Classifying all Feynman integral geometries for two-loop particle scattering

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Center for Quantum Mathematics, University of Southern Denmark

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2509.xxxxx with P. Bargieła, H. Frellesvig, R. Marzucca, R. Morales, M. Wilhelm and T. Yang

Centre for Quantum Mathematics

Motivation: QFT and Precision Physics

High-energy theory motivation $\Longrightarrow$ precision predictions for new experiments!

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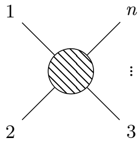
Scattering amplitude $\mathcal{A}$



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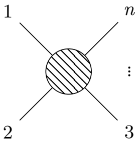
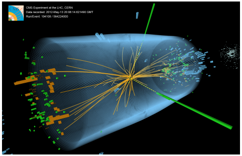


Colliders:

$\sigma \sim \int d(\text{phase space})\, |\mathcal{A}|^2$



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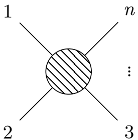

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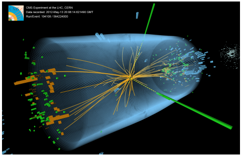

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[Talks by Emil, Gang, Nabha,...]

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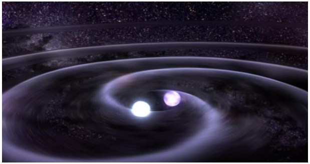

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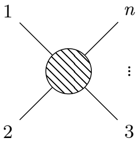

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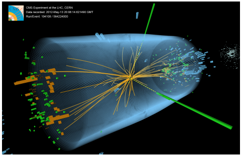

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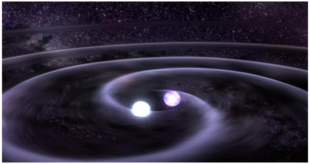

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Understanding Scattering Amplitudes $\Rightarrow$ better predictions

Motivation: Scattering Amplitudes and Particle Phenomenology

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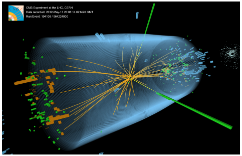

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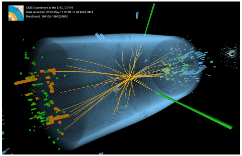

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- Main observable: cross section σ

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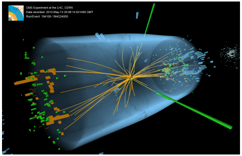

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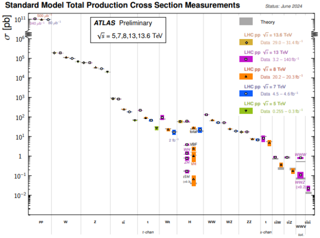

Image: [ATLAS-PHYS-PUB-2024-011 (2024)]

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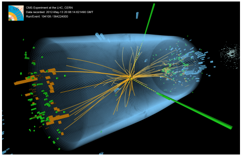

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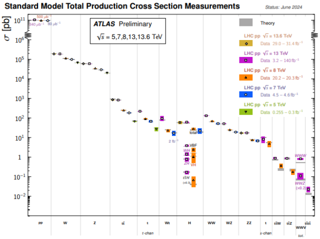

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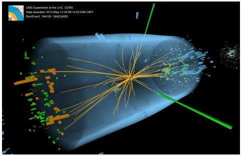

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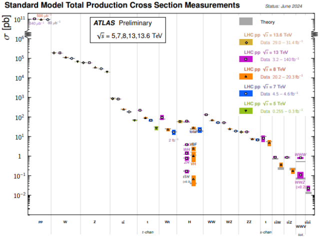

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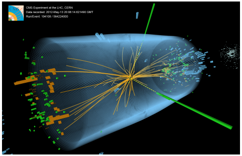

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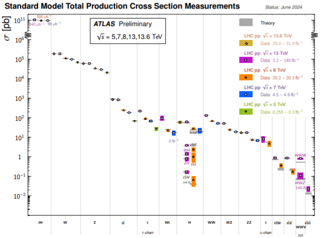

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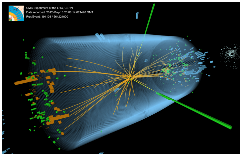

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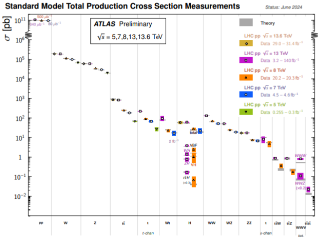

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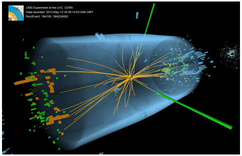

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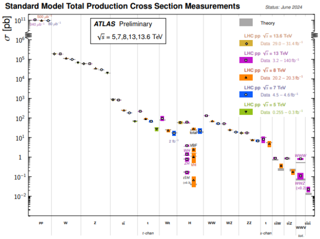

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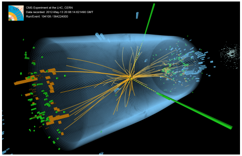

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$\Longrightarrow$ **New experiments** (HL-LHC) require higher theoretical precision!

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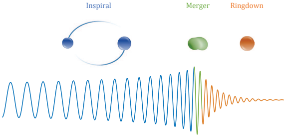

Image: [1610.03567]

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- Study GW emission during BH/NS mergers

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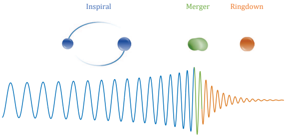

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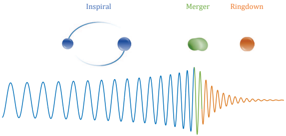

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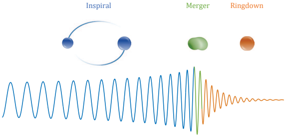

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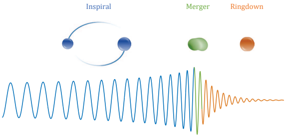

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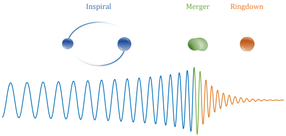

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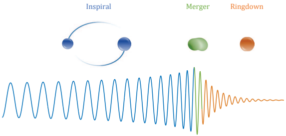

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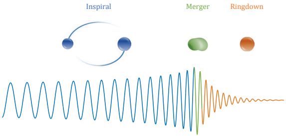

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→ See Hjalte's talk!

Perturbation theory

→ Series expansion in interaction strength

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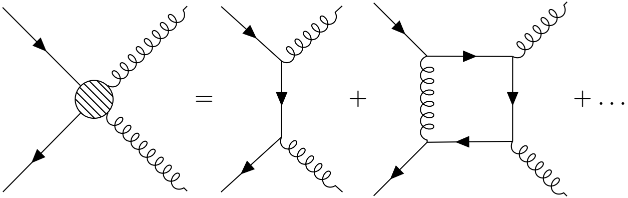

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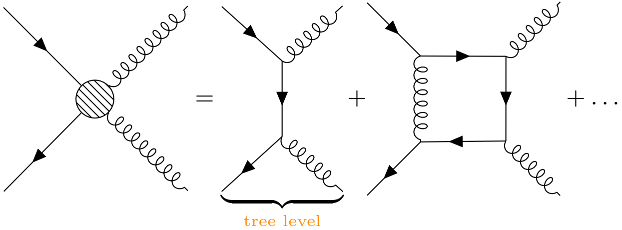

tree level leading order in perturbation theory

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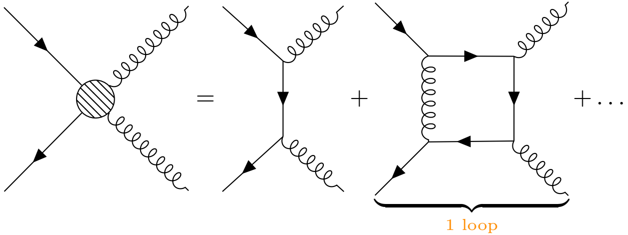

1 loop next-to-leading order in perturbation theory

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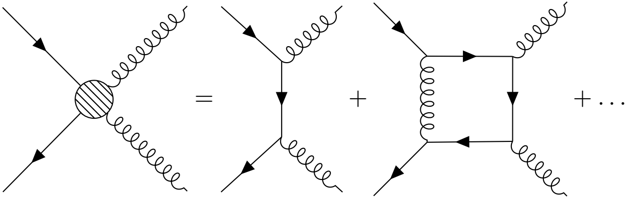

... higher orders in perturbation theory

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Feynman Integrals: loops $\sim$ integrals over loop momenta

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$\implies$ more loops = higher precision!

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Feynman Integrals: loops ∼ integrals over loop momenta

$\Longrightarrow$ more loops = higher precision!
Current state of the art: ∼ 1-4 loop for particle pheno, ∼ up to 4 loops for gravity.

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Feynman Integrals: loops $\sim$ integrals over loop momenta

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$\longrightarrow$ Integrals become complicated because of underlying geometry!

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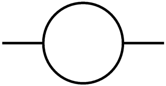
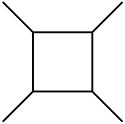
At one loop

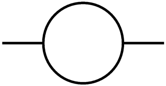
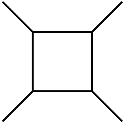
 $\overset{D=2}{\sim} \log(\dots),$  $\overset{D=4}{\sim} \mathrm{Li}_2(\dots) + \dots$

Classical polylogarithms

$$\mathrm{Li}_n(x) \equiv \int_0^x \frac{dt}{t} \mathrm{Li}_{n-1}(t) \quad \text{with} \quad \mathrm{Li}_1(x) \equiv -\log(1-x) = \int_0^x \frac{dt}{1-t}$$

Multiple polylogarithms

At one loop

 $\overset{D=2}{\sim} \log(\dots)$,  $\overset{D=4}{\sim} \mathrm{Li}_2(\dots) + \dots$

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$\Rightarrow$ **General 1-loop**: Multiple polylogarithms (MPLs) [Chen (1977)], [Goncharov (1995)], ...

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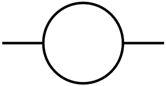
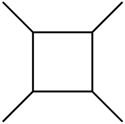
 $\overset{D=2}{\sim} \log(\dots),$  $\overset{D=4}{\sim} \mathrm{Li}_2(\dots) + \dots$

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$\Longrightarrow$ **Beyond 1-loops**: space of integrals richer than MPLs!

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When, and how, do integrals beyond multiple polylogarithms occur?

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What is the geometry of $y^2 = P(x_2)$?

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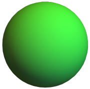
- $\deg P(x_2) \leq 2 \Longrightarrow$ Rationalizing $\sqrt{P(x_2)} \to \sqrt{(\dots)^2}$

  $\hookrightarrow$ MPL $\Longrightarrow$ Integral over Riemann sphere



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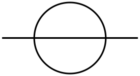
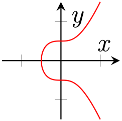
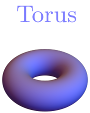
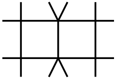


[Broadhurst, Fleischer, Tarasov (1993)], ..., [Adams, Bogner, Weinzierl (2013)], ...

Elliptic curve

$\deg P(x_2) = 3, 4 \implies$ 

$=$ 



[Caron-Huot, Larsen (2012)], ..., [Morales, Spiering, Wilhelm, Yang, Zhang (2022)]

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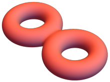
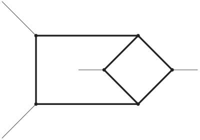
g-torus → hyperelliptics

$\deg P(x_2) \geq 5 \implies$



2-torus



[Georgoudis, Zhang (2015)], [Marzucca, McLeod, Page, Pögel, Weinzierl (2023)]

Calabi-Yau manifolds

In general: Feynman (parameter) integrals ≥ 2 variables

$$I = \int_0^1 \frac{dx_1 \cdots dx_n}{{x_1}^2 - P(x_2, \ldots, x_n)} = \int_0^1 dx_2 \cdots dx_n \left(\frac{\log(1 - 1/y)}{2y} - (- \to +) \right)$$

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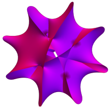

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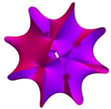
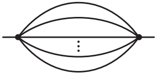
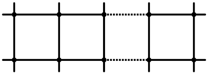


Examples:



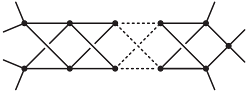
CY $(L-1)$-fold

[Broadhurst, Fleischer, Tarasov (1993)], ..., [Bönisch, Duhr, Fischbach, Klemm, Nega (2021)], [Pögel, Wang, Weinzierl (2023)],...



CY $(L-1)$-fold

[Bourjaily, He, McLeod, von Hippel, Wilhelm (2018)], ..., [Cao, He, Tang (2023)]



CY $(2L-2)$-fold

[Bourjaily, McLeod, von Hippel, Wilhelm (2018)], [Lairez, Vanhove (2022)]

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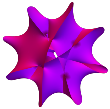
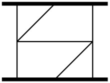
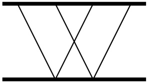


Gravity Examples:



CY 3-fold

[Frellesvig, Morales, Wilhelm (2023)]



CY 3-fold

[Klemm, Nega, Sauer, Plefka (2024)]

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Detecting Geometries: Leading Singularities

Leading singularity (LS) $\sim$ maximally iterated discontinuity [Cachazo (2008)],...

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Advantage: Easier to calculate while keeping geometric information!

$\hookrightarrow$ Baikov representation [Baikov (1997)], [Frellesvig, Papadopoulos (2017)]

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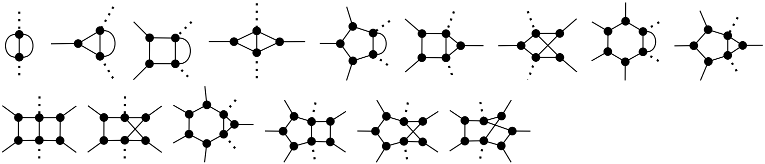
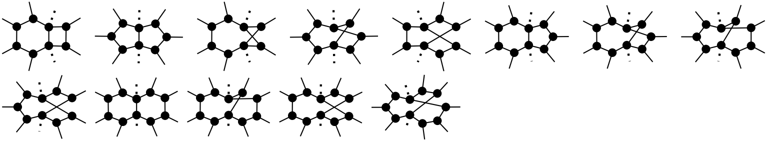
All two-loop Feynman integrals

In 't Hooft Veltman scheme → 84 basis diagrams in $D = 4 - 2\varepsilon$ [Bargieła, Yang, (2025)]





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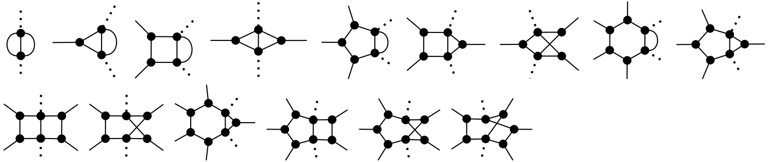
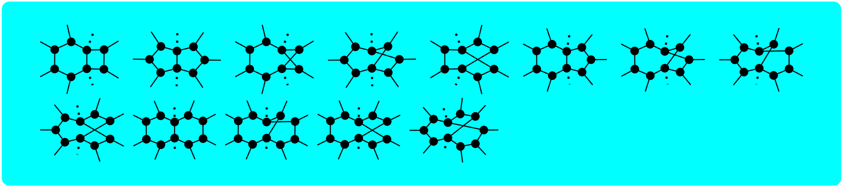


39 evanescent integrals: **only contribute at 3 loops** in $D = 4$!



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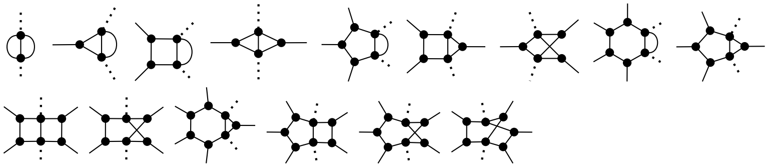

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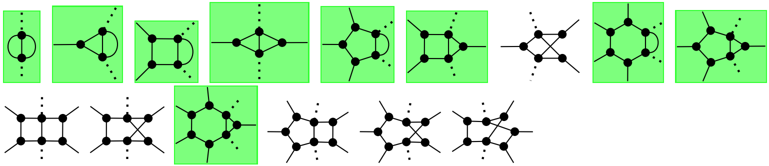

Algebraic LS immediately from Baikov!

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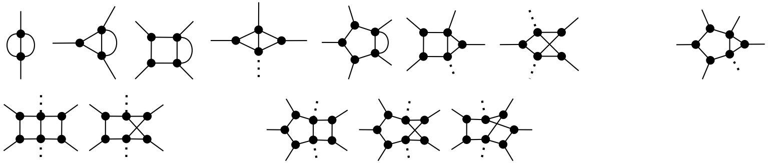

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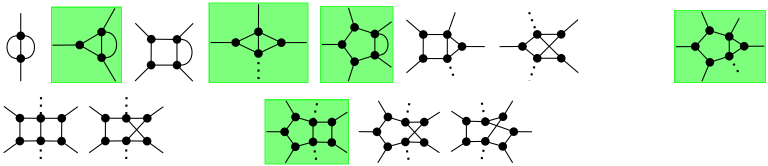
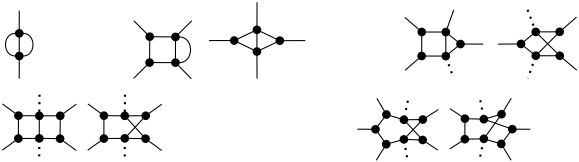


$$\mathrm{LS}(I) \propto \int \frac{dx}{P_i(x)\sqrt{\mathbf{P_2(x)}}} \rightarrow \text{algebraic LS!}$$

All two-loop Feynman integrals



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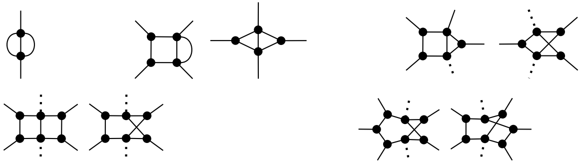
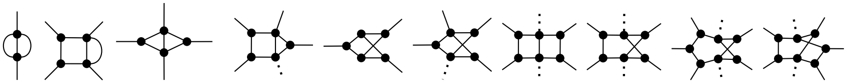


Let's look closer at what is left over!

Classification of diagrams

21 non-evanescent diagrams remain:



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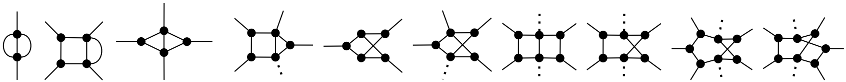

Some studied before with special/massless kinematics!
We study ALL with **generic** kinematics!

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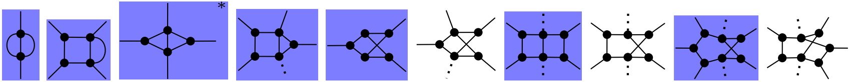
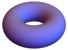


Elliptic curves: $\mathrm{LS}(I) \propto \int \frac{dx}{\sqrt{P_4(x)}}$ 

* Cross-checks pending

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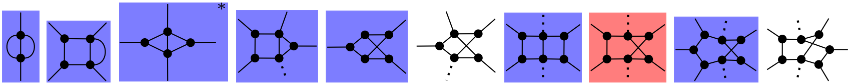
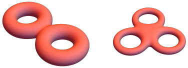


Hyperelliptic curves: $\mathrm{LS}(I) \propto \int \frac{dx}{\sqrt{P_{6,8}(x)}}$



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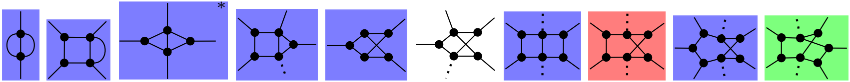

MPLs: $\mathrm{LS}(I)$ is algebraic

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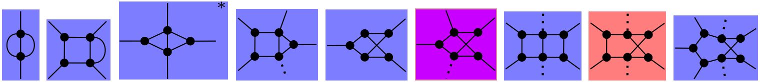
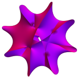


K3: $\quad \mathrm{LS}(I) \propto \int \frac{dx_1\, dx_2}{\sqrt{P_6(x_1, x_2)}}$



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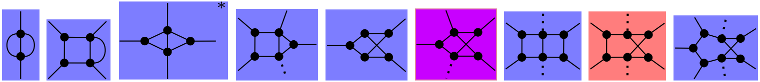

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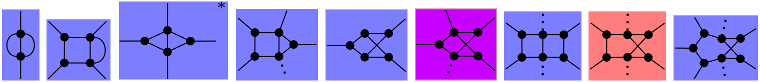
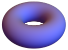


Elliptic: 12 × 

$$\mathrm{LS}(I) \propto \int \frac{dx}{\sqrt{P_4(x)}}$$

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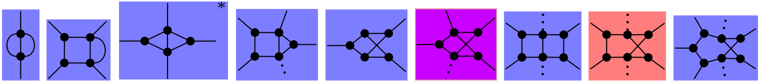
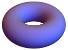
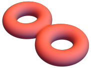


Elliptic: 12 ×  $\mathrm{LS}(I) \propto \int \frac{dx}{\sqrt{P_4(x)}}$

Hyperlliptic: 1 ×  2 ×  $\mathrm{LS}(I) \propto \int \frac{dx}{\sqrt{P_{6,8}(x)}}$

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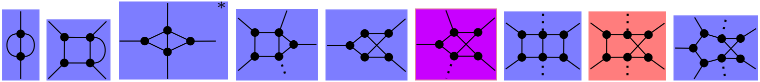
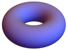
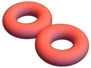
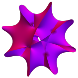


Elliptic: 12 ×  $\text{LS}(I) \propto \int \frac{dx}{\sqrt{P_4(x)}}$

Hyperlliptic: 1 ×  2 ×  $\text{LS}(I) \propto \int \frac{dx}{\sqrt{P_{6,8}(x)}}$

K3: 2 ×  $\text{LS}(I) \propto \int \frac{dx_1\, dx_2}{\sqrt{P_6(x_1, x_2)}}$

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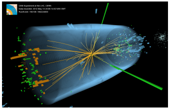
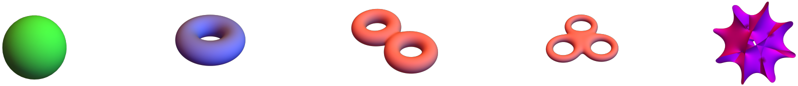
Conclusion

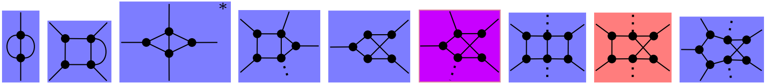
1. Precision physics requires high loop orders
   $\rightarrow$ non-trivial Feynman integral geometries.



2. **Simplest case**: polylogs. **Higher loop orders**: elliptic curves, higher genus curves, CY manifolds ...



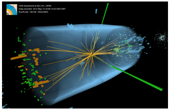
3. Leading Singularities = powerful tool for classifying geometries.
4. Classified the geometries of **ALL** 2-loop integrals in $D = 4$, for generic kinematics $\oplus$ understand degeneration!

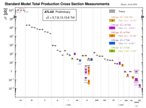


5. **Determined whole space of functions!**

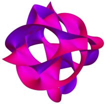
Next steps

1. Specify relevant kinematics for pheno calculations: Les Houches Wishlist [2504.06689]?





2. Calculate higher loop orders $\Longrightarrow$ Structural results



3. Calculate integrals on geometries via differential equations [Kotikov (1991)], [Henn (2013)]
$\Longrightarrow$ LS results are the first step for canonization [Duhr, Maggio, Nega, Sauer, Tancredi (2025)], [Bree, Gasparotto, Matijašić, Mazloumi, Melnichenko, Pögel, Teschke, Wang, Weinzierl, Wu, Xu (2025)]!

Next steps

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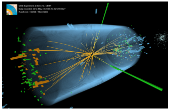

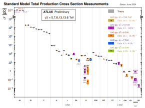

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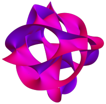

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Thank you!