# Boundary energy-momentum tensors for asymptotically flat spacetimes

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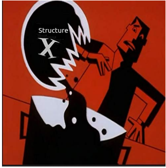
# The hunt for "Structure X"

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A HOLOGRAPHIC DESCRIPTION
FOR $\Lambda = 0$, IF THERE
REALLY IS SUCH A THING,
MUST INVOLVE NOT C.F.T.
BUT SOMETHING ELSE –
CALL IT "STRUCTURE X"
AS WE DON'T KNOW WHAT
IT IS. WHATEVER IT IS,

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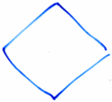


0 + + + + +

A TOO-NAIVE VERSION
OF STRUCTURE X WOULD
BE A FIELD THEORY
ON NULL INFINITY WITH
ITS PECULIAR DIFFERENTIAL
GEOMETRY.

[Witten, '98]

# Motivation

- In AdS/CFT, the boundary energy-momentum tensor (EMT) plays a crucial rôle

⇒ Defined via procedure known as holographic renormalisation [Balasubramnian, Kraus, '99; de Haro, Solodhukin, Skenderis, '00]

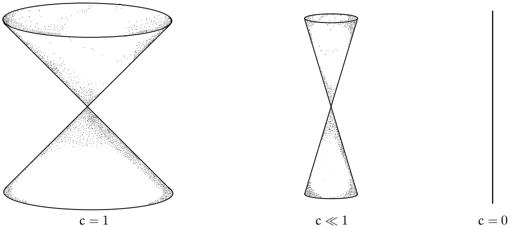
Goal: Do the same for asymptotically flat spacetimes

- The boundary geometry on $\mathscr{I}^+$ in asymptotically flat spacetime is conformally Carrollian [Duval, Gibbons, Horvarthy, '14]

⇒ Must understand near-boundary on-shell description of gravity with arbitrary carrollian data at $\mathscr{I}^+$

# A taste of Carrollian symmetry & geometry

- Physically $c \to 0$; hence "ultra-local"




- Carroll boosts act as

$$t \to t' = t + b_i x^i \,, \qquad \vec{x} \to \vec{x}' = \vec{x}$$

| Metric $g_{\mu\nu}$ replaced by $(v^\mu, h_{\mu\nu})$ satisfying $v^\mu h_{\mu\nu} = 0$ |
|---|

- A *conformal* Carrollian structure transforms as

$$\delta v^\mu = -\Lambda_D v^\mu \,, \quad \delta h_{\mu\nu} = 2\Lambda_D h_{\mu\nu} \,, \quad \delta\tau_\mu = \Lambda_D + \lambda_\mu \,, \quad \delta h^{\mu\nu} = -2\Lambda_D + 2^{(\mu}\lambda^{\nu)}$$

[Duval, Gibbons, Horvathy, '14; Hartong, '15; Figueroa-O'Farrill, EH, Prohazka, Salzer, '21; Hansen, Obers, Oling, Søgaard, '21]

# Plan

1. AF spacetimes and Carroll-covariant Bondi–Sachs gauge
2. Constraints, variations, and Ward identities
3. Holographic renormalisation

# Carrollian geometry at $\mathscr{I}^+$ from Penrose construction

Let $(\mathcal{M}, g)$ be an AF $(d+2)$-dimensional Lorentzian manifold:

- Write

$$g = -2UV + E^a E^a \qquad (a = 1, \ldots, d)$$

  $U$, $V$ null; $E^a$ spacelike

- Treat $\mathscr{I}^+$ as a Penrose b'dary $\Rightarrow$ split $x^M = (r, x^\mu)$ [Penrose, '63]

- Partially fix local $\mathrm{SO}(d+1,1)$ & diffeos to get

$$\begin{aligned} g_{rr} &= 0\,, \quad g_{r\mu} = -V_\mu\,, \quad g_{\mu\nu} = -SV_\mu V_\nu + \Pi_{\mu\nu}\,, \\ g^{rr} &= S\,, \quad g^{r\mu} = U^\mu\,, \qquad g^{\mu\nu} = \Pi^{\mu\nu} \qquad (S = \mathcal{O}(r)) \end{aligned}$$

- Penrose construction fixes boundary conditions:

$$V_\mu\big|_{r\to\infty} = \tau_\mu\,, \;\; r^{-2}\Pi_{\mu\nu}\big|_{r\to\infty} = h_{\mu\nu}\,, \;\; U^\mu\big|_{r\to\infty} = v^\mu\,, \;\; r^2\Pi^{\mu\nu}\big|_{r\to\infty} = h^{\mu\nu}$$

See also [Hartong, '15]

# Carroll-covariant Bondi–Sachs gauge

This gauge corresponds to

$$\boxed{g_{rr} = 0\,, \qquad \Gamma^{\rho}_{rr} = 0\,, \qquad \Gamma^{\mu}_{\mu r} = d\, r^{-1}}$$

⇒ This fixes $V_\mu = e^\beta \tau_\mu$ ($\beta = \mathcal{O}(r^{-1})$) and $h^{\mu\nu}\Pi_{\mu\nu}$

☞ In "standard" Bondi gauge, metric fully gauge fixed to

$$\tau = du \qquad \text{and} \qquad h = ds^2_{\mathbb{CS}^d} \qquad (\mathscr{I}^+ \cong \mathbb{R} \times \mathbb{CS}^d)$$

- We, however, want *arbitrary* boundary data $(\tau, h)$, so we can vary with respect to it

When $d \geq 2$, Einstein's equations constrain the boundary geometry

$$K_{\mu\nu} := -\frac{1}{2}\pounds_v h_{\mu\nu} = \frac{1}{d} K h_{\mu\nu}$$

but this "issue" can be dealt with

See also [Geiller, Zwikel, '22 & '24]

# Solving Einstein's equations asymptotically

Einstein's equations allow to solve for $\beta$, $S$, $h^\rho_\mu v^\sigma \Pi_{\rho\sigma}$ ($h^\mu_\nu = e^\mu_a e^a_\nu$), and $\pounds_v \left( h^\rho_{\langle\mu} h^\sigma_{\nu\rangle} \Pi_{\rho\sigma} \right)$

$$\boxed{\Pi_{\mu\nu} = r^2 h_{\mu\nu} + r\left(C_{\mu\nu} - 2\tau_{(\mu} a_{\nu)}\right) + \mathcal{O}(1)\,, \quad S = \frac{2}{d} K r + \mathcal{O}(1)\,, \quad \beta = \mathcal{O}(r^{-2})}$$

with $C_{\mu\nu}$ the shear & $a_\mu = \pounds_v \tau_\mu$

- Residual transformations $\xi^\mu = \chi^\mu + r^{-1} h^{\mu\nu}\lambda_\nu + \mathcal{O}(r^{-2})$ and $\xi^r = r\Lambda_D + \mathcal{O}(1)$ turn into Carroll transformations

$$\begin{aligned}
\delta\tau_\mu &= \pounds_\chi \tau_\mu + \Lambda_D \tau_\mu + \lambda_\mu\,, \\
\delta h_{\mu\nu} &= \pounds_\chi h_{\mu\nu} + 2\Lambda_D h_{\mu\nu}\,, \\
\delta C_{\mu\nu} &= \pounds_\chi C_{\mu\nu} + 2h^\rho_{\langle\mu} h^\sigma_{\nu\rangle}(\mathcal{D}_\rho \lambda_\sigma + a_\rho \lambda_\sigma)
\end{aligned}$$

- When $d = 1$, the $r$-expansion truncates

Agrees with [Barnich, Troessaert, '10; Geiller, Zwikel, '22]

# On-shell near-boundary metric for $d = 2$

We denote by $X^{(n)}$ the coefficient of $r^{-n}$ in the $r$-expansion of $X$

- The on-shell metric expands near $\mathscr{I}^+$ as

$$ds^2 = -2dr\left(\tau_\mu + \mathcal{O}(r^{-2})\right)dx^\mu + \left(r^2 h_{\mu\nu} + rg^{(-1)}_{\mu\nu} + g^{(0)}_{\mu\nu} + \mathcal{O}(r^{-1})\right)dx^\mu dx^\nu$$

where

$$\begin{aligned} g^{(-1)}_{\mu\nu} &= -K\tau_\mu\tau_\nu - 2\tau_{(\mu}a_{\nu)} + C_{\mu\nu}\,, \\ g^{(0)}_{\mu\nu} &= -(S^{(0)} - a^2)\tau_\mu\tau_\nu - 2\tau_{(\mu}P^{(0)}_{\nu)} + h^\rho_\mu h^\sigma_\nu \Pi^{(0)}_{\rho\sigma} \end{aligned}$$

- The free data not determined by Einstein's equations consists of $S^{(1)}$ and $h^\rho_\mu v^\sigma \Pi^{(1)}_{\rho\sigma} \Rightarrow$ generalised Bondi mass and angular momentum aspect

# Constrained boundary variations

Consider the variation of a boundary action functional $S[\tau, h, C]$

$$\delta S[\tau, h, C] = \int d^{d+1}x\, e \left( T^{\mu}\delta\tau_{\mu} + \frac{1}{2}T^{\mu\nu}\delta h_{\mu\nu} + \frac{1}{2}S^{\mu\nu}\delta C_{\mu\nu} \right)$$

Together, the responses form a **EMT-news complex**

In $d = 2$, the boundary geometry is constrained by $K_{\mu\nu} = K h_{\mu\nu}/2$. Must either

- Solve constraint $\Rightarrow h_{\mu\nu} = H^2(\partial_\mu X \partial_\nu X + \partial_\mu Y \partial_\nu Y)$; or
- Add Lagrange multiplier term $S_{\text{LM}} = \dfrac{1}{2}\displaystyle\int d^{d+1}x\, e \chi^{\mu\nu} K_{\mu\nu}$ where $\chi^{\mu\nu}$ is spatial and STF. Implies no distinction between $T^{\mu\nu} + t^{\mu\nu}$ where $t^{\mu\nu} = \frac{1}{2}h^{\mu\rho}h^{\nu\sigma}\mathcal{L}_v\chi_{\rho\sigma} - v^{(\mu)}h^{\nu)\sigma}(\mathcal{D}_\rho - a_\rho)\chi^{\rho}{}_{\sigma}$

# Ward identities

The Ward identities are obtained by varying the boundary action functional under the local symmetries of the boundary structure:

$$\chi^\mu : \quad 0 = -e^{-1}\partial_\mu(e[T^\mu{}_\nu + S^{\mu\rho}C_{\rho\nu}]) + T^\mu\partial_\nu\tau_\mu + \frac{1}{2}T^{\mu\rho}\partial_\nu h_{\mu\rho} + \frac{1}{2}S^{\mu\rho}\partial_\nu C_{\mu\rho}\,,$$

$$\Lambda_D : \quad 0 = T^\mu\tau_\mu + T^{\mu\nu}h_{\mu\nu} + \frac{1}{2}S^{\mu\nu}C_{\mu\nu}\,,$$

$$\lambda_\mu : \quad 0 = h^\mu_\rho T^\rho - (\mathcal{D}_\rho - a_\rho)S^{\rho\mu}$$

In the above, $T^\mu{}_\nu = T^\mu\tau_\nu + T^{\mu\rho}h_{\rho\nu}$ is the EMT

- When there are extra fields, such as the shear, the energy flux no longer vanishes in a Carroll-invariant theory
- These Ward identities can (and will!) be *anomalous*
- The conservation equations $U^\mu R_{\mu\nu}\big|_{\mathcal{O}(r^{-d})}$ give the Bondi loss equations and can be written in the form of the Ward identities above

# A well-posed variational principle

- Demand well-posed* variational principle for large $r$; regularise on cut-off hypersurface $r = \Lambda$

$$\delta S_{\text{EH}} = \dots + \int_{r=\Lambda} d^3 E\, J^r \qquad (E := \sqrt{-g} = e r^d e^\beta)$$

⇒ Need extrinsic GHY-type boundary term, but cut-off surface *indefinite* [Parattu, Chakraborty, Padmanabhan, '16]

$$S_{\text{ext}} = 2\int_{r=\Lambda} d^{d+1}x\sqrt{-g}\left(\delta^M_P + V^M N_P\right)\nabla_M N^P\,, \qquad N_M = \partial_M r \text{ and } V^M = -\delta^M_r$$

$$\Rightarrow \delta\left(S_{\text{EH}} + S_{\text{ext}}\right) = \dots + 2\int_{r=\Lambda} d^3x\, E\Big(\overbrace{\tilde{\mathcal{T}}^\mu \delta V_\mu + \frac{1}{2}\tilde{\mathcal{T}}^{\mu\nu}\delta\Pi_{\mu\nu}}^{=\mathcal{O}(r^{-1})} + \overbrace{\frac{1}{2} d r^{-1} E^{-1}\delta\left(ES\right)}^{\mathcal{O}(1)}\Big)$$

☞ The leading term *contains variations of the free data* $S^{(1)}$ ⇒ must be cancelled by adding

$$S_{\text{norm}} = -2\int_{r=\Lambda} d^{d+1}x\, E r^{-1} S$$

# Holographic renormalisation for AF spacetimes

- $\delta\left(S_{\mathrm{EH}}+S_{\mathrm{ext}}+S_{\mathrm{norm}}\right)$ *still* diverges as $r\to\infty$, but cured by

$$S_{\mathrm{int}}=-\int_{r=\Lambda}d^3x\,E\,r\,R[C]$$

- In addition to this, we add finite counterterms that make the on-shell action Weyl invariant (*improvements*)

⇒ This leads to

$$\delta\left(S_{\mathrm{EH}}+S_{\mathrm{ext}}+S_{\mathrm{norm}}+S_{\mathrm{int}}\right)\Big|_{r\to\infty}=\int d^3xe\left(T^\mu\delta\tau_\mu+\frac{1}{2}T^{\mu\nu}\delta h_{\mu\nu}+\frac{1}{2}S^{\mu\nu}\delta C_{\mu\nu}\right)$$

which happens because $\mathcal{T}^{\mu\nu}=\mathcal{O}(r^{-3})$

☞ The explicitly computed currents satisfy (cf. Ward identities)

$$\mathcal{A}^\mu_{\mathrm{B}}=h^\mu_\rho T^\rho-\left(\mathcal{D}_\rho-a_\rho\right)S^{\rho\mu}$$
$$0=T^\mu\tau_\mu+T^{\mu\nu}h_{\mu\nu}+\frac{1}{2}S^{\mu\nu}C_{\mu\nu}$$

# Carroll-covariant Bondi loss equations

The diffeo Ward identity turns into the Bondi loss equations

$$0 = -\left(\pounds_v - \frac{3}{2}K\right)(\tau_\mu T^\mu) - \frac{1}{4}N^{\rho\sigma}N_{\rho\sigma} + (\mathcal{D}_\mu + a_\mu)\left(T^\rho h^\mu_\rho\right) \ ,$$

$$0 = -(\pounds_v - K)\, P_\kappa + h_{\kappa\sigma}\mathcal{D}_\mu \tilde{T}^{\mu\sigma} + \frac{1}{2}h^\mu_\kappa\,(\partial_\mu + 3a_\mu)\left(T^{\rho\sigma}h_{\rho\sigma} + \frac{1}{2}N^{\rho\sigma}C_{\rho\sigma}\right)$$
$$+ \frac{1}{4}h_{\kappa\sigma}\mathcal{D}_\mu\left(N^{\mu\lambda}C_\lambda{}^\sigma - N^{\sigma\lambda}C_\lambda{}^\mu\right) + T^\sigma h^\mu_\sigma F_{\mu\kappa} - \frac{1}{4}N^{\mu\sigma}h^\nu_\kappa\,(\mathcal{D}_\nu + a_\nu)\, C_{\mu\sigma}$$

where $\tilde{T}^{\mu\nu} := h^\mu_{\langle\rho}h^\nu_{\sigma\rangle}T^{\rho\sigma}$ and $P_\mu = T^{\rho\sigma}\tau_\rho h_{\sigma\mu}$

- Covariant Weyl-invariant news tensor defined by

$$N_{\mu\nu} = -\mathcal{L}_v C_{\mu\nu} - \frac{1}{2}KC_{\mu\nu}$$

- Related to improved response via $S^{\mu\nu} = \frac{1}{2}h^{\mu\rho}h^{\nu\sigma}N_{\rho\sigma}$

# Take home messages

1. Boundary described by conformal Carrollian geometry, with shear on same footing as Carrollian structure

2. There exists a generalised version of Bondi–Sachs gauge that does not fix the boundary geometry, but Einstein's equations impose a constraint

3. Holographic renormalisation can be done in asymptotically flat spacetimes and reveals special role of shear

Thank you for your attention