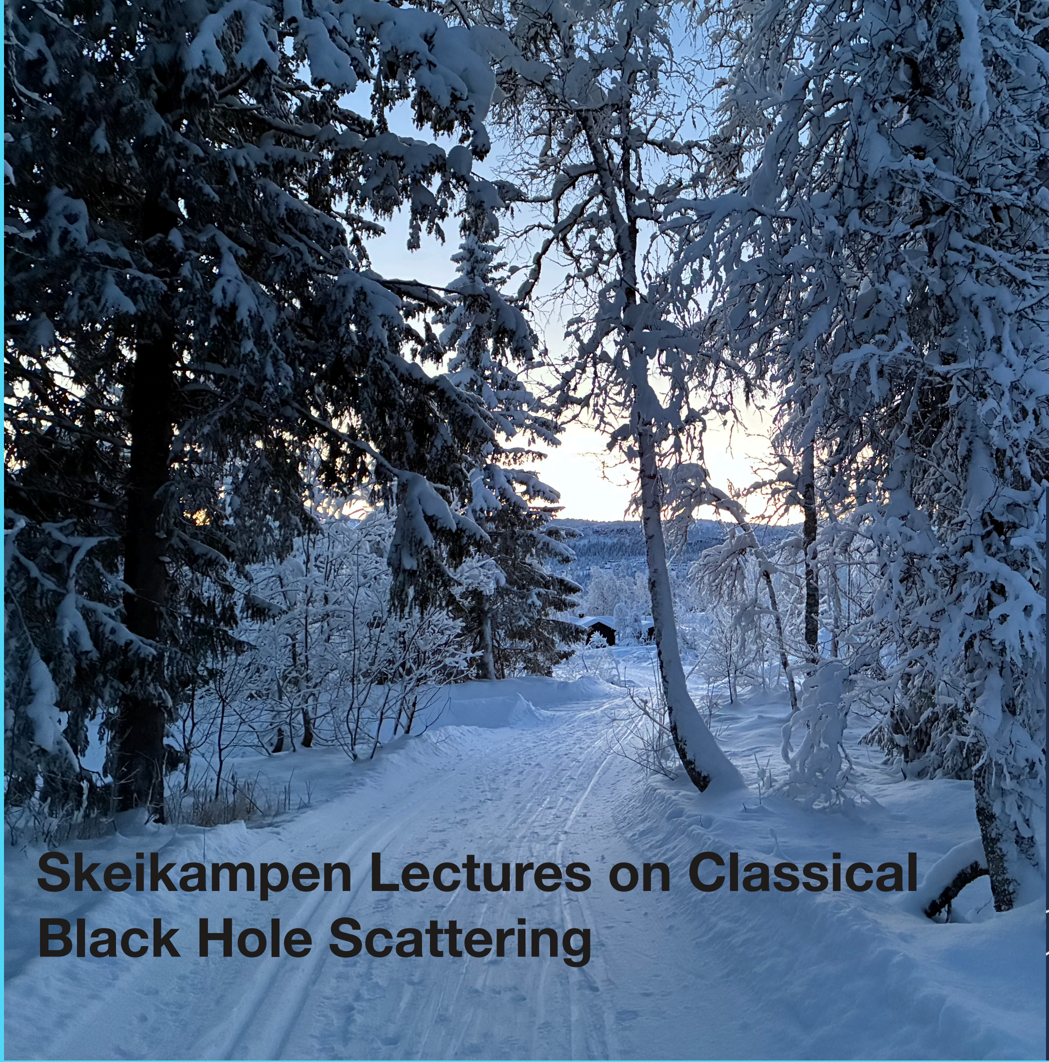
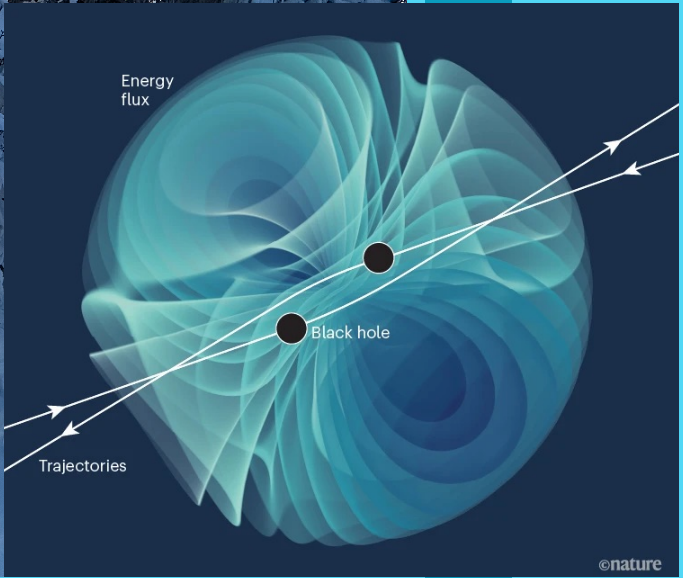
Skeikampen Lectures on Classical Black Hole Scattering





PLAN:

1) WORLDLINE EFT & GRAVITATIONAL 2-BODY PROBLEM

2) BH/NS SCATTERING: WORLDLINE QFT APPROACH:

[Diagrammar; Observables; GSF expansion; Feynman integral structure]

3) IMPULSE Δp @ 1PM & 2PM

4) HIGH PM WORKFLOW: INTEGRAND, IBPs, DE, BOUNDARIES

5) SPIN

6) GEODESIC MOTION FROM WQFT

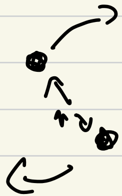
7) SCATTERING WAVEFORM @ LO.

REFERENCES: PhD thesis of GUSTAV UHRE JAKOBSEN 2308.04388

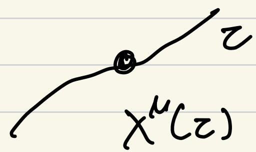
Physics Reports (2026?)

0) MOTIVATION → INTRO SLIDES.

1) WORLDLINE EFFECTIVE FIELD THEORY & THE GRAVITATIONAL 2-BODY PROBLEM



$Gm \ll r$ Consider BHs/NSs as point particles



1) $$S_{BH} = -m \int ds = -m \int_{-\infty}^{\infty} d\tau \sqrt{\dot{x}^\mu \dot{x}^\nu g_{\mu\nu}}$$

Polyakov-trick:

$$\boxed{S_{BH} = \tilde{S}_{BH} = -\frac{m}{2} \int d\tau \left(e^{-1} \dot{x}^2 + e \right)}$$

$e(\tau)$

$$\frac{\delta \tilde{S}_{BH}}{\delta e} = 0 = 1 - e^{-2} \dot{x}^2 \Rightarrow \underline{e = \sqrt{\dot{x}^2}}$$

Reinsert e in $S_{BH}[e=\sqrt{\dot{x}^2}] = S_{BH}$

$\boxed{\text{Proper time gauge:} \quad e=1}$ $\Rightarrow$ $\boxed{S_{BH} = -\frac{m}{2}\int d\tau\, \dot{x}^2}$

Constraint $\underline{1 = g_{\mu\nu}\, \dot{x}^\mu(\tau)\, \dot{x}^\nu(\tau)}$

Constraint is preserved under dynamics

$$\frac{d}{d\tau}\left(g_{\mu\nu}\, \dot{x}^\mu \dot{x}^\nu\right) \underset{\text{E.o.m}}{=} 0$$

<u>E.o.M:</u> $\boxed{\ddot{x}^\mu + \Gamma^\mu{}_{\rho\sigma}\, \dot{x}^\rho \dot{x}^\sigma = 0}$

2) $S_{GR} = \frac{1}{16\pi G} \int d^4x \sqrt{-g}\, R$ $\rightarrow$ EINSTEIN EQS.

$\Rightarrow$ $S = S_{BH1} + S_{BH2} + S_{GR} + S_{G.F.}$

WORLDLINE EFT DESCRIPTION OF BHs/NSs

$$S = -\frac{m}{2}\int d\tau\, g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + \begin{pmatrix}\text{SPIN}\\ \text{DOFS}\end{pmatrix} + \begin{pmatrix}\text{TIDAL}\\ \text{EFFECTS}\end{pmatrix} + \frac{1}{16\pi G}\int d^4x\sqrt{g}\,R$$

$$C_{E^2}\int d\tau\,(R_{\mu\dot{x}\nu\dot{x}})^2 + C_{B^2}(\tilde{R}_{\mu\dot{x}\nu\dot{x}})^2$$

SPINLESS SITUATION:

$$R_{\mu\nu\rho\sigma}\dot{x}^\nu = R_{\mu\dot{x}\rho\sigma}$$

3) WQFT DO EXPANSION IN "G"

$$\left(\frac{GM}{b}\right)^{\#}$$

Background field expansion:

$$x^\mu(\tau) = b^\mu + v^\mu\cdot\tau + z^\mu(\tau)$$

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \sqrt{G}\,h_{\mu\nu}(x)$$

Solve for z^μ & $h_{\mu\nu}$ by quantising them!

$$\langle \mathcal{O} \rangle_{\text{WQFT}} = \int \mathcal{D}[z^\mu, h_{\mu\nu}] \; \mathcal{O}[z,h] \; e^{-\frac{i}{\hbar} S[z,h]} \Big/ Z$$

$\Rightarrow$ TREE-LEVEL ONE POINT FUNCTIONS SOLVE

$\hbar \to 0$ CLASSICAL EQUATIONS OF MOTION

$$\langle z^\mu(\tau) \rangle = \sum_{n=1}^{\infty} G^n \, z^\mu_{(n)}(\tau) \qquad \sqrt{G} \langle h_{\mu\nu}(x) \rangle = \sum_n G^n \, h_{(n)\mu\nu}$$

CLASSICAL TRAJ. WAVEFORM

$$\langle x \rangle = \int dx \; x \; p(x)$$

WHY 1-POINT FUNCTIONS @ TREE LEVEL

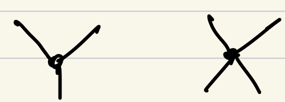
ϕ: Scalar Field

$$Z = \int \mathcal{D}\phi \; e^{-\frac{i}{\hbar} S[\phi]} = \int \mathcal{D}\phi \sum \frac{g^n \phi^n}{\hbar^n} \, e^{-\frac{i}{\hbar}\int \phi \partial^2 \phi}$$

$$S[\phi] = \int \frac{1}{2} \phi \partial^2 \phi + g_1 \phi^3 + g_2 \phi^4$$

$$\frac{1}{2}\int d^4x \, \partial_\mu \phi \, \partial^\mu \phi$$

$$\langle \phi \phi \rangle_{g=0} \sim \frac{\hbar}{p^2} \qquad x \xrightarrow{\;p\;} y$$

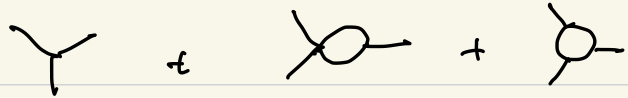


$\sim g_1/\hbar$ $\qquad$ $\sim g_2/\hbar$

Hence if we look at connected n-point fncts

$$\langle \phi_1 \dots \phi_n \rangle_{con} = \sum_{l=0}^{\infty} (\hbar)^{l+n-1} \left(\text{graphs with } n\text{-external legs and } n\text{-loops} \right)$$

Makes sense only for $l=0$ & $n=1$ $\Rightarrow$ TREE-LEVEL 1-PT FCT.

3-pt: $\langle \phi\phi\phi \rangle =$ 

TREE   1-LOOP

$$\hbar^3 \frac{1}{\hbar} + \hbar^3 \frac{1}{\hbar^2}\cdot\hbar^2 \quad \hbar^3 \frac{1}{\hbar^3}\hbar^3 = \hbar^2 + \hbar^3$$

$$\boxed{\langle \phi_1 \dots \phi_N \rangle_{\mathrm{DISCON}} \overset{\hbar\to 0}{=} \langle \phi_1 \rangle^{\mathrm{TREE}}_{\mathrm{CON}} \langle \phi_2 \rangle^{\mathrm{TREE}}_{\mathrm{CON}} \cdots \langle \phi_N \rangle^{\mathrm{TREE}}_{\mathrm{CON}}} + O(\hbar) + O(1/\hbar)$$

Schwinger-Dyson-Eq!

$$0 = \left\langle \frac{\delta S(\phi)}{\delta \phi} \right\rangle_{\hbar \to 0} \longrightarrow 0 = \frac{\delta S[\langle \phi \rangle^{\mathrm{TREE}}]}{\delta \phi}$$

FEYNMAN DIAGRAMATICS

Worldline deflection:

$$S_{WL}^{(0)} = -\frac{m}{2}\int d\tau \left[\eta_{\mu\nu}\, \dot{x}^{\mu}(\tau)\, \dot{x}^{\nu}(\tau) + 1 \right]$$

$\hookrightarrow \dot{x}^{\mu}(\tau) = v^{\mu} + \dot{z}^{\mu}(\tau)$

$$= -\frac{m}{2}\int d\tau \left[1 + 2\, \dot{z}\cdot v + \dot{z}^2 \right]$$

$\hookrightarrow z(\infty) - z(-\infty) \to 0$

$\underbrace{}_{}$ DROP $\Leftrightarrow$ $(w \sim iw)$

F.T:

$$z(\tau) = \int_{\omega} e^{-i\omega\cdot\tau}\, \tilde{z}(\omega)$$

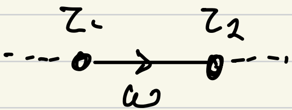
$$= -\frac{m}{2}\int d\tau \; \dot{z}^2$$

$$= -\frac{m}{2}\int_{\omega} \omega^2\, \tilde{z}(\omega)\cdot \tilde{z}(-\omega)$$

$$\boxed{\left\langle z^{\mu}(\omega)\; z^{\nu}(-\omega) \right\rangle = -\frac{i}{m}\, \frac{\eta_{\mu\nu}}{(\omega + i0^{+})^2}}$$

PROPAGATOR

E.O.M: $m\,\partial_\tau^2 X^\nu = \delta(\tau)$ $\Leftrightarrow$ $i\,m\,\omega^2 X^\mu(\omega) = 1$

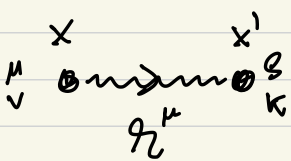


$$= -i\,\frac{\eta^{\mu\nu}}{m}\int_\omega \frac{e^{i\omega(\tau_1-\tau_2)}}{(\omega+i0^+)^2} = \frac{i\eta^{\mu\nu}}{2m}\Big(|\tau_1-\tau_2| + (\tau_1-\tau_2)\Big)$$

this is the retarded Green's function for $m\,\partial_\tau^2$ on $\mathbb{R}$

$\Box h_{\mu\nu}(x) = 0$

GRAVITON PROPAGATOR:



$$= \int_k \frac{i\,P_{\mu\nu\rho\kappa}\,e^{ik\cdot(x-x')}}{(k^0+i0^+)^2 - \vec{k}^2} \underset{D=4}{=} \frac{\delta[(x-x')^2]}{4\pi|\vec{x}-\vec{x}'|}\,P_{\mu\nu\rho\kappa}$$

$p^\rho p^\kappa$ $\quad P_{\mu\nu\rho\kappa} = \frac{1}{2}\left(\eta_{\mu\rho}\eta_{\nu\kappa} + \eta_{\nu\rho}\eta_{\mu\kappa}\right) - \frac{1}{D-2}\eta_{\mu\nu}\eta_{\rho\kappa}$

VERTEX RULES:

a) WORLDLINE VERTICES

: "quantum" fields

$$\boxed{S_{WL}^{WT} = -\frac{m\kappa}{2}\int_\tau h_{\mu\nu}[x(\tau)]\,\dot{x}^\mu(\tau)\,\dot{x}^\nu(\tau)}$$

$$= \frac{m\kappa}{2}\int_\tau h_{\mu\nu}[x(\tau)]\left(v^\mu v^\nu + 2v^\mu \dot{z}^\nu(\tau) + \dot{z}^\mu(\tau)\,\dot{z}^\nu(\tau)\right)$$

$$\hookrightarrow\; h_{\mu\nu}[x(\tau)] = \int_k e^{ik\cdot(b+v\cdot\tau+z(\tau))}\,h_{\mu\nu}(-k)$$

$\int_k = \int \frac{dk}{(2\pi)}$ $\quad \hat{\delta}(x) = 2\pi\delta(x)$

$$z^\mu(\tau) = \int_\omega e^{i\omega\tau}\,z^\mu(\omega)$$

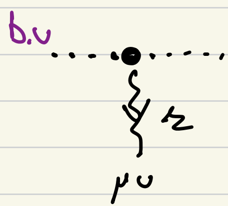
$$= \sum_{n=0}^{\infty} \frac{i^n}{n!}\int_k e^{ik(b+v\tau)}\,\underbrace{[k\cdot z(\tau)]^n\,h_{\mu\nu}(-k)}_{\hookrightarrow\ \text{F.T.}}$$

$$= \sum_{i=0}^{\infty} \frac{i^n}{n!} \int_{r,\omega_1\ldots\omega_n} e^{ir\cdot b}\, e^{i(r\cdot v + \sum_i \omega_i)\cdot\tau} \left(\prod_{i=1}^{n} r\cdot z(-\omega_i)\right) h_{\mu\nu}(-r)$$

Insert this to get GRAVITON-WL VERTICES:

n=0 :

$$S_{\rm INT}\Big|_{z^0} = -\frac{mK}{2}\int_r e^{ir\cdot b}\,\delta(r\cdot v)\, h_{\mu\nu}(-r)\, v^\mu v^\nu$$

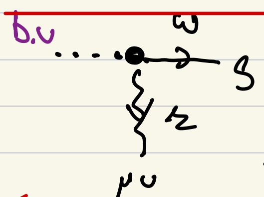


$$= -i\frac{mK}{2}\, e^{ir\cdot b}\,\delta(r\cdot v)\, v^\mu v^\nu$$

$$\kappa^2 = 32\pi G$$

n=1 :

$$S_{\rm INT}\Big|_{z^1} = -i\frac{mK}{2}\int_{r,\omega} e^{ir\cdot b}\,\delta(r\cdot v+\omega)\, h_{\mu\nu}(-r)\, z^\rho(-\omega)\left(2\omega\, v^{(\mu}\delta^{\nu)}_\rho + v^\mu v^\nu r_\rho\right)$$

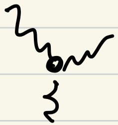
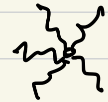


$$= \frac{mK}{2}\, e^{ir\cdot b}\,\delta(r\cdot v+\omega)\left(2\omega\, v^{(\mu}\delta^{\nu)}_\rho + v^\mu v^\nu r_\rho\right)$$

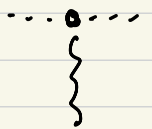
b) GRAVITON VERTICES

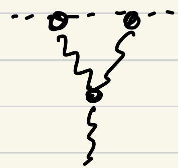
$$S_{EH} = \frac{1}{16\pi G}\int d^D x \sqrt{-g}\, R = \sum_{n=2}^{\infty} \sqrt{G}^{\,n-2}\ \text{“}\partial^2 h^n\text{”}$$

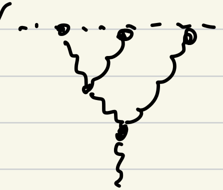
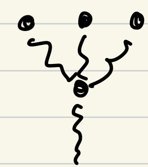
$g = \eta + \sqrt{G}\, h$

 $\sim \sqrt{G}$  $\sim G$  $\sim G^{3/2}$ ...

FUN WITH A SINGLE WORLDLINE

$\sqrt{G}\,\langle h_{\mu\nu}\rangle =$ 

$+$ 

$+ \Big($  $+$  $\Big) + \dots$

G ✓ $\qquad G^2 \qquad G^3$

$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$

Yields PM expansion of Schwarzschild (harmonic coordinates)

AICHELBURG-SEXL FROM WQFT

Looks at massless point particle $(e \to e \cdot m)$

$$S = -\frac{1}{2}\int d\tau \left[e^{-1} \dot{x}^2 - e\,\cancel{m}^2 \right] \xrightarrow[m=0]{} \underline{-\frac{1}{2}\int d\tau\, e^{-1} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}$$

E.O.M for e: $\boxed{0 = \dot{x}^2}$ $\quad e = 1/p \quad\quad x^\mu(\tau),\ e^{-1}(\tau)$

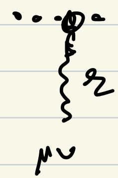
Energy of particle

$$S_{\text{MASSLESS}} = -\frac{p}{2}\int d\tau\, g_{\mu\nu}[x]\, \dot{x}^\mu \dot{x}^\nu$$

Background field exp. $\quad x^\mu(\tau) = b^\mu + p \cdot \tau + z^\mu(\tau)$

$$p^2 = 0$$

$$\langle h_{\mu\nu}(x)\rangle = \int_r e^{ir\cdot x}$$



$$= \int_r e^{ir\cdot x}\left(-\frac{iK}{2}\,\delta(r\cdot P)\; P^\rho P^\kappa\right) i\,\frac{P_{\rho\kappa,\mu\nu}}{r^2}$$

$$= \frac{K}{2}\, P^\mu P^\nu \int_r e^{ir\cdot x}\,\delta(P\cdot r)\,\frac{1}{r^2}$$

Light-cone coords: $r^\mu = n^\mu (r\cdot\bar n) + \bar n^\mu (r\cdot n) + r_\perp^\mu$ $\qquad P^\mu = P\, n^\mu$

$x^+ = n\cdot x \qquad x^- = \bar n\cdot x \qquad\qquad n\cdot\bar n = 1 \quad n^2 = 0 = \bar n^2$

$$\int_r e^{ir\cdot x}\,\delta(P\cdot r)\,\frac{1}{r^2} = P^{-1}\int \frac{d^{D-2}r_\perp\, \cancel{dr}\, d\bar r}{(2\pi)^D}\, e^{i(\bar r\cdot x^- + \cancel{r\cdot x^+} - r_\perp\cdot x_\perp)}\, \frac{\delta(\cancel{r})}{\cancel{2r\bar r} - r_\perp^2}$$

$$= - \delta(x^-)\, p^{-1} \underbrace{\int \frac{d^{D-2}k_\perp}{(2\pi)^{D-2}} \frac{e^{-ik_\perp x_\perp}}{k_\perp^2}}$$

$$D = 4 - 2\varepsilon$$

$$= \frac{\Gamma(-\varepsilon)}{4\pi} \left(\pi x_\perp^2\right)^{\varepsilon} \underset{\varepsilon \to 0}{=} \log\left(x_\perp^2 / L^2\right)$$

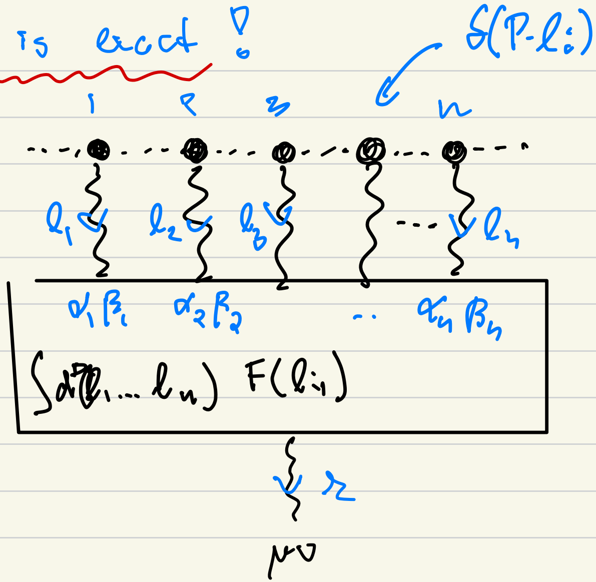
$$\Rightarrow \quad \kappa \langle h_{\mu\nu} \rangle = 4\, G\, P^{-1} \delta(x^-) \log\left(\frac{x_\perp^2}{L^2}\right) P^\mu P^\nu$$

TAKE $\varepsilon \to 0$:

$$\boxed{ds^2 = 2\, dx^+ dx^- - d\vec{x}^2 + 4\, P G\, \delta(x^-) \log\left[\frac{\vec{x}^2}{4L^2}\right] (dx^-)^2}$$

AICHELBURG - SEXL SHOCKWAVE METRIC (1971)

This is exact !

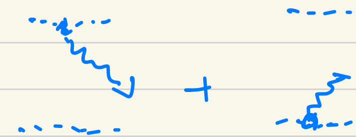


$P^2=0 \qquad P\cdot l_i=0$

$r=\sum_i l_i$

$$\Omega_{\alpha_1\beta_1\ldots\alpha_n\beta_n;\mu\nu}\, p^{\alpha_1}p^{\beta_1}\cdots p^{\alpha_n}p^{\beta_n} \neq$$

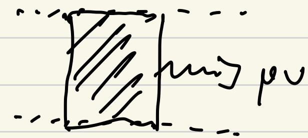
NOW THE 2-BODY PROBLEM



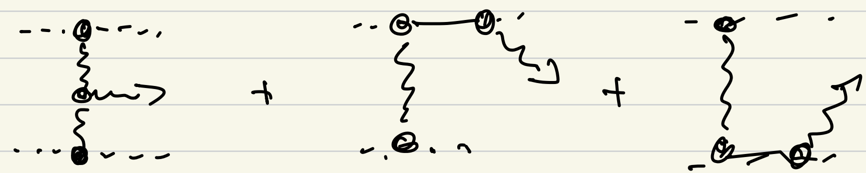
OBSERVABLES:

I WAVEFORM

$$\langle h_{\mu\nu}\rangle^{\text{TREE}} =$$



$$=$$



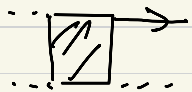
$$G^{3/2}$$

2 IMPULSE:

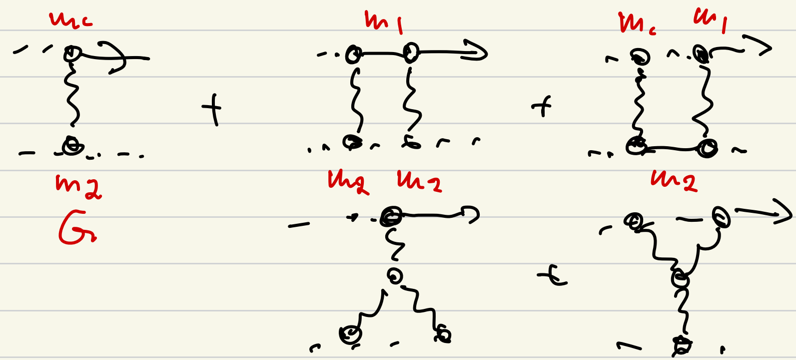
$$\Delta p_1^\mu = m_1 \left\langle \dot{x}_1^\mu \right\rangle \Big|_{\tau=-\infty}^{\tau=\infty} = m_1 \int d\tau \left\langle \ddot{x}_1(\tau) \right\rangle$$

$$\ddot{x}_1 = \ddot{z}_1$$

$$\underset{\text{FT.}}{=} -m_1\,\omega^2 \left\langle z_1^\mu(\omega) \right\rangle \Big|_{\omega \to 0}$$

$\Delta p_1^\mu =$ 

$\omega = 0$

$=$ 

G^2

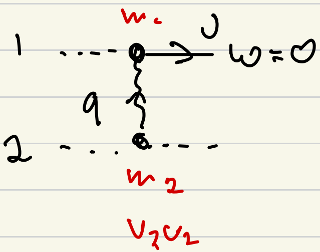
$+ \ G^3 \ + \ G^4 \ + \ G^5$ (STATE OF THE ART) $+ \ldots$

# DIAGS.	26	201	426 + (613)

SELF FORCE EXPANSION: m_1, m_2 $\qquad m_1 \gg m_2$

IMPULSE @ 1PM

$$b = b_2 - b_1$$



$$= 8\pi i G m_1 m_2 \int_q e^{iq\cdot b}\, \delta(v_2\cdot q)\,\delta(q\cdot v_1)\, \frac{P_{\mu\nu; v_2 v_2}\, q^\mu}{q^2}$$

$$P_{\nu q; v_2 v_2} = q^\nu (2\gamma^2 - 1)$$

$$\gamma = v_1 \cdot v_2 = \frac{1}{\sqrt{1 - \vec{v}^2/c^2}}$$

$$\Delta p_1^\mu = 4\pi G m_1 m_2 (2\gamma^2 - 1)\, \partial_{b_1^\mu} \underbrace{\int_q \frac{\delta(q\cdot v_1)\,\delta(q\cdot v_2)}{q^2}\, e^{iq\cdot b}}_{= c + \frac{1}{2\pi\sqrt{\gamma^2-1}} \log |b|}$$

$$= 2 G m_1 m_2 \frac{2\gamma^2 - 1}{\sqrt{\gamma^2 - 1}} \frac{b^\mu}{|b|^2}$$

SCATTERING ANGLE:

$$|\Delta p_1^\mu| = 2 p_\infty \sin(\theta/2)$$

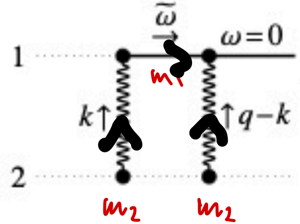
$$p_\infty = m_1 m_2 \sqrt{\gamma^2 - 1} \,/E$$

$$\Rightarrow \quad \boxed{\frac{\theta}{\Gamma} = \frac{GM}{|b|} \frac{2(2\gamma^2 - 1)}{\gamma^2 - 1}}$$

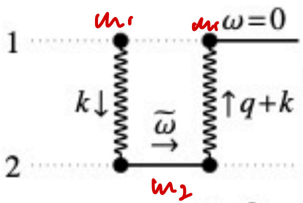
$$\Gamma = \sqrt{1 + 2\nu(\gamma - 1)} \qquad \nu = \frac{m_1 m_2}{m_1 + m_2}$$

$$M = m_1 + m_2$$

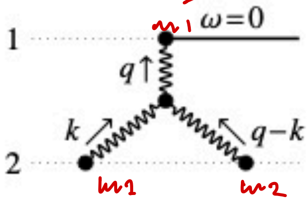
2PM IMPULSE



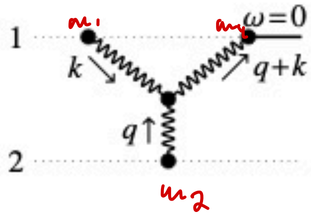
$$= i16\pi^2 G^2 m_1 m_2^2 \int_{k,q} \frac{(q^\mu - k^\mu)\delta(q\cdot v_1)\delta(q\cdot v_2)\delta(k\cdot v_2)}{k^2(k-q)^2(k\cdot v_1 + i0^+)^2} e^{iq\cdot b} \times \left[8\gamma^2(k\cdot v_1)^2 - (2\gamma^2-1)^2 k\cdot(q-k)\right],$$



$$= i16\pi^2 G^2 m_1^2 m_2 \int_{k,q} \frac{(q^\mu + k^\mu)\delta(q\cdot v_1)\delta(q\cdot v_2)\delta(k\cdot v_1)}{k^2(k+q)^2(k\cdot v_2 + i0^+)^2} e^{iq\cdot b} \times \left[8\gamma^2(k\cdot v_2)^2 - (2\gamma^2-1)^2 k\cdot(k+q)\right].$$

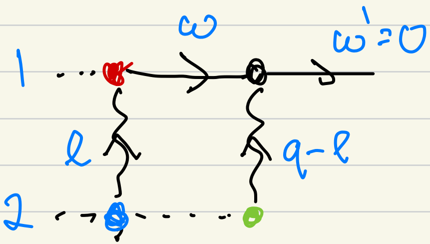


$$= -i64\pi^2 G^2 m_1 m_2^2 \int_{k,q} \frac{q^\mu \delta(q\cdot v_1)\delta(q\cdot v_2)\delta(k\cdot v_2)}{k^2 q^2 (k-q)^2} e^{iq\cdot b} \times \left[\gamma^2 q^2 + (k\cdot v_1)^2 + (2\gamma^2-1) k\cdot(k-q)\right],$$



$$= i64\pi^2 G^2 m_1^2 m_2 \int_{k,q} \frac{(k^\mu + q^\mu)\delta(q\cdot v_1)\delta(q\cdot v_2)\delta(k\cdot v_2)}{k^2 q^2 (k+q)^2} e^{iq\cdot b} \times \left[\gamma^2 q^2 + (k\cdot v_2)^2 - (2\gamma^2-1) k\cdot(k+q)\right].$$

EXAMPLE:



$$\delta(q\cdot v_2) \qquad \omega = \ell\cdot v_1 \qquad \delta(q\cdot v_1)$$

$$\int_{q,\ell,\omega} e^{iq\cdot b}\, \frac{\delta(\ell\cdot v_2)\ \overbrace{\delta[(q-\ell)\cdot v_2]}^{=\,\delta(q\cdot v_2)}\ \delta(\ell\cdot v_1-\omega)\ \overbrace{\delta(\omega-(q-\ell)\cdot v_1)}^{=\,\delta(q\cdot v_1)}}{(\omega+i0^+)^2\ \ell^2\ (q-\ell)^2}$$

$$v_1^2 = 1 = v_2^2$$

$$\gamma = v_1\cdot v_2$$

$$= \int_q e^{iq\cdot b}\, \delta(q\cdot v_1)\, \delta(q\cdot v_2)\ \underbrace{\int_\ell \frac{\delta(\ell\cdot v_2)}{\ell^2\,(q-\ell)^2\,[\ell\cdot v_1+i0^+]^2}}_{|q|^\alpha\ f[\gamma,\,\varepsilon]}$$

@ n-PM ORDER WE HAVE THE GENERAL STRUCTURE

$$\tilde{J}_{\vec{n}} = \int_q \delta(q\cdot v_1)\,\delta(q\cdot v_2)\, e^{-iq\cdot b} \int_{\ell_1\ldots\ell_{n-1}} \frac{\text{num}}{D_1^{n_1}\cdots D_j^{n_j}}\, \underbrace{\delta(\ell_1\cdot V_*)\cdots\delta(\ell_{n-1}\cdot V_*)}$$

$$V_* = \begin{cases} v_1 \\ v_2 \end{cases}$$

MIRRORS

$m_1^{\#}\quad m_2^{\#}$

WITH

$$D_i = \begin{cases} (r^0 + i0^+)^2 - \vec{r}^{\,2} & \\ r\cdot V_* + i0^+ & \end{cases} \qquad r^\mu = \sum_{a=1}^{n-1} c_a \ell_a^\mu + d\, q^\mu$$

$\gamma = v_1\cdot v_2$

$$\tilde{I}_{n_1\ldots n_j}[1] = \int_{\ell_1\ldots\ell_{n-1}} \frac{1}{D_1^{n_1}\cdots D_j^{n_j}}\, \delta(\ell_1\cdot V_*)\cdots\delta(\ell_{n-1}\cdot V_*) = \frac{1}{|q|^\alpha}\, I_{n_1\ldots n_j}[\gamma, \varepsilon]$$

2PM - INTEGRAL

$$I_{n_1 n_2 n_3}[1] = \int_\ell \frac{f(\ell\cdot v_2)\quad [1]}{[\ell^2]^{n_1}\,[(\ell-q)^2]^{n_2}\,[\ell\cdot v_1 + i0^+]^{n_3}} = |q|^{d-1-2n_1-2n_2-n_3}\, I_{\vec{n}}[\gamma, \epsilon]$$

$$[I] = d-1-2n_1-2n_2-n_3$$

Symmetries:

1) $I_{n_1 n_2 n_3} = I_{n_2 n_1 n_3}$ $\qquad \ell \to \ell + q$

2) $I_{0 n_2 n_3} = 0 = I_{n_1 0 n_3}$ $\qquad$ Scaleless

TENSOR RED.

$$\hat{v}_1^\mu = \frac{\gamma v_2^\mu - v_1^\mu}{\gamma^2 - 1} \qquad \hat{v}_i \cdot v_j = \delta_{ij}$$

$$I_{\vec{n}}[\ell^\mu] = c_1\, \hat{v}_1^\mu + c_2\, \hat{v}_2^\mu + c_q\, q^\mu/|q|^2$$

$$I_{\vec{n}}[\ell \cdot v_2] = 0 = c_2 \qquad I_{n_1 n_2 n_3}[\ell \cdot v_1] = I_{n_1 n_2 n_3 - 1}[1] = c_1$$

$$I_{n_1 n_2 n_3}[\ell \cdot q] = c_q \qquad \ell \cdot q = \frac{1}{2}\left[\ell^2 - (\ell - q)^2 + q^2\right]$$

$$\parallel$$

$$\frac{1}{2} I_{n_1 - 1 n_2 n_3}[1] - \frac{1}{2} I_{n_1 n_2 - 1 n_3}[1] + \frac{q^2}{2} I_{n_1 n_2 n_3}[1]$$

HIGHER RANKS:

$$I_{\vec{n}}[\ell^\mu \ell^\nu] = \sum_{ij} c_{ij} \, v_i^{(\mu} v_j^{\nu)} + \sum_i c_i \, v_i^\mu q^\nu + c \, q^\mu q^\nu$$

$$+ d \, \eta^{\mu\nu}$$

$$\Rightarrow \quad I_{112}[\ell \cdot (\ell - q)(q - \ell)^\mu] = -\frac{q^2}{4}\left[I_{112} \, q^\mu - 2 I_{111} v_1^\mu\right]$$

SCALAR INTEGRAL @ 1-LOOP CAN BE PERFORMED WITH STD. QFT TECHNIQUES:

$$I_{n_1 n_2 n_3} = |q|^{d-1-2n_1-2n_2-n_3} \, (-i)^{2n_1+2n_2+n_3} \, \frac{2^{n_1} (4\pi)^{\frac{1-d}{2}}}{\left[\gamma^2-1\right]^{n_3/2}}$$

$$\Gamma_{n_1 n_2 n_3}\left[3-2\epsilon\right]$$

FINAL F.T.

$$\int_q e^{iq\cdot b} \, \delta(q\cdot u_1)\, \delta(q\cdot u_2)\, |q|^n = \frac{1}{\sqrt{\gamma^2-1}} \, \frac{2^n}{\pi^{(d-2)/2}} \, \frac{\Gamma\left(\frac{d-2+n}{2}\right)}{\Gamma\left(-n/2\right)}$$

$$\propto \quad |P_{12} \cdot b|^{2-d-n}$$

$$P_{12}^{\mu\nu} = \eta^{\mu\nu} - \sum_{i=1}^{2} \hat{v}_i^{\mu} \hat{v}_i^{\nu}$$

$b = b_2 - b_1$

$$\boxed{\begin{aligned}\Delta p_1^{\mu} = \frac{G m_1 m_2 b^{\mu}}{|b|^2} \left[\frac{2(2\gamma^2-1)}{\sqrt{\gamma^2-1}} + \frac{3\pi}{4} \frac{5\gamma^2-1}{\sqrt{\gamma^2-1}} \frac{G(m_1+m_2)}{|b|} \right] \\ + \left(m_2 \hat{v}_1^{\mu} - m_1 \hat{v}_2^{\mu} \right) 2 G^2 m_1 m_2 \frac{(2\gamma^2-1)^2}{(\gamma^2-1)|b|^2} + \mathcal{O}(G^3)\end{aligned}}$$

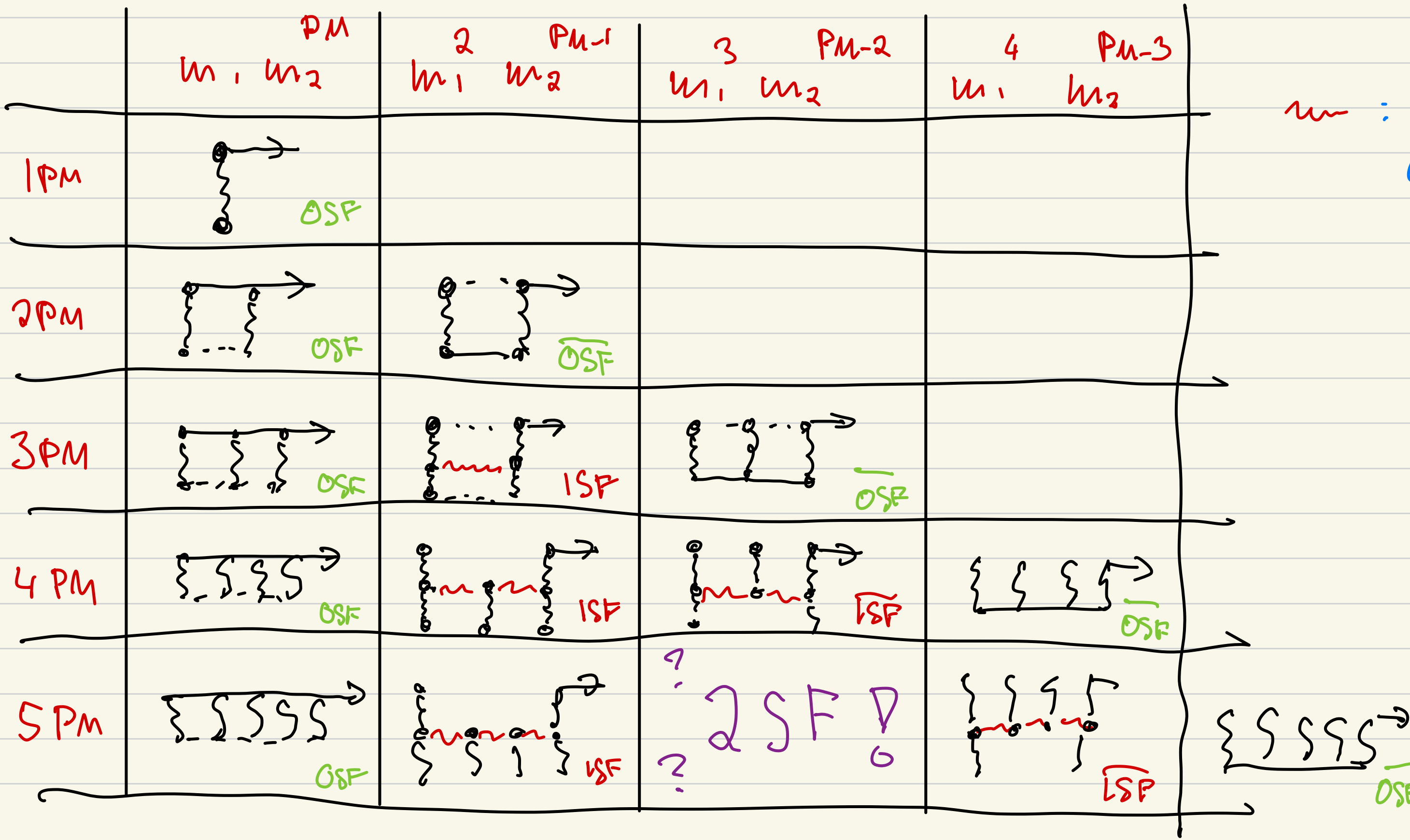
$\Delta p_1 = -\Delta p_2$ upon $1 \leftrightarrow 2$ SWAP

$\Leftrightarrow$ CONSERVATIVE

$p_1^2 = (p_1 + \Delta p_1)^2 = m^2$

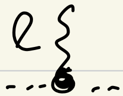
$\hookrightarrow \quad p_1 \cdot \Delta p_1 = \Delta p_1^2$

HIGHER PM ORDERS : PM VS. SF ORDERS



: ACTIVE GRAVITON

$i0^+$ reboot



$$\sim \int_\ell \frac{\delta(\ell\cdot v)}{\ell^2} = \int_\ell \frac{\delta(\ell^0)}{(\ell^0)^2-\vec{\ell}^{\,2}} = \int_{\vec{\ell}} \frac{1}{-\vec{\ell}^{\,2}} = \text{POTENTIAL}$$

RADIAL GRAVITON $\quad (\ell^0, \vec{\ell}) \sim (v, v)$

POTENTIAL " $\quad (\ell^0, \vec{\ell}) \sim (v, 1)$

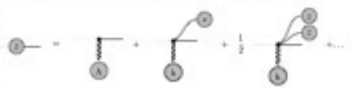
HIGH PRECISION WQFT COMPUTATIONS: WORKFLOW

[Driesse,Jakobsen,Klemm,Mogull,Nega,JP,Sauer,Usovitsch]

$$\Delta p_i^\mu \quad f_{\mu\nu}(u,\hat{x}) \quad \Delta S_i^\mu$$

Generate graphs & Insert Feynman rules

Berends-Giele recursion



(5PM: 6g vertex)

↓

Reduction & planarization to scalar integral family

$$\int \prod_i d^D l_i \frac{l_{i_1}^{\mu_1} \ldots l_{i_n}^{\mu_n}}{\prod_i D(l_i)}$$

↓

$i0^+$ Integration by parts & Master Integrals

Kira 3.0

Reduced symmetries due to retarded prop.

→

Solve masters by Diff. Eq. technique

Single scale integrals

$$x = \gamma - \sqrt{\gamma^2 - 1}$$

$$\gamma = v_1 \cdot v_2$$

↑

$i0^+$ Fix boundary conditions in static limit (PN) $\gamma \to 1$

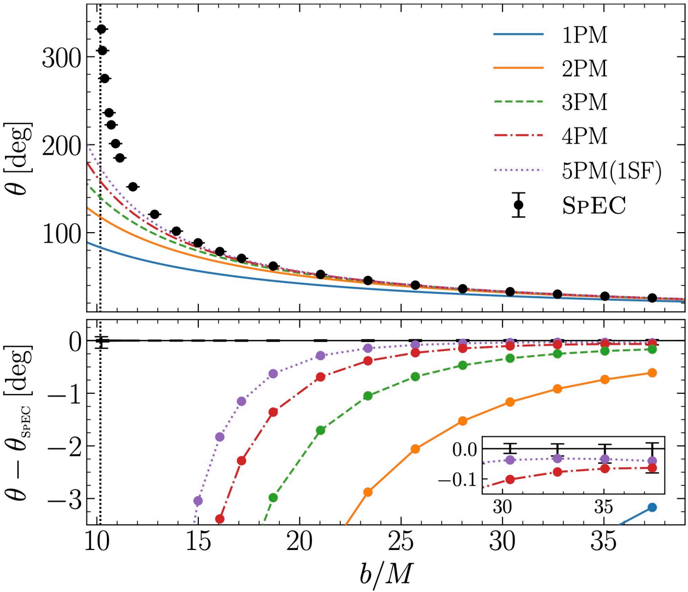
Method of regions (potential & radiation gravitons)

↑

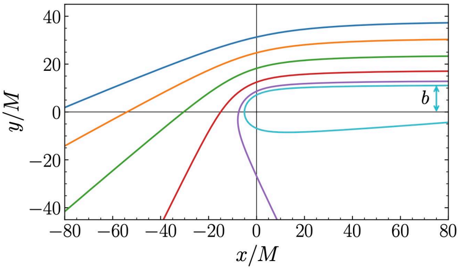
SCATTERING ANGLE: COMPARISON TO NUMERICAL RELATIVITY

[Long, Pfeiffer, Kidder, Scheel, 2511.10196]

Scattering angle



Trajectories:



Parameters:

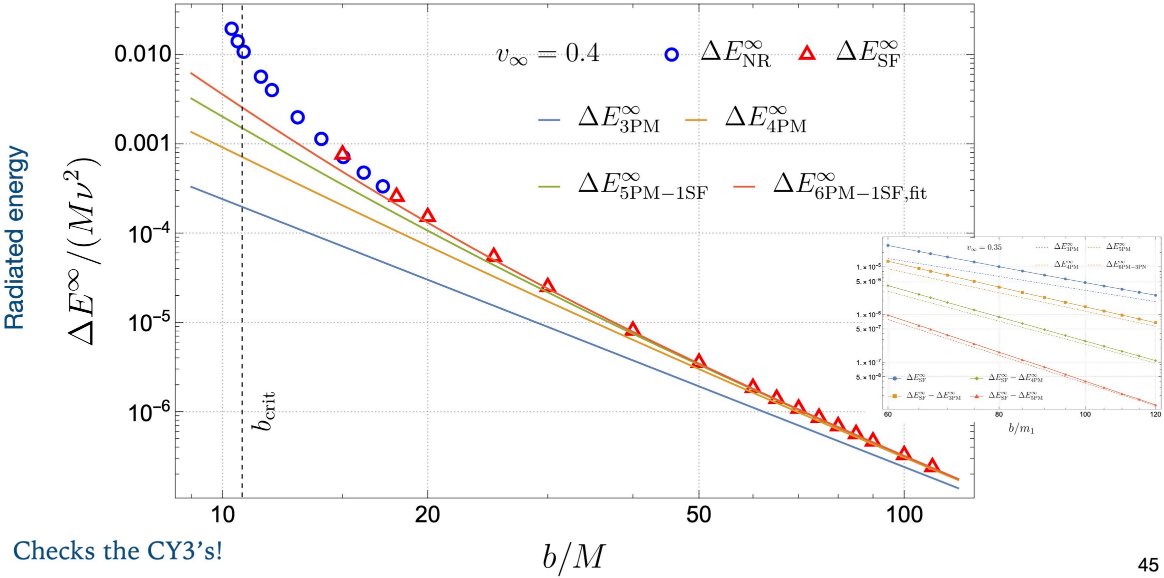
$$m_1 = m_2$$

$$v = 0.296c$$

$$M = m_1 + m_2$$

RADIATED ENERGY: COMPARISON TO SEMI-NUMERICAL SELF-FORCE

[Warbuton, 2512.02274]



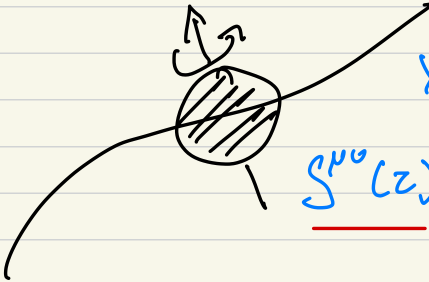
Checks the CY3's!

SCATTERING SPINNING BLACK HOLES: STATE-OF-THE-ART

Higher orders in spin are **suppressed by physical PM counting**: $a_i^\mu = G m_i \chi_i^\mu\,,\qquad |\chi_i| < 1$

	S^0	S^1	S^2	S^3	S^4	S^5
1PM (tree level)	G	G^2	G^3	G^4	Hoogeveen, Jakobsen, JP `25	G^6
2PM (1 loop)	G^2	G^3	G^4	G^5	Guevara, Ochirov, Vines '18 Chen, Chung,Huang, Kim '21 Haddad, Jakobsen, **Mogull** JP '24	Bern, Kosmopoulos, Luna, Roiban, Teng '22; Aoude, Haddad, Helset '23; Bautista '23; Chen, Wang '24; Bohnenblust, Cangemi, Johansson, Pichini '24
3PM (2 loops)	Bern, Cheung, Parra-Martinez, Ruf, Herrmann, Roiban, Shen, Solon, Zeng, Kälin, Liu, Porto, Di Vecchia, Heissenberg, Russo, Veneziano, Travaglini, Brandhuber, Damgaard, Planté, Vanhove, Bjerrum-Bohr	**Jakobsen, Mogull '22** Akpinar, Cordero, Kraus, Ruf, Zeng '24	**Jakobsen, Mogull '22** Akpinar, Cordero, Kraus, Ruf, Zeng '24	Akpinar, Cordero, Kraus, Smirnov, Zeng '25 Haddad, Jakobsen, **Mogull**, JP `25	Akpinar, Cordero, Kraus, Smirnov, Zeng '25 Haddad, Jakobsen, **Mogull**, JP `25	G^8
4PM (3 loops)	Dlapa, Kälin, Liu, Neef, Porto, Damgaard, Hansen, Planté, Vanhove, Bern, Parra-Martinez, Roiban, Ruf, Shen Solon, Zeng	**Jakobsen, Mogull, JP, Sauer, Xu '23**	G^6	G^7	G^8	G^9
5PM (4 loops)	**Driesse, Jakobsen, Klemm, Mogull, Nega, JP, Sauer, Usovitsch '24**	G^6	G^7	G^8	G^9	G^{10}

SPIN ON THE WORLD LINE



$S^{\mu\nu}(\tau)$: SPIN TENSOR

$S^{\mu\nu} = -S^{\nu\mu}$

HAMILTONIAN PERSPECTIVE

(van Holten, 2015)

PHASE SPACE: $\{x^a, P_b, S^{ab}\}$

- FLAT SPACE POISSON STRUCTURE:

$$\boxed{\begin{aligned} \{x^a, P_b\} &= \delta^a_b \\ \{S^{ab}, S^{cd}\} &= \eta^{bd} S^{ac} + 3\text{ more} \end{aligned}}$$

- CURVED SPACE-TIME:

$$\Pi_\mu := P_\mu + \frac{1}{2}\omega_{\mu|ab} S^{ab} \qquad S^{\mu\nu} = e^\mu_a e^\nu_b S^{ab}$$

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$$

$$\{X^\mu, \pi_\nu\} = \delta^\mu_\nu \qquad \{S^{\mu\nu}, S^{\rho\kappa}\} = g^{\nu\kappa} S^{\mu\rho} + 3 \text{ MORE}$$

$$\{S^{\mu\nu}, \pi_\rho\} = 2\, \Gamma^{[\mu}{}_{\rho\kappa} S^{\nu]\kappa} \;\hat{=}\; D_\rho S^{\mu\nu} \qquad \partial_\rho S^{\mu\nu} = 0$$

$$\{\pi^\mu, \pi^\nu\} = -\frac{1}{2} R_{\mu\nu;ab} S^{ab}$$

<u>LAGRANGIAN DESCRIPTION ?</u> TRADITIONAL WAY : Body field $\Lambda^\mu{}_a$

Better solution: Take $S^{\mu\nu}$ as composite (inspired by string theory)

INTRODUCE A COMPLEX VECTOR $\boxed{\alpha^a(\tau)}$, $\bar{\alpha}^a(\tau) = (\alpha^a(\tau))^\dagger$

Postulate:

$$\boxed{\{\alpha^a, \bar{\alpha}^b\} = -\frac{i}{m}\eta^{ab}}$$

Spin-tensor:

$$\boxed{S^{ab} := 2mi\, \bar{\alpha}^{[a}\alpha^{b]}}$$

$$S^{ab}_{-\infty} \sim \bar{\alpha}^{[a}_{-\infty}\alpha^{b]}_{-\infty}$$

SPINNING PARTICLE:

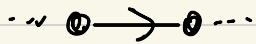
$$\boxed{S_{BH/NS} = -m\int d\tau \Big[\frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + i\bar{\alpha}_\mu D_\tau\alpha^\mu + \frac{1}{2}R_{\bar{\alpha}\alpha\bar{\alpha}\alpha} \quad C_E \cdot R_{\bar{\alpha}\dot{x}\alpha\dot{x}}\,\bar{\alpha}\cdot\alpha + \mathcal{O}(R^{n\geq 1}, S^{n\geq 3})\Big]}$$

$$D_\tau\alpha^\mu = \dot{\alpha}^\mu + \Gamma^\mu{}_{\dot{x}\alpha}$$

BACKGROUND FIELD EXP.

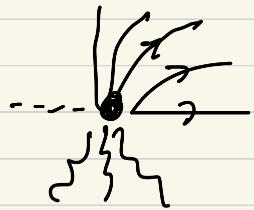
$$\alpha^\mu(\tau) = \alpha^\mu_{-\infty} + \alpha'^\mu(\tau)$$

PROGATOR:



$$\boxed{\langle \alpha^{\mu}(\omega)\,\bar{\alpha}^{\nu}(-\omega)\rangle_0 = \frac{-i\eta^{\mu\nu}}{m(\omega+i0^{+})}}$$

NEW VERTICES:



$R^1 S^n$ $\quad$ $n \leq 4$

$\mathcal{L}_{(R^1S^s,n)}$	$s=2$	$s=3$	$s=4$
$n=1$	$\frac{1}{m^2}(S\cdot S)^{\mu\alpha}R_{\mu\dot z\alpha\dot z}$	$\frac{1}{m^3}S^{\mu\nu}(S\cdot S)^{\sigma\alpha}R_{\mu\nu\alpha\dot z;\sigma}$	$\frac{1}{m^4}(S\cdot S)^{\rho\sigma}(S\cdot S)^{\mu\alpha}R_{\mu\dot z\alpha\dot z;\rho\sigma}$
$n=2$	$\frac{1}{m^2}S^{\mu\nu}S^{\alpha\beta}R_{\mu\nu\alpha\beta}$	$\frac{1}{m^3}S^{\dot z\sigma}S^{\mu\nu}S^{\alpha\beta}R_{\mu\nu\alpha\beta;\sigma}$	$\frac{1}{m^4}(S\cdot S)^{\rho\sigma}S^{\mu\nu}S^{\alpha\beta}R_{\mu\nu\alpha\beta;\rho\sigma}$

$R^2 S^n$ $\quad$ $n \leq 4$

$\mathcal{L}_{(R^2S^s,n)}$	$s=0$	$s=2$	$s=4$
$n=1$	$\lambda^4 R_{\mu\dot z\nu\dot z}R^{\mu\dot z\nu\dot z}$	$\frac{\lambda^2}{m^2}R_{\mu\dot z\alpha\dot z}R^{\mu}{}_{\dot z\beta\dot z}(S\cdot S)^{\alpha\beta}$	$\frac{1}{m^4}(R_{\mu\dot z\nu\dot z}(S\cdot S)^{\mu\nu})^2$
$n=2$	$\lambda^4 R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$	$\frac{\lambda^2}{m^2}R_{\mu\nu\alpha\dot z}R^{\mu\nu}{}_{\beta\dot z}(S\cdot S)^{\alpha\beta}$	$\frac{1}{m^4}R_{\mu\nu\sigma\dot z}R_{\alpha\beta\rho\dot z}S^{\mu\nu}S^{\alpha\beta}(S\cdot S)^{\sigma\rho}$
$n=3$		$\frac{\lambda^2}{m^2}R_{\mu\dot z\nu\sigma}R^{\mu\dot z\nu\dot z}(S\cdot S)^{\dot z\sigma}$	$\frac{1}{m^4}(R_{\mu\nu\alpha\beta}S^{\mu\nu}S^{\alpha\beta})^2$
$n=4$		$\frac{\lambda^2}{m^2}R_{\mu\nu\alpha\beta}R^{\mu\nu}{}_{\sigma\dot z}S^{\dot z\sigma}S^{\alpha\beta}$	$\frac{1}{m^4}R_{\mu\dot z\nu\dot z}R^{\mu}{}_{\alpha\beta\dot z}(S\cdot S)^{\dot z\nu}(S\cdot S)^{\alpha\beta}$

$\dot z = \dot x$ $\quad$ λ: Length scale.

STRUCTURE OF HIGHER DIM. OPERATORS

$R^n S^s$

$$S_{NM} = -m \int d\tau \sum_A C_A \mathcal{L}_A \qquad A \in (R^n, S^s; n)$$

WILSON-COEF.

DECOMPOSITION OF SPIN-TENSOR:

PAULI-LUBANSKI VECTOR S^μ

SSC-VECTOR: Z^μ

$$\boxed{S^{\mu\nu} = \varepsilon^{\mu\nu\rho\sigma} \dot{x}_\rho S_\sigma + 2 Z^{[\mu} \dot{x}^{\nu]}} \quad (*)$$

$\#$ DoF: $3+3=6$

SPIN-SUPPLEMENTARY CONDITION!

$$\boxed{Z^\mu = 0}$$

CONSTRAINT!

INVERT (+): $S^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\dot{x}_\nu S_{\rho\sigma}$ $Z^\mu = S^{\mu\nu}\dot{x}_\nu$

NEEDS TO HAVE $\boxed{\frac{d}{d\tau}\left[S^{\mu\nu}\dot{x}_\nu\right]=0}$

TRANSLATE SSC $Z^\mu=0$ TO $\bar{\alpha}^\mu, \alpha^\nu$: $\boxed{\frac{d}{d\tau}\left(\alpha\cdot\dot{x}\right)=0}$

CONSTRAINS THE WILSON COEFF:

$(R^1, S^{2,3,4})$: 3 free coeff. $(R^2, S^{2,3,4})$: 6 free coeff

<u>FOR KERR-BH:</u> (R^1, S^4) all coeffs fixed !

$(R^2, S^{4\ldots 6})$?