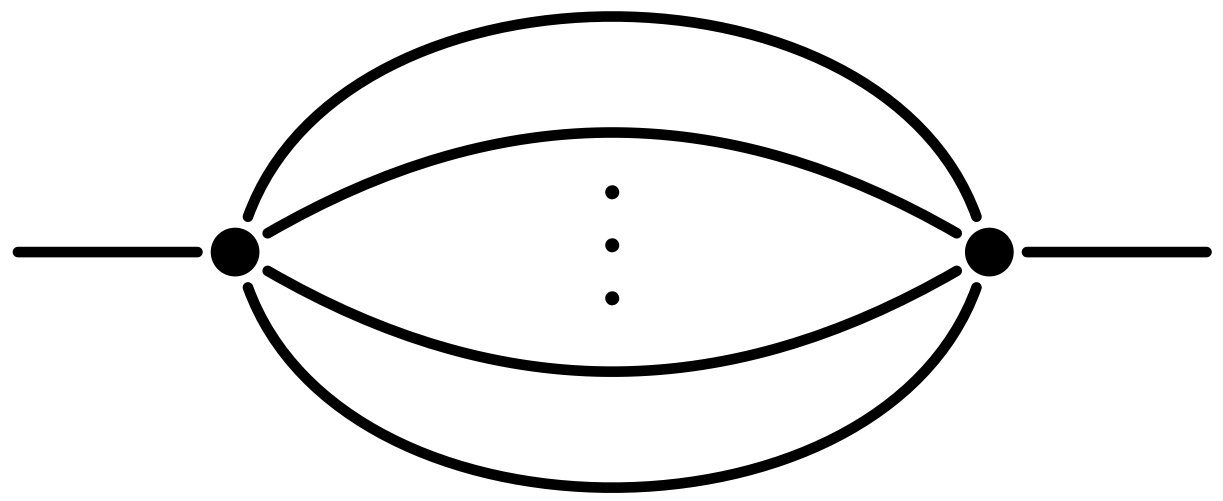
Single scale +
n-fold Calabi–Yau integral +
degree (n+1) Picard–Fuchs operator
$\rightarrow$
Calabi–Yau operator
"non-trivial" Calabi–Yau
$\rightarrow$
ε-factorised DEQ from Ansatz
Solve via series expansion

Conclusions

Banana integrals: Simplest example of Calabi–Yau integrals

Simplification: Equal-mass = single scale

Single scale integral
n-fold Calabi–Yau,
degree (n+1) Picard–Fuchs operator



Ansatzing allows to find ε-factorised form algorithmically

Use information from theory of Calabi–Yau operators

Calabi-Yau 2-fold

Picard–Fuchs is symmetric square of elliptic curve

Modular forms

Calabi-Yau ($\geq$3)-fold

Not relatable to elliptics
Function space unknown

q-expansion

Conclusions

- Beyond polylogs, control of geometry is crucial for evaluation of Feynman Integrals
- Integrals beyond elliptic ones are relevant to collider phenomenology today!
- Identification of “simplest” geometry not trivial
- There exists a wealth of mathematical knowledge for geometries associated that can be applied to Feynman integrals (algebraic curves studied since 19th century)