Resurgence in TQFT

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The WRT-TQFT and quantum Chern-Simons

- Let M be a closed oriented 3-manifold. Let $k \in \mathbb{Z}_+$. The $\mathrm{SU}(2)_k$ Witten-Reshektihin-Turaev invariant of M is a complex-valued topological invariant

$$\mathrm{WRT}(M, k) \in \mathbb{C}.$$

- The WRT-invariant extends to a full TQFT and is the mathematical model of the partition function of $\mathrm{SU}(2)$ quantum Chern-Simons

$$\int_{A \in \mathcal{A}_{\mathrm{SU}(2)}(M)} \exp(2\pi i (k-2)\,\mathrm{CS}_M(A))\;\mathcal{D}(A)$$

The space of solutions to the Euler-Lagrange equation $\delta\,\mathrm{CS}_M = 0$ is equal to the moduli space $\mathcal{M}_{\mathrm{Flat}}(M)$ of flat $\mathrm{SU}(2)$-connections

$$\mathrm{Crit}(\mathrm{CS_M}) := \mathrm{CS}_M(\mathcal{M}_{\mathrm{Flat}}(M)) \subset \mathbb{R}/\mathbb{Z}.$$

Conjecture 1 (The asymptotic expansion conjecture)

For each $S \in \mathrm{Crit}(\mathrm{CS_M})$ *there exists a Puiseux series*

$$W_S \in \bigcup_{m=1}^{\infty} \mathbb{C}((x^{1/m}))$$

such that following asymptotic expansion holds

$$\mathrm{WRT}(M,k) \sim \sum_{S\in \mathrm{Crit}(\mathrm{CS_M})} e^{2\pi i k S} W_S(k^{-1}), \quad \textit{as } k \to +\infty.$$

Status: The AEC is central to quantum topology and it is open in general but proven for:

- Lens spaces, torus bundles by work of Jeffrey, Garoufalidis, Andersen–Jørgensen.
- Classes of mapping tori of surface diffeomorphisms by work of Andersen, Andersen–Jørgensen–Himpel–Martens–McLellan, Charles, Andersen–M, Ios.
- Surgeries on the figure 8 knot due to work of Charles–Marche, and Andersen–M (unpublished) building on Andersen-Hansen.
- **New: Seifert fibered integral homology spheres** by work of Andersen–Han–Li–M–Sauzin-Sun as explained later today. .

The AEC means: For every $S \in \mathrm{Crit}(\mathrm{CS_M})$ there exists positive integers N_S and m_S and a sequence $(W_{S,n})_{n=-N_S}^{\infty} \subset \mathbb{C}$ such that

$$W_S(x) = \sum_{n=-N_S}^{\infty} W_{S,n} x^{n/m_S}.$$

Let $m := \max(m_S \mid S \in \mathrm{Crit}(\mathrm{CS_M}))$. For every positive integer L there exists a real constant $C_L > 0$ depending on L such that

$$\left| \mathrm{WRT}(M,k) - \sum_S e^{2\pi i k S} \sum_{n=-N_S}^{L} W_{S,n} k^{-n/m_S} \right| \leq C_L k^{-\frac{L+1}{m}}.$$

Divergence: often, as expected from pertubation theory, the tail $R_L(k)$ (defined below) does not converge!

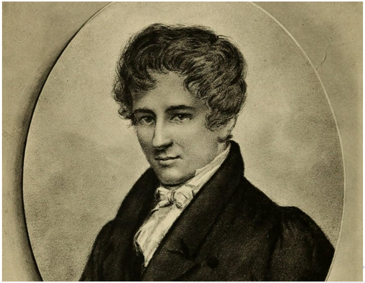
$$R_L(k) := \sum_S e^{2\pi i k S} \sum_{n=L+1}^{\infty} W_{S,n} k^{-n/m_S}.$$

Divergent series - noget Fandenskap

"Divergente Rækker er i det Hele noget Fandenskap, og det er en Skam at man vover at Grunde nogen Demonstration derpaa. Man kan faae frem hvad man vil naar man bruger dem, og det er dem som har gjort saa megen Ulykke og saa mange Paradoxer. Kan det tænkes noget skrækkeligere end at sige at

$$0 = 1^n - 2^n + 3^n - 4^n + \text{etc}$$

hvor n er et heelt Tal." - Niels Henrik Abel



Resurgence: is a theoretical framework used to decode the information stored in the divergent tail of an asymptotic expansion. Resurgence was independently developed by Dingle and Ecalle.

ReNewQuantum: is an ERC synergy project, lead by J.E. Andersen (SDU), B. Eynard (IPHT Saclay), M. Kontsevich (IHES), M. Marino (Université de Genevè), aimed at advancing theories of resurgence and geometric and topological recursion.

Applications: includes

- Foundations of QFT and pertubation theory
- Topological Strings and Topological Quantum Field Theory

Poincaré's example. Let $w = e^s, \Re(s) < 0$. Let z be an indeterminate and consider the functional equation

$$\varphi(z) - w\varphi(z+1) = z^{-1}. \tag{1}$$

Formal solution. For $n \geq 0$ let $a_n(w) = (-1)^n \sum_{m=0}^{\infty} m^n w^m$. The unique formal solution to (1) is

$$\tilde{\varphi}(z) = \sum_{n=0}^{\infty} a_n(w) z^{-n-1} \in z^{-1}\mathbb{C}[[z^{-1}]].$$

Goal: Define a holomorphic function $\varphi(z)$, which solves (1) and is a resummation of $\tilde{\varphi}(z)$ in the sense that $\varphi(z) \sim \tilde{\varphi}(z)$ as $z \to \infty$, i.e.

$$\forall L \in \mathbb{Z}_+, \exists C_L > 0 : \left| \varphi(z) - \sum_{m=0}^{L} a_n(w) z^{-n-1} \right| \leq C_L |z^{-L-2}|.$$

The Borel transform $\mathcal{B} : z^{-1}\mathbb{C}[[z^{-1}]] \to \mathbb{C}[[\zeta]]$ is given by

$$\mathcal{B}\left(\sum_{n=0}^{\infty} c_n z^{-n-1}\right) = \sum_{n=0}^{\infty} \frac{c_n}{n!}\zeta^n.$$

1. Write $\hat{\phi} := \mathcal{B}(\tilde{\phi})$. If $\tilde{\phi}(z) \in z^{-1}\mathbb{C}[[z^{-1}]]$ converges at $z = \infty$, then $\hat{\phi}(\zeta)$ is the germ of an entire holomorphic function.

2. Assume that $\hat{\phi}(\zeta)$ converges and admits an analytic continuation along a ray $e^{i\theta}\mathbb{R}_+$, such that for some $C > 0$:

$$\hat{\phi}(\zeta) = \mathcal{O}(\exp(C|\zeta|)), \quad \forall \zeta \in e^{i\theta}\mathbb{R}_+.$$

Then the Laplace transform $\mathcal{L}_\theta(\hat{\phi})$ give in (2) is holomorphic on a half-plane and admits the asymptotic expansion (3)

$$\mathcal{L}_\theta(\hat{\phi})(z) := \int_{e^{i\theta}\mathbb{R}_+} e^{-z\zeta}\hat{\phi}(\zeta)d\zeta, \quad \forall z : \mathrm{Re}(ze^{i\theta}) > C, \quad (2)$$

$$\mathcal{L}_\theta(\hat{\phi})(z) \sim \tilde{\phi}(z), \quad \text{as } z \to \infty. \quad (3)$$

Consider again the formal solution $\tilde{\varphi}$ and set $\hat{\varphi} = \mathcal{B}(\tilde{\varphi})$. The series φ defined below is convergent on $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ and is a holomorphic solution to the functional equation (1)

$$\varphi(z) = \sum_{m=0}^{\infty} \frac{w^m}{z+m}, \quad \varphi(z) - w\varphi(z+1) = z^{-1}.$$

1. The series $\hat{\varphi}$ is the germ of the meromorphic function

$$\zeta \mapsto (1 - e^{s-\zeta})^{-1}, \quad \text{poles: } \omega_k := s + 2\pi i k, k \in \mathbb{Z}.$$

2. The function $\hat{\varphi}$ is of sub-exponential growth in each ray $e^{i\theta}\mathbb{R}_+$ with $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and for such θ it holds that

$$\mathcal{L}_\theta(\hat{\varphi})(z) = \varphi(z).$$

In particular, $\tilde{\varphi}$ is divergent and φ is a resummation of $\tilde{\varphi}$.

Stokes phenemonon

The Borel transform $\hat{\varphi}$ is of sub-exponential growth on each ray $e^{i\theta'}$ with

$$\theta' \in [\arg(\omega_k), \arg(\omega_{k+1})] \subset \left(\frac{\pi}{2}, \frac{3\pi}{2}\right),$$

and $\mathcal{L}_{\theta'}(\hat{\varphi})$ is a solution to the functional equation equation (1). Further, we have the following family of resurgence relations

$$\varphi(z) - \mathcal{L}_{\theta'}(\hat{\varphi})(z) = 2\pi i \frac{e^{-\omega_k z}}{1 - e^{-2\pi i z}}.$$

Summary: starting from a formal solution to a problem, we obtain via Borel-Laplace resummation a holomorphic solution. There are resurgence relations between different such solutions.

The Ohtsuki series $W_{M,0}(x) \in \mathbb{Z}[[x]]$

By work of Habiro the WRT invariant of a homology 3-sphere M is an algebraic integer

$$\mathrm{WRT}(M,k) \in \mathbb{Z}[e^{2\pi i/k}].$$

The Ohtsuki series is a power series valued topological invariant "associated with the contribution of the trivial flat connection"

$$W_{M,0}(h) \in \mathbb{Z}[[h]].$$

Assume $k = p^e$ is an odd prime power. By work of Rozansky the Ohtsuki series converges p-adically the WRT invariant

$$W_{M,0}(e^{2\pi i/k} - 1) = \mathrm{WRT}(M,k).$$

Fix $r \geq 3$ and let $p_1, ..., p_r \in \mathbb{Z}_+$ be pairwise coprime. Set $P = p_1 \cdots p_r$. Let $M = M(p_1, ..., p_r)$ be the associated Seifert fibered integral homology 3-sphere. Consider the Ohtsuki series

$$W_0(x) \in \mathbb{Z}[[x]].$$

Let $\mathrm{Crit}(\mathrm{CS_M})^{\mathbb{C}} = \mathrm{CS}_M(\mathcal{M}^{\mathrm{Irr}}_{\mathrm{Flat}}(M, \mathrm{SL}(2,\mathbb{C}))$. Set $c = \sqrt{2\pi i P}$. Define the function G and the sequence $\left(\chi_m\right)_{m=1}^{\infty} \subset \mathbb{Z}$ by

$$G(z) := (z^P - z^{-P})^{2-n} \prod_{j=1}^{n} (z^{\frac{P}{p_j}} - z^{-\frac{P}{p_j}}) = (-1)^n \sum_{m=1}^{\infty} \chi_m z^m.$$

Theorem 1 (Joint with Andersen, 2022)

$$\mathcal{B}(W_0)(\zeta) = \frac{4c}{\pi i \sqrt{\zeta}} G\left(e^{\frac{c\sqrt{\zeta}}{P}}\right),$$

$$\mathrm{Crit}(\mathrm{CS_M})^{\mathbb{C}} = \frac{i}{2\pi} \mathcal{P}(\mathcal{B}(W_0)) \mod \mathbb{Z}.$$

Consider the BPS q-series invariant $\hat{Z}(M,q) \in q^{\delta}\mathbb{Z}[[q]]_0, \delta \in \mathbb{Q}$.

Theorem 2 (Joint with Andersen, 2022)

Set $q = e^{2\pi i \tau}, \tau \in \mathbb{H}$. The BPS q-series invariant is a Borel-Laplace resummation of the Okhsuki series

$$\hat{Z}(M,q) = \sum_{m=1}^{\infty} \chi_m q^{\frac{m^2}{4P}} = v.p.\frac{\lambda}{\sqrt{\tau}} \int_{i\mathbb{R}_+} e^{-\frac{\xi}{\tau}} \mathcal{B}(W_0)(\xi)\,\mathrm{d}\,\xi.$$

Further, it holds that $\lim_{q \to \zeta_k} \hat{Z}(M,q) = \mathrm{WRT}(M,k)$.

Building on this result and by applying resurgence, we proved

Theorem 3 (Joint with Andersen, Han, Li, Sauzin, Sun 2024)

The asymptotic expansion conjecture holds for M.

These results builds on Lawrenze–Rozansky 99, Lawrence–Zagier 99 and Gukov–Mariño–Putrov 16