# The Brief Femtouniverse

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What can we do to solve the problems of confinement and the mass gap in quantum chromodynamics (QCD) using mathematical methods?

At large bare coupling on a lattice, there is confinement and a mass gap. Unfortunately, the large-bare-coupling approximation is too far from the continuum limit at zero bare coupling. At best, this a yields a quark model. This criticism also applies to string-theory-motivated large-bare-coupling approaches.

The action of the **CORRECT** large-bare-coupling effective theory must include those marginal or irrelevant operators produced by an integration over short-distance degrees of freedom.

The original femtouniverse is a picture of the bag in QCD. QCD is weakly coupled below a distance of $1\ Fm = 10^{-13}\ cm$. Above this distance, chromoelectric flux is suppressed.

Bjorken (1979)

Hansson, Johnson and Petersen (1982)

We consider a small-diameter region of Euclidean spacetime, instead of space, which we shall call brief femtouniverse.

Alternative approach: toroidal femtouniverse, periodic boundary conditions (possibly twisted):

Lüscher (1982)

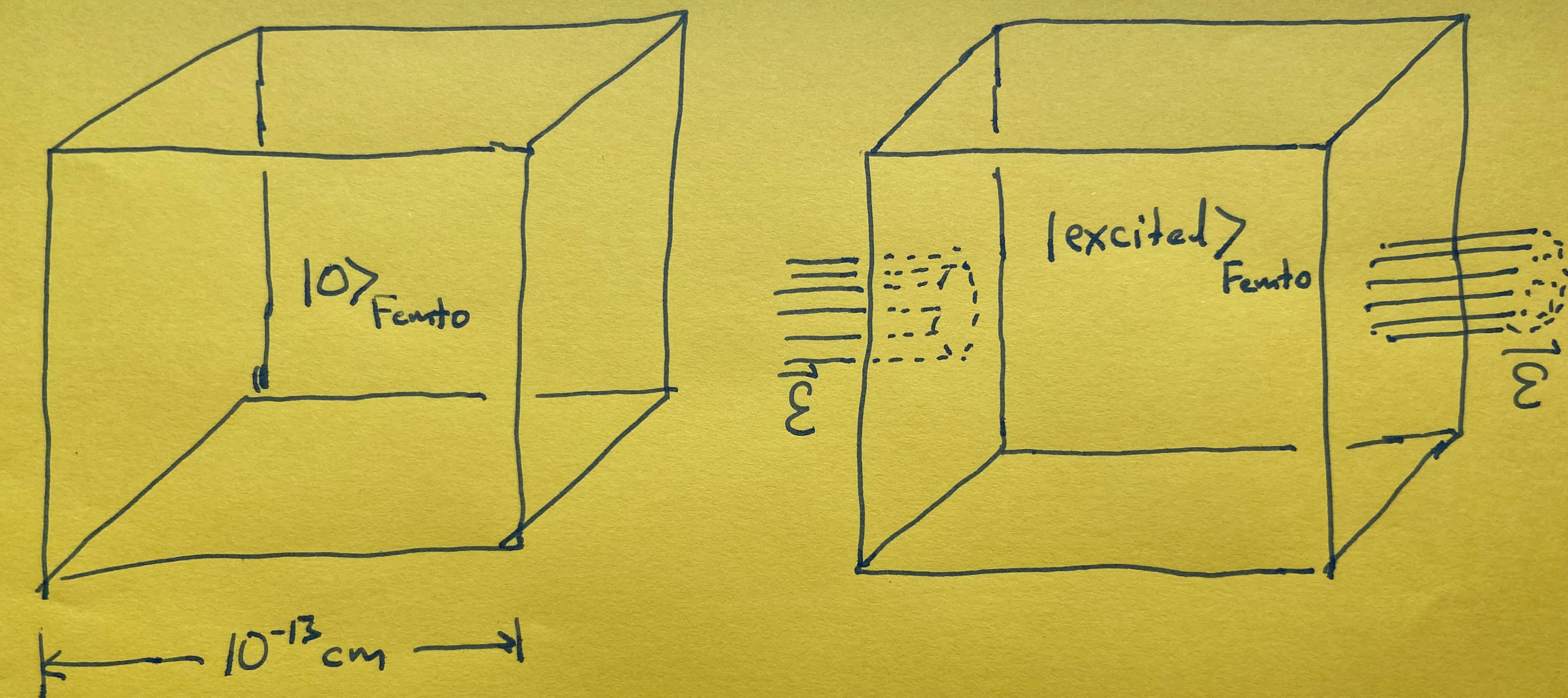
Lüscher and Münster (1984)

van Baal (1980's - 2000's)

Kovtun, Ünsal and Yaffe (2007)

Bergner, González-Arroyo and Soler (2025)

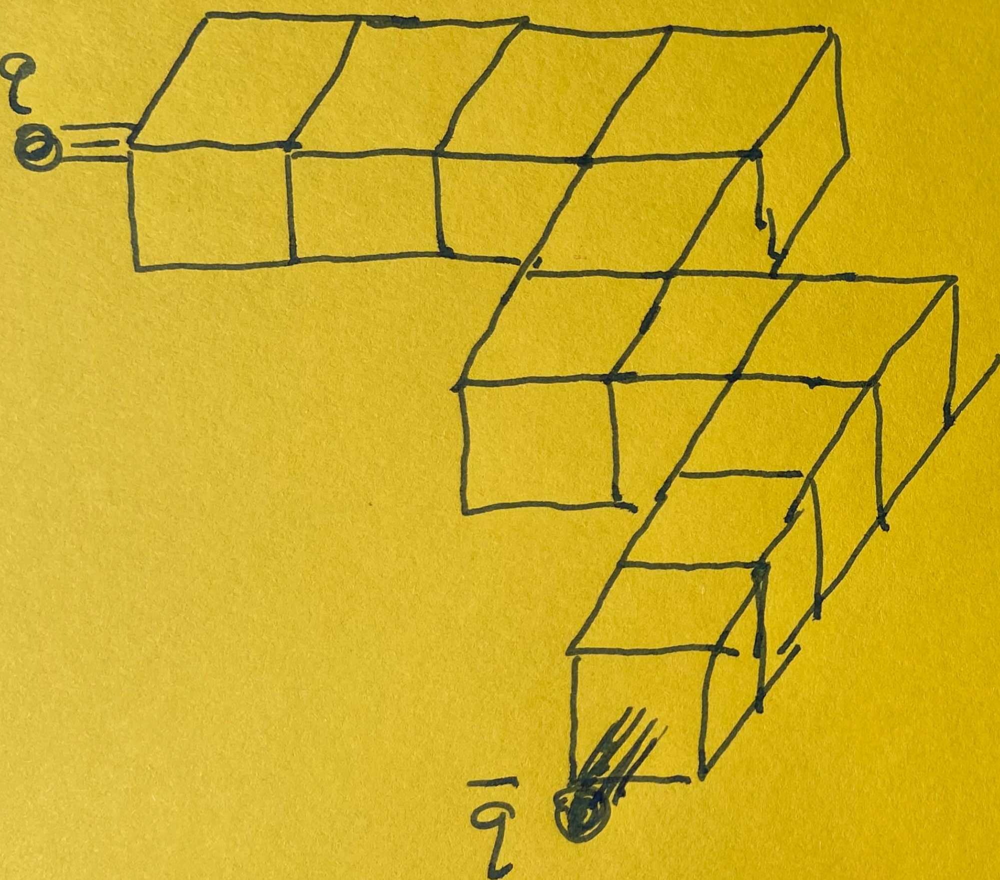
Bjorken ('79)




$E = 0$

$E = \Delta E > 0$

# String of Excited Cubes




String Tension

$$= \frac{\Delta E}{10^{-13}\text{cm}}$$

Instead of p.b.c., use Dirichlet b.c.. Functionally integrate within femtouniverses (FU) covering spacetime, to find effective theory on $(d-1)$-dimensional interfaces.

**Femtouniverse RG (FURG)**

Ideally, FU tile, but they can overlap, as a first approximation.

1. Consider arbitrary Dirichlet boundary conditions on one FU.
2. Find smallest action $S_0$ config. inside the FU, consistent with the boundary data. These lie in a lower-dimensional manifold $M$, parametrized by $\varphi_1,\ \varphi_2,\ldots\ \varphi_k$.
3. Approximate the functional integral by steepest descents,

$$I = \int d^n x\, f(x) e^{-S(x)} \approx C \int_M d^k\varphi\ \sqrt{\det(G)(\varphi)}\ f[x(\varphi)]\ \frac{e^{-S_0}}{\sqrt{\det'(\delta^2 S)}},$$

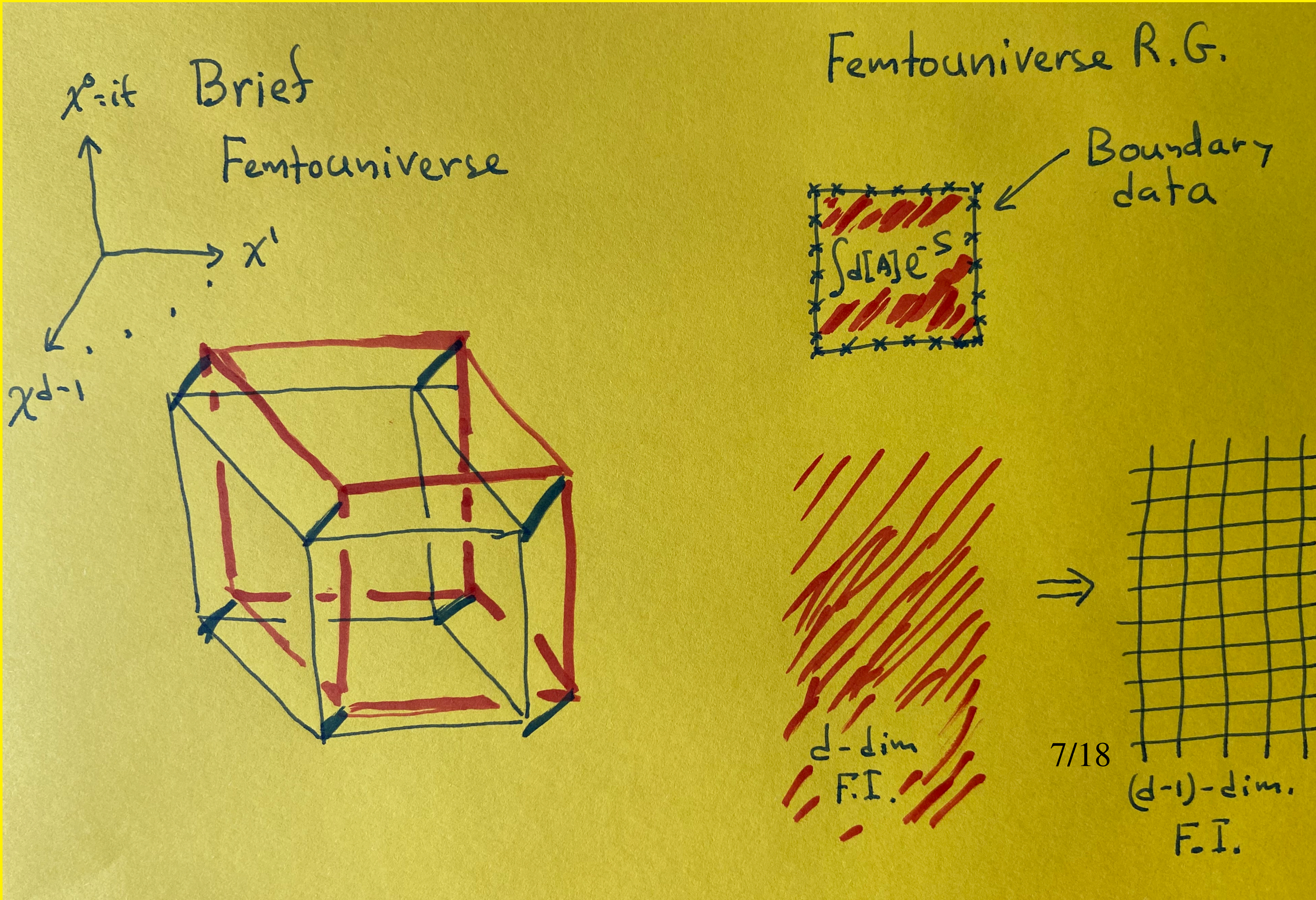
$$G_{ij}(t) = \sum_{l=1}^{k} \frac{\partial x_l(\varphi)}{\partial \varphi_i}\frac{\partial x_l(\varphi)}{\partial \varphi_j}, \quad \delta^2 S_{ij} = \left.\frac{\partial^2 S(x)}{\partial x_i \partial x_j}\right|_{x(\varphi)\in M}, \quad \left.\frac{\partial S(x)}{\partial x_j}\right|_{x(\varphi)\in M} = 0.$$

4. Identify boundary data of intersecting/overlapping FU.
5. This yields a model with d.o.f. on the codimension-1 boundary, to which we wish to apply strong-coupling expansions or other nonperturbative approximation methods.
6. Difficult to implement 5., because the functional integration is over a $(d-1)$-dimensional region. We therefore pick a finite number of points at the boundary, with values of the field (spins) assigned, and

   ***interpolate***

   to a solution of the field equations inside the FU. **Result: an effective lattice model.**

**Summary:** FURG is a real-space (block-spin) RG transformation, via semiclassics.


Brief
Femtouniverse
$x^0 = it$
$x^1$
$x^{d-1}$
Femtouniverse R.G.
Boundary
data
$\int d[A] e^{-S}$
d-dim
F.I.
(d-1)-dim.
F.I.

Yang-Mills in $d = 4$ is still too hard. We apply FURG to the O(3) nonlinear sigma model ($\mathbb{CP}^1$ model) for $d = 2$. **This work is not finished yet.**

$$Z = \int [d^3 n] \delta(\vec{n} \cdot \vec{n} - 1) e^{-S},$$

$$S = S_{\rm NLSM} + S_\Theta = \frac{1}{2g_0} \int d^2x \ \vec{n} \cdot (-\partial^2) \vec{n} + \frac{\mathrm{i}\Theta}{4\pi} \int d^2x \ \vec{n} \cdot (\partial_0 \vec{n} \times \partial_1 \vec{n}) \ .$$

Spin-1 AF chains, $\mathrm{A_2Cu_3O(SO_4)_3}$ where $\mathrm{A_2 = Na_2, NaK, K_2}$ , with $T < 3\ K$ (Haldane correspondence).

**Background field method:**

Integrate over "fast" degrees of freedom between $r$ (UV regulator) and FU size $R$ (IR regulator). The remaining "slow" degrees of freedom lie in the boundary data.

$\overline{\phi}$, $\phi$ are fluctuation fields. Gauge field $A_\mu = \vec{e}_c \cdot \partial_\mu \vec{e}_b$, where $\vec{n}, \vec{e}_1, \vec{e}_2$ are a system of orthonormal vector fields.

$$\begin{aligned} Z(\text{boundary data}) &= \int [d\phi] e^{-S_{\text{NLSM}}} \\ &\approx \exp - \int_{|x|\le R} d^2x \left[ \frac{1}{2g(r)} \vec{n}_{\text{cl}} \cdot (-\partial^2) \vec{n}_{\text{cl}} - \frac{1}{2\pi} \ln \frac{r}{L} \; A^\mu A_\mu \right] \\ &\times \det{}^{-1} \left[ -\mathscr{D}^2 - \frac{1}{2} \vec{n}_{\text{cl}} \cdot (-\partial^2) \vec{n}_{\text{cl}} \right]_{r,R}, \end{aligned}$$

$$\langle \overline{\phi}(x) \phi(y) \rangle_{r,R} = \frac{2}{-\partial^2} \delta^2(x-y) \bigg|_{r,R} = -\frac{1}{2\pi} \ln \frac{R^2 |x-y|^2 + R^3 r}{R^4 - 2R^2 x \cdot y + |x|^2 |y|^2 + R^3 r}.$$

Leading order in pert. theory:

$$Z(\text{boundary data}) = e^{-\int d^2x\, \mathscr{L}_{\text{eff}\, r,R}},$$

where

$$\begin{aligned}\mathscr{L}_{\text{eff}\, r,R} \;=\; & \left[\frac{1}{2g(r)} - \frac{1}{2\pi}\ln\frac{R}{r}\right] \vec{n}_{\text{cl}}\cdot(-\partial^2_{r,R})\vec{n}_{\text{cl}} - \frac{1}{2\pi}\ln\frac{R}{L}\; A^\mu A_\mu \\ - \; & \frac{1}{2\pi}\ln\frac{(R^2-|x|^2)^2+R^3r}{R^4}\left[\vec{n}_{\text{cl}}\cdot(-\partial^2_{r,R})\vec{n}_{\text{cl}} + A^\mu A_\mu\right].\end{aligned}$$

Asymptotic freedom:

$$\frac{1}{2g(R)} = \frac{1}{2g(r)} - \frac{1}{2\pi}\ln\frac{R}{r}.$$

$$\beta(g) = -\frac{\partial g(R)}{\partial \ln R} = -\frac{1}{\pi}g(R)^2.$$

**Interpolation**

Stereographic coordinates for orthogonormal system of vectors:

$$\vec{n} \;=\; \frac{1}{\overline{w}w+1}(\overline{w}+w,\, \mathrm{i}\overline{w}-\mathrm{i}w,\, \overline{w}w-1), \;\; \vec{e}_1 = \frac{1}{2\sqrt{\overline{w}w}}(-\mathrm{i}\overline{w}+\mathrm{i}w,\, \overline{w}+w,\, 0),$$

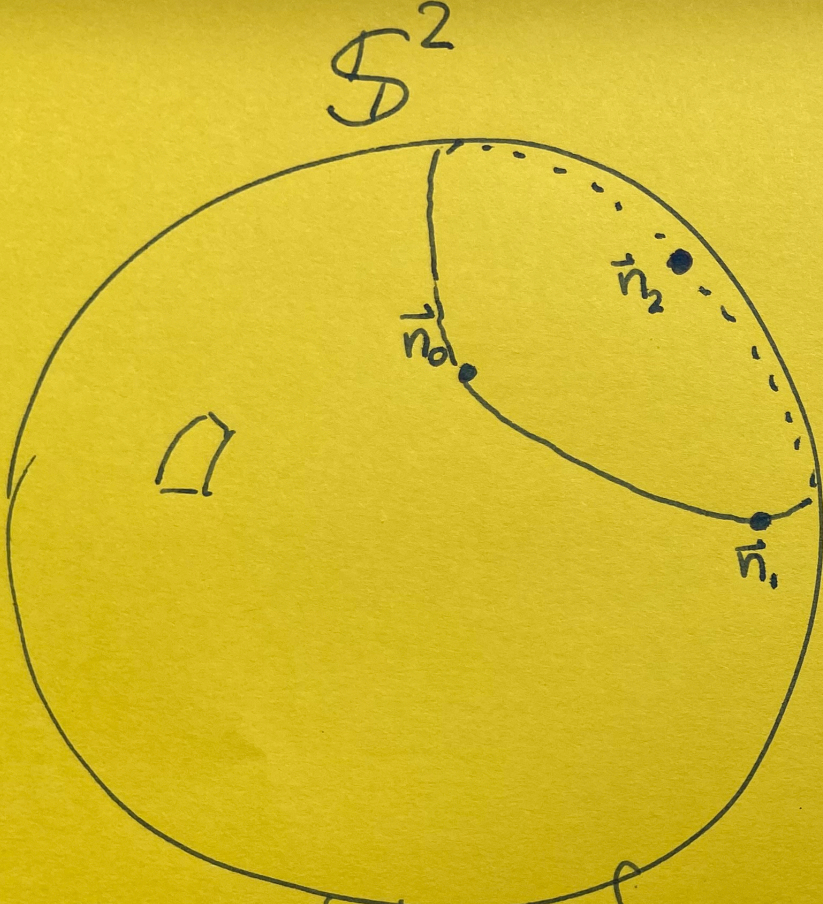
$$\vec{e}_2 \;=\; \frac{1}{2(\overline{w}w+1)\sqrt{\overline{w}w}}[-(\overline{w}w-1)(\overline{w}+w),\; -\mathrm{i}(\overline{w}w-1)(\overline{w}-w),\; 4\overline{w}w].$$

Note $w$, $\overline{w}$ are the fields in the $\mathbb{CP}^1$ formulation.

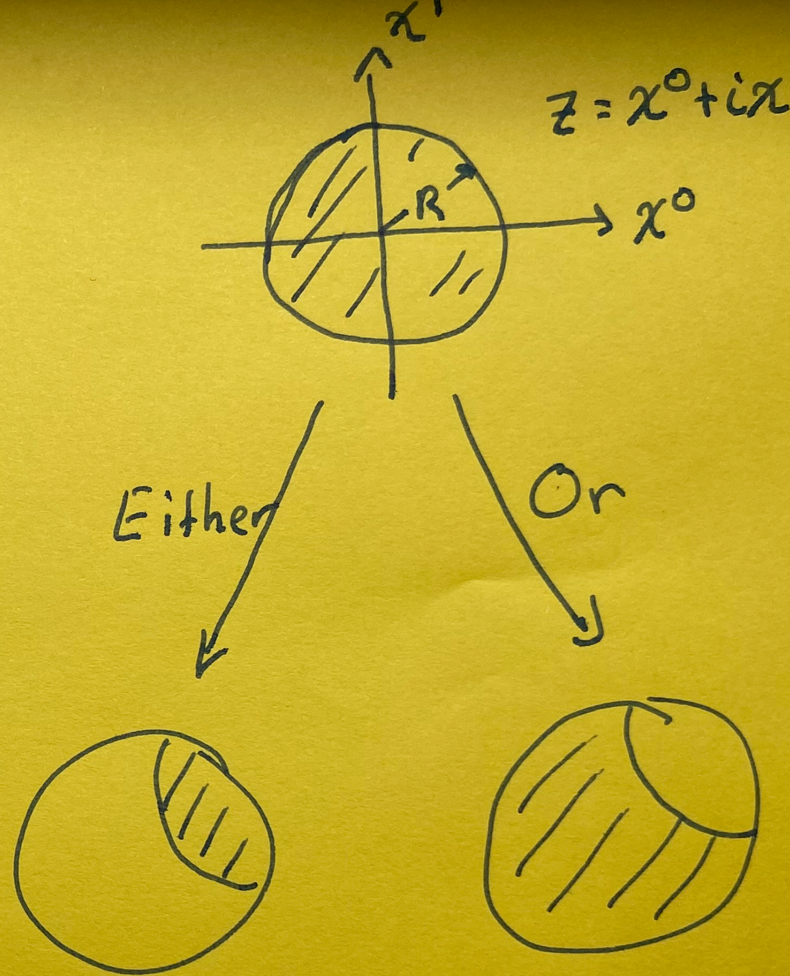
Given any three spins $\vec{n}_0$, $\vec{n}_1$ and $\vec{n}_2$, located at three equidistant points of the circle (disc boundary), we can interpolate to self-dual $(w = w(z))$ or anti-self-dual solutions $(w = w(\bar{z}))$ inside the disc.

Central point: $\vec{n}_{\text{center}} = \frac{\vec{N}}{N}$,

$$\vec{N} = \vec{n}_0 \times \vec{n}_1 + \vec{n}_1 \times \vec{n}_2 + \vec{n}_2 \times \vec{n}_0.$$




Interpolation from three spins




Depending on $\vec{n}_0, \vec{n}_1, \vec{n}_2$ and whether $\vec{n}$ is self-dual or anti-self-dual.

From $\vec{n}_0$, $\vec{n}_1$ and $\vec{n}_2$, we can reconstruct the angles $\theta$, $\varphi_1$ and $\varphi_2$ in the parametrization, after a rotation,

$$\begin{aligned}\vec{n}_{\text{center}} &= (0,0,1),\ \ \vec{n}_0 = (\sin\theta, 0, \cos\theta),\\ \vec{n}_1 &= (\sin\theta\cos\varphi_1, \sin\theta\sin\varphi_1, \cos\theta),\\ \vec{n}_2 &= (\sin\theta\cos\varphi_2, \sin\theta\sin\varphi_2, \cos\theta).\end{aligned}$$

The circle is the locus of:$(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$, fixed $\theta$, arbitrary $\varphi \in [0, 2\pi)$. In stereographic coordinates, $w = \cot\frac{\theta}{2}e^{\mathrm{i}\varphi}$, $\overline{w} = \cot\frac{\theta}{2}e^{-\mathrm{i}\varphi}$.

## The interpolation gives: Instantons!

The instanton partition function was written as a Coulomb gas many decades ago.

Fateev, Frolov and Schwarz

Berg and Lüscher

There remain questions about instanton-antiinstanton configurations, unitarity and how the mass gap forms.

**In our approach there can be instantons in some FU and anti-instantons in others.**

Recall

$$I = \int d^n x\, f(x) e^{-S(x)} \approx C \int_M d^k\varphi\, \sqrt{\det(G)(\varphi)}\, f[x(\varphi)]\, \frac{e^{-S_0}}{\sqrt{\det'(\delta^2 S)}},$$

$$G_{ij}(t) = \sum_{l=1}^{k} \frac{\partial x_l(\varphi)}{\partial \varphi_i}\frac{\partial x_l(\varphi)}{\partial \varphi_j}, \quad \delta^2 S_{ij} = \left.\frac{\partial^2 S(x)}{\partial x_i \partial x_j}\right|_{x(\varphi)\in M}, \quad \left.\frac{\partial S(x)}{\partial x_j}\right|_{x(\varphi)\in M} = 0.$$

## Instantons (continued):

Instanton solution is a $q \longrightarrow 1$ map:

$$w(z) = \cot\frac{\theta}{2} \prod_{j=1}^{q} \frac{a^{(j)} z + b^{(j)} R}{\overline{b}^{(j)} z + \overline{a}^{(j)} R},$$

Angles $\varphi_1^{(j)},\ \varphi_2^{(j)} \in [0, 2\pi)$, with $j = 1, \ldots, q$, and

$$\sum_{j=1}^{q} \varphi_1^{(j)} = \varphi_1 + 2\pi q_1, \quad \sum_{j=1}^{q} \varphi_2^{(j)} = \varphi_2 + 2\pi q_1 + 2\pi q_2,$$

for integers $q_0,\ q_1,\ q_2 = 0, 1, 2\ldots,$ with $q = q_0 + q_1 + q_2 + 1$,
$T_1^{(j)} = \tan[\varphi_1^{(j)}/2]$, $T_2^{(j)} = \tan[\varphi_2^{(j)}/2]$, and

$$\begin{aligned} a^{(j)} &= \sqrt{3}\left[T_2^{(j)} - T_1^{(j)}\right] + 2T_1^{(j)} T_2^{(j)} - \mathrm{i}\left[T_1^{(j)} + T_2^{(j)}\right], \\ b^{(j)} &= \sqrt{3}\left[T_2^{(j)} - T_1^{(j)}\right] - 2T_1^{(j)} T_2^{(j)} + \mathrm{i}\left[T_1^{(j)} + T_2^{(j)}\right]. \end{aligned}$$

The manifold $M$ of saddle-points is parametrized by $\varphi_{1,2}^{j}$.

The classical action of the instanton is given by $S_0 = S_{\mathrm{NLSM}} + S_\Theta$,

$$S_{\mathrm{NLSM}} = \frac{2\pi q}{g_0} \begin{cases} 1-\cos\theta, & \varphi_1 < \varphi_2, \\ 1+\cos\theta, & \varphi_1 > \varphi_2 \end{cases},$$

$$S_\Theta = -\frac{\mathrm{i}\Theta q}{2} \begin{cases} 1-\cos\theta, & \varphi_1 < \varphi_2, \\ 1+\cos\theta, & \varphi_1 > \varphi_2 \end{cases}.$$

There are similar formulas for antiinstantons.

The metric on $M$ is

$$G_{\{i,a\}\{j,b\}} = \int_{\bar{z}z<R^2} \frac{d\bar{z}dz}{2\mathrm{i}[\overline{w}(\bar{z})w(z)+1]^2} \frac{\partial \overline{w}(\bar{z})}{\partial \phi_a^{(i)}} \frac{\partial w(z)}{\partial \phi_b^{(j)}}.$$

The fluctuation operator:

$$\delta^2 S = (-\mathscr{D}^2 - F_{01})^2, \quad \mathscr{D}_\mu = \partial_\mu - \mathrm{i}A_\mu, \; A_\mu = -\frac{\mathrm{i}}{2}\frac{\overline{w}w-1}{\overline{w}w+1}\partial_\mu \ln\frac{w}{\overline{w}}.$$

The work ahead is in the determinants of these two operators and the integration over moduli $\{\varphi_{1,2}^{(j)}\}$. In the effective lattice model, not only must the spins be matched between FU, but the integers $q_{0,1,2}$ must be matched at edges (links).

An improvement is to put spins at vertices of equilateral triangles, which tile. Repeat the above with Dixonian elliptic function $\mathrm{sm}(\zeta)$, which conformally maps an eq. triangle into the unit disc:

$$\zeta = \int_0^{\mathrm{sm}(\zeta)} \frac{du}{(1-u^3)^{3/2}}.$$

For example

$$\langle\overline{\phi}(x)\phi(x')\rangle_{r,R} = \frac{2}{-\partial^2}\delta^2(x-x')\Big|_{r,R} = -\frac{1}{2\pi}\ln\frac{|\mathrm{sm}(z/R)-\mathrm{sm}(z'/R)|^2+r/R}{|1-\overline{\mathrm{sm}(z/R)}\mathrm{sm}(z'/R)|^2+r/R},$$

$z = x^0 + \mathrm{i}x^1$, $z' = x^{0\,\prime} + \mathrm{i}x^{1\,\prime}$, and the triangle has side $\frac{\sqrt{3}}{6\pi}\Gamma\left(\frac{1}{3}\right)^3 R$.

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**and THANK YOU!**