Exercise 1: Interactions of relativistic protons

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1. The relation expressing the energy threshold condition for the two-particle interaction $a+b \to c+d$ is:
$$E_a E_b - \vec{p}_a \vec{p}_b c^2 \geq \Delta m\, c^4 (m_a + m_b + \Delta m/2) + m_a m_b\, c^4 \tag{1}$$
where $\Delta m = m_c + m_d - (m_a + m_b)$, and $m_i, E_i, \vec{p}_i$ are the rest mass, total energy, and momentum, respectively, of the particle i.
   (a) Calculate the energy threshold for the interaction $p + \gamma \to e^- + e^+$.
   (b) Calculate the energy threshold for the interaction ${}^A_Z X + \gamma \to {}^A_Z X + \pi^0$.
   (c) An ultra-high-energy cosmic ray (UHECR) proton and an iron nucleus are traveling through the Universe and are interacting with photons from the cosmic microwave background (CMB) via the photopair process. Using the expression you derived above, calculate the minimum energy of the UHECR proton/iron nucleus for a head-on collision with the CMB photons. Comment on your results. [$m_e c^2 \approx 0.5$ MeV, $m_p c^2 \approx m_n c^2 \approx 1$ GeV, $m_{\pi^0} c^2 \approx 135$ MeV, $\epsilon_{CMB} \approx 6 \times 10^{-4}$ eV].

2. The mean free path for energy losses of an UHECR proton with Lorentz factor γ_p due to $p\gamma$ interactions with CMB photons is given by:
$$\frac{1}{\bar{\ell}(\gamma_p)} = \frac{1}{2\gamma_p^2} \int_{\bar{\epsilon}_{th}}^{\infty} \mathrm{d}\epsilon\, \bar{\epsilon}\, \hat{\sigma}(\bar{\epsilon}) \int_{\bar{\epsilon}/(2\gamma_p)}^{\infty} \mathrm{d}\epsilon' \frac{n_{CMB}(\epsilon')}{\epsilon'^2} \tag{2}$$
where $\hat{\sigma} = 70\, H(\bar{\epsilon} - \bar{\epsilon}_{th})\, \mu$b is an approximate expression for the effective cross section, $\bar{\epsilon}_{th} \simeq 400$ and $H(x)$ is the Heaviside function. All energies in the expression above are normalized to $m_e c^2$.
   (a) The CMB photon number density can be approximated as a power law with a sharp cutoff: $n_{CMB}(\epsilon) = n_0 \epsilon$, for $\epsilon \leq \epsilon_{\rm CMB}$, where $\epsilon_{CMB} \simeq 6 \times 10^{-4}$ eV. Calculate the normalization n_0 if you know that the local CMB energy density is $u_{CMB} \approx 0.25$ eV cm^{-3}.

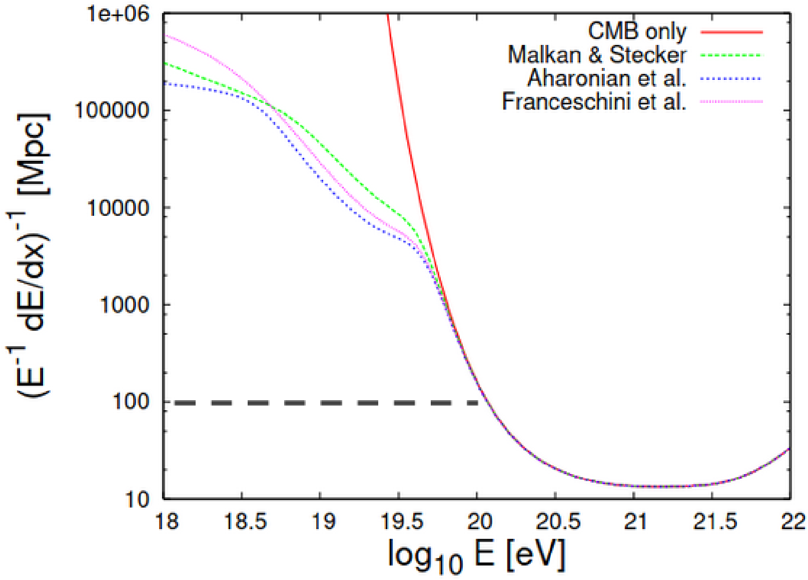

Figure 1: Mean free path for $p\gamma$ interactions of UHECR protons with ambient photon fields.

(b) Make a plot of $\bar{\ell}(E_p)$ versus E_p (*Hint:* think about the upper limits of integration). How does the approximate solution found in the previous question compare with the accurate result shown in Fig. 1? Comment on your results.