Perturbative TQFTs

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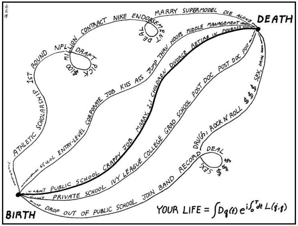
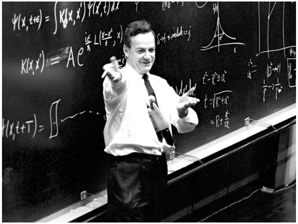
Motivation: Feynman path integral

In Quantum Mechanics, we have the *Feynman path integral*

$$\Psi(x_0, x_1) = \int_{\gamma:\ \substack{\gamma(0)=x_0 \\ \gamma(1)=x_1}} e^{\frac{i}{\hbar}S[\gamma]} D\gamma$$



The Path Integral Formulation of Your Life



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2. an *action functional* $S_M \colon F_M \to \mathbb{R}$ given by

$$S_M[\phi] = \int_M \mathcal{L}[\phi, \partial\phi, \dots]$$

where $\mathcal{L} \colon F_M \to \mathrm{Dens}(M)$ is the Lagrangian density.

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Classical physics are understood by solving the Euler-Lagrange equations, $EL = \{\phi \in F_M, \delta S[\phi] = 0\}$.

Motivation: Functional integrals

In Quantum Field Theory, we are interested in the partition function

$$Z_M = \int_{\phi \in F_M} e^{\frac{i}{\hbar} S_M[\phi]} D\phi$$

and expectation values of observables

$$\langle O \rangle = \frac{1}{Z} \int_{\phi \in F_M} e^{\frac{i}{\hbar} S_M[\phi]} O(\phi) D\phi$$

Mathematical problem: Spaces F_M usually don't have sensible integration theories.

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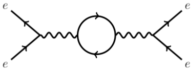
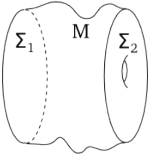


(a) Approach I



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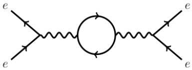

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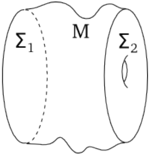

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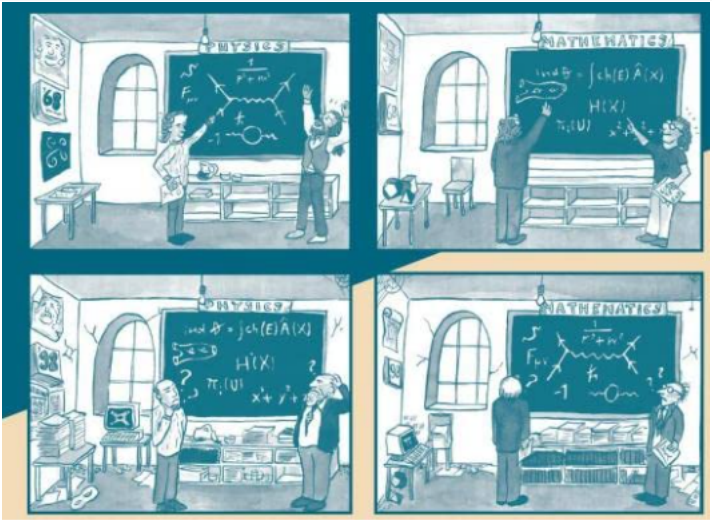
Comic of the situation



Chern-Simons theory: the drosophilia melanogaster

Chern-Simons theory is the 3d gauge theory with

$$F_M = \Omega^1(M, \mathfrak{g}), \quad S_M = \int_M \frac{1}{2}\langle A, dA\rangle + \frac{1}{6}\langle A, [A, A]\rangle$$

where $\mathfrak{g}, \langle\cdot,\cdot\rangle$ is a quadratic Lie algebra. The partition function

$$Z_M = \int_{F_M} e^{\frac{ik}{2\pi} S_{CS}(A)} DA$$

and Wilson loop observables for $\gamma\colon S^1 \to M$

$$\langle O_\gamma \rangle_M = \frac{1}{Z}\int_{F_M} e^{\frac{ik}{2\pi} S_{CS}[A]} \left(\mathrm{tr}_R \operatorname{Pexp} \int_\gamma A\right)$$

were studied by Witten in 1989.

Different Chern-Simons invariants

- Witten: For $M = S^3$, $\langle O_\gamma \rangle = J_{K(\gamma)}$ is the Jones polynomial of the knot K.
- Witten: Z_M are topological invariants of 3-manifolds.[1]
- Reshitikhin-Turaev described a *functorial* construction of similar invariants $Z^k_{RT,M}$ based on quantum groups (Approach II).
- Fröhlich-King, Bar-Natan, Kontsevich, Axelrod-Singer, & many others studied the perturbative quantization $Z^{\mathrm{pert}}_{CS,M}$ of Chern-Simons (Approach I).

[1] Witten considered framed 3-manifolds due to a 1-loop anomaly

Research Goals

Asymptotic Expansion Conjecture (AEC) [Witten, Andersen, ...]

Prove that as $k \to \infty$, $Z^k_{RT,M} \sim Z^{\text{pert}}_{CS,M}$.

Wide Open Question (WOQ)

What is the analogous result for the Reshetikhin-Turaev functorial field theory?

Goal

Develop new (general) tools to combine perturbative QFT and functorial QFT to prove the WOQ and use WOQ to prove AEC.

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Figure: Illustration of the gluing axiom

Cobordism category

Reformulate in categorical language: Consider category **Cob** where

- Objects are $d-1$-dimensional manifolds Σ,
- Morphisms from Σ_1 to Σ_2 are d-manifolds ∂M with boundary $\partial M = \Sigma_1 \sqcup \Sigma_2$.

This category carries a symmetric monoidal structure ("tensor product") given by the disjoint union.

Definition

A TQFT is a symmetric monoidal functor

$$Z\colon \mathbf{Cob} \to \mathbf{Vect}.$$

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- The coefficients $a_k(p)$ are given by sums over k-loop Feynman diagrams.

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- Only BRST cohomology is physically relevant.

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- Extend the action functional S_M to $\mathcal{S}_M \in \mathcal{O}(\mathcal{F}_M)[[\hbar]]$ satisfying $\Delta \mathcal{S}_M = 0$ (the quantum master equation).
- *Choose* a gauge-fixing Lagrangian $\mathcal{L} \subset \mathcal{F}_M$ such that $\mathcal{S}_M$ restricted to $\mathcal{L}$ has non-degenerate critical points.

BV formalism II

Then one can prove that for a family of Lagrangians $\mathcal{L}_t$ one has that

$$\frac{d}{dt}\int_{\mathcal{L}_t} e^{\frac{i}{\hbar}\mathcal{S}_M}\mu^{\frac{1}{2}} = 0.$$

In particular, one has that

$$Z = \int_{F_M} e^{\frac{i}{\hbar}S_M[\phi]}\mathcal{D}[\phi] = \int_{\mathcal{L}} e^{\frac{i}{\hbar}\mathcal{S}_M}\mu^{\frac{1}{2}}$$

when both sides are defined. Otherwise we use the perturbative expansion of the RHS as a definition.

What are classical TFTs?

For me, perturbative = semiclassical. Understand classical first.

- From a local d-dimensional field theory (F, S), can extract a symplectic manifold F_Σ associated to a $d-1$-dimenionsal manifold - the phase space.
- If $\Sigma \subset \partial M$, there is a map $\pi\colon F_M \to F_\Sigma$ ("restriction")
- If $M\colon \Sigma_1 \to \Sigma_2$ is a cobordism, then

$$(\pi_1 \times \pi_2)(EL) \subset F_{\Sigma_1} \times F_{\Sigma_2}$$

  is a relation.
- In good cases, this is a canonical relation (lagrangian).

The symplectic category

- This suggests "enhancing" the category of symplectic manifolds by allowing canonical relations as morphisms. Simply denote this category **Symp**.
- This idea was already studied Hörmander, Sniatycki-Tulczyjew, Weinstein, Guillemin-Sternberg. . . before TQFTs
- The main problem is that the composition of relations needs a transversality assumption
- Ignoring these problems, define classical TFTs as functor

$$F\colon \mathbf{Cob} \to \mathbf{Symp}$$

Perturbative TQFTs - sketch of a definition

Let $F\colon \mathbf{Cob} \to \mathbf{Symp}$ be a classical TFT.

Data

A *perturbative TQFT quantizing F* assigns

- to $d-1$-dimensional manifold Σ and a polarization $\mathcal{P}$ of F_Σ a $O_{F_\Sigma}[[\hbar]]$-module $\mathcal{H}^{\mathcal{P}}_\Sigma$
- to a d-dimensional manifold M with boundary an element $\tilde{Z}_M \in H^{\mathrm{top}}(EL_M) \otimes \mathcal{H}^{\mathcal{P}}_{\partial M}$.

Perturbative TQFTs - sketch of a definition

Here is an attempt to formalize the gluing axiom. Suppose $M = M_1 \cup_\Sigma M_2$.
Then we require that there is an open set of pairs of polarizations $\{(\mathcal{P}_1, \mathcal{P}_2)\}$ of F_Σ with pairings

$$\langle \cdot, \cdot \rangle_{\mathcal{P}_1, \mathcal{P}_2, \Sigma} \colon \mathcal{H}_\Sigma^{\mathcal{P}_1} \times \mathcal{H}_\Sigma^{\mathcal{P}_2} \to \mathbb{C}[[\hbar]]$$

and

$$\pi_* \langle \tilde{Z}_{M_1}, \tilde{Z}_{M_2} \rangle = \tilde{Z}_M$$

where $\pi \colon F_{M_1} \times F_{M_2} \to F_M$ and π_* denotes pushforward of forms.

Perturbative TQFTs - remarks

- The module $\mathcal{H}_\Sigma$ should be constructed as a "infinite level limit" of geometric quantization of F_Σ, the reduced phase space of the theory.
- The partition function is a top form on F_M. For a closed manifold M, the idea is that the integral of this top form

$$Z_M = \int_{EL_M} \tilde{Z}_M$$

(if it exists) is the formal power series describing the semiclassical asymptotics of the non-perturbative theory.

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Notice: $\iota^{\partial}_{Q}\omega^{\partial} = dS^{\partial}$ for $S^{\partial} = \iota_E \iota_Q \omega$ (E Euler vector field of M).

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- degree -1 symplectic form ω_M
- a degree zero function $\mathcal{S}_M$
- a degree $+1$ vector field Q_M,
- a surjective submersion $\pi\colon \mathcal{F}_M \to \mathcal{F}^{\partial}_{\partial M}$ satisfying $(Q_M)^2 = 0$, $\delta\pi(Q_M) = Q_{\partial M}$ and

$$\iota_Q \omega + \delta \mathcal{S}_M = \pi^* \alpha^{\partial}.$$

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- Compatible with cutting and gluing of manifolds by taking fibered products $\mathcal{F}_{M_1} \times_{\mathcal{F}_\Sigma} \mathcal{F}_{M_2}$
- Call corresponding functor $\mathcal{F} : \mathbf{Cob} \to \mathbf{BFV}$ a *BFV functor*.

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- $\Omega^{\mathcal{P}}$ is a coboundary operator which is a quantisation of S^{∂}.

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$$(\hbar^2\Delta + \Omega)\widehat{Z}_M = 0. \qquad \text{(mQME)}$$

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- The state $\widehat{Z}_M \in \mathrm{Dens}^{\frac{1}{2}}\mathcal{M}_M \otimes \widehat{\mathcal{H}}_{\partial M}$ (computed via Feynman diagrams) is a half-density on $\mathcal{M}_M$ with values in $\widehat{\mathcal{H}}_{\partial M}$.
- It is a cocycle for the BV-BFV operator:

$$(\hbar^2\Delta + \Omega)\widehat{Z}_M = 0. \qquad \text{(mQME)}$$

Conjecture

The $(\hbar^2\Delta + \Omega)$ cohomology gives a perturbative TQFT.

Reshetikhin-Turaev theories

A. Mathematically, RT invariants Z^k_{RT} of a 3-manifold $M = S^3_L$ are given by certain combinations of colored Jones polynomials of L, depending on a level $k \in \mathbb{N}$.
B. By now it is understood that there is a 3-2-1 extended TQFT producing the RT invariants with $Z^k_{RT}(S^1) = \mathbf{C}$, a modular tensor category $\mathbf{C}$.
C. Physically, these TQFTs correspond to Anyon models and form the basis for topological quantum computation

The main conjecture

Conjecture - How to prove WOQ

- There is a perturbative TQFT Z^{∞}_{RT} such that

$$Z^{k}_{RT,M} \sim_{k\to\infty} Z^{\infty}_{RT,M}$$

- This perturbative TQFT coincides with the perturbative TQFT defined by the $(\hbar^2\Delta + \Omega)$-cohomology of the perturbative quantization of the Chern-Simons BFV functor

Let me present some evidence of this fact.

Some evidence for part A

- Jeffrey, Freed-Gompf, Rozansky, Andersen and many others worked on the asymptotic behaviour of Z^k_{RT}.
- In particular, Andersen-Himpel and Andersen-Mistegaard show that on finite order mapping tori the asymptotics are given by integrals of top forms over the moduli space of flat connections
- For manifolds with boundary, Andersen and many collaborators show that the state spaces of Z^k_{RT} can be computed via geometric quantization of moduli spaces of flat connections at level k.
- This is evidence that one can indeed understand the asymptotics of RT as a perturbative TQFT, but no results for manifolds with boundary exist.

Evidence for part B

- Rozansky proved a certain class of manifolds M that

$$Z^k_{RT,M} \sim_{k\to\infty} \sum_{x\in S_{CS}(M)} e^{ikx} T_x \left(1 + \sum_{l=1}^{\infty} r_{x,l} k^{-l}\right)$$

(the AEC at first order)

- For $x = 0$, r_0 can be identified with the LMO invariant

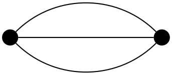
This gives a conjectural chain of equalities

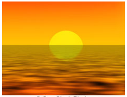
$$(Z^{\infty}_{RT})_{x=0} = Z_{LMO} \stackrel{?}{=} Z_{KKTL} = Z_{BC} \stackrel{?}{=} Z^{pert}_{BV}$$

Theta invariants

Theta invariants

At level 1 (2-loops) the first equality is known, and the invariant known as theta (or sunset) invariant:





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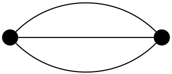

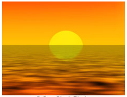

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Theorem

[Kuperberg-Thurston '99, Lescop 2000's] The theta invariant equals the Casson invariant up to a framing correction.

State on solid torus: Feynman diagrams I

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1-point diagrams:

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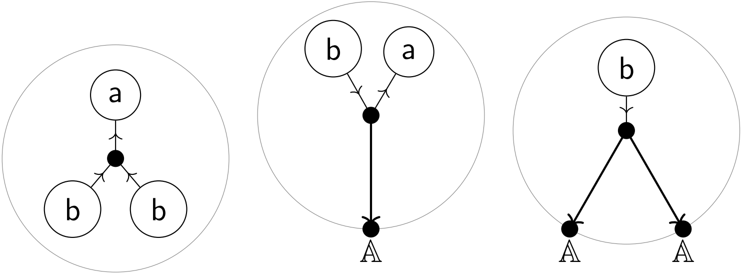

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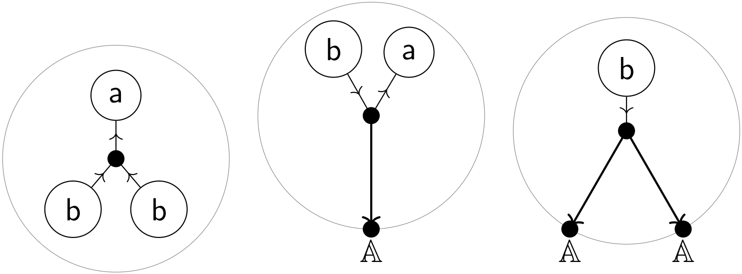

Rules: e.g. Last one gives

$$\psi_\Gamma = g_i^{jk} \int_{C_2(\partial M)} \left(\pi_{1,*} \mathrm{b}_1^i \eta_{12}\eta_{13}\right) \mathbb{A}_j \mathbb{A}_k$$

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- In particular, diffeomorphism type independent of m, n and only depends on $q \mod p$.

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Theorem [W., Cattaneo-Mnev-W.]

The theta invariant (2-loop contribution to the state) of $L_{(p,q)}$ is

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Agrees with Results by Kuperberg-Thurston-Lescop since the Casson(-Walker-Lescop) invariant of $L(p,q)$ is $s(q,p)$.