Lectures on "beyond" linear GR
on "beyond" GR

Refs:

(A)
- Damour 1982 "Surface effects in BH physics"
- arxiv: 1101.2451 NS → Einstein
- " : 1802.05286 Gal / Carrol
- " : 1903.09654 Carroll Physics
- " : 2212.06175 Non Linear BH...
- " : 2310.15427 F-G & BDNK

To speak of non-linear science as like refering to the bulk of biology as the study of non-elephant animals (S. Ulam)

Context

What to expect in observables?
perturbative approaches

. regime assumptions
First pples understanding
. Using right intuition?

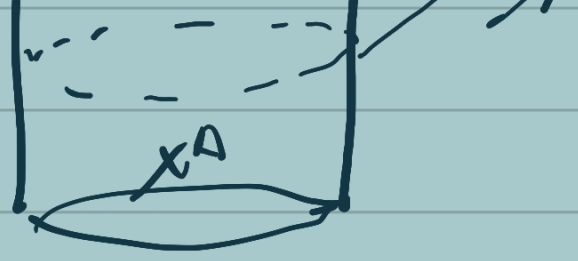
strategy:
Back told" efforts in a new light
Connect with physics we know better or with different tods

Consider different surfaces in asymptot. dS $(\Lambda<0)$

$\Lambda<0$ $\Lambda=0$
- Timelike / Null at ∞
- Timelike at r_c
- Null (e.g. $\mathcal{H}$)

Introduce a null foliation,

$$ds^2 = -A\,du^2 - 2B\,du\,dr + r^2 h_{AB}(dx^A - U^A du)(dx^B - U^B du)$$



- <u>Note</u>: 1) since $u = \text{const}$ is null this is a "more rigid" choice for coords

2) Einstein Eqs have a particular structure.

•) $E_{AB} - \frac{1}{2} g_{AB} E_{CD} g^{CD} = 0 \Rightarrow \dot{h}' \sim h''$

•) $E^u{}_a = 0 \quad \left\{ \begin{array}{l} A_{,r} \ldots \\ B_{,r} \ldots \end{array} \right. \quad U^A{}_{,rr} \ldots$

•) $E^r{}_a \rightarrow$ constraints that are propagated by Bianchi ids.

$\boxed{\text{Aside}}$ in a Cauchy foliation...

$\hat{n}^a$ $\quad E_{ab} n^a n^b \rightarrow$ HAMILT. CONST

$E_{ab} n^a \perp^b_c \to$ Mom Const.

$E_{ab} \perp^a_c \perp^b_d \to (\gamma_{cd})^{-1} \gamma_{ef}$ "

(very roughly).

$\Rightarrow$ Bianchi identities $\Rightarrow$

if $\{H=0, M_i=0\}_{t=0}$

Evol Eqns $\Rightarrow$ $\dot{H} \sim F_H(H, M, EE)$

$\dot{M}_j \sim F_M(H, M, EE)$

In the null foliation.

$\to$ if $\{R^r_a = 0\}_{at\ r=R}$

$\to$ propagated through

$\boxed{\text{Aside 2}}$ Relativistic Hydrodynamics & "c limits"

Starting point: stress-energy tensor

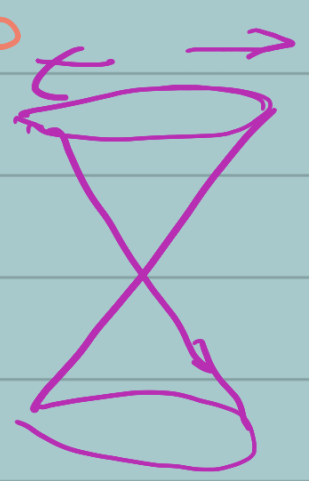
$$T^{ab} = (\epsilon + p)\frac{u^a u^b}{c^2} + p\, g^{ab} + \tilde{\tau}^{ab} + \frac{u^{(a} q^{b)}}{c^2}$$

ϵ : energy density
p : pressure
u^a : fluid velocity

$\nabla_a T^{ab} = 0$
$u^a u_a = -c^2$

Now, let's take limits:

Galilean limit $c \to \infty$



"open the cone"

- $u^0 = \gamma c = c + \ldots$
- $u^i = \gamma v^i = v^i$
- $\epsilon = (e + c^2)\rho_0$

Now, how about the equations?

- $c\nabla_\mu T^{\mu 0} = c^2 A + \mathcal{E} + O(1/c^2)$
- $\nabla_\mu T^{\mu}{}_i = M_i + O(1/c^2)$

$A = 0 \to$ continuity equation

$\mathcal{E} = 0 \rightarrow$ energy conservation

$\mathcal{M}_i = 0 \rightarrow$ mom. conservation

Take for simplicity, flat spacetime

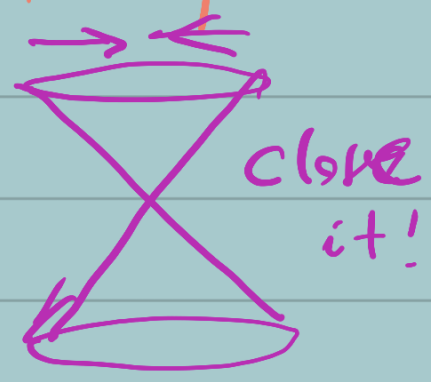
- $\mathcal{C} \simeq \partial_t \rho + \partial_i (\rho v^i)$
- $\mathcal{E} \simeq \partial_t e + v^i \partial_i e \quad \ldots\ldots$
- $\mathcal{M}_i \simeq \partial_t (\rho v_i) + \partial_j (\rho v^j v_i) + \ldots$

$\Rightarrow$ Navier-Stokes eqn

$\rightsquigarrow$ Galilean Limit $(c \to \infty)$

$\boxed{\text{Carrollian limit}}$ $c \to 0$

$v \to 0$ faster than $c \rightarrow v^i = c^2 \beta^i$



$$\rightsquigarrow \begin{cases} u^0 \simeq c \\ u^i \simeq c^2 \beta^i \end{cases}$$

$$\rightarrow \begin{cases} c \nabla_a T^a{}_0 = \frac{1}{c^2} \mathcal{F} + \mathcal{E} + \mathcal{O}(c^2) \\ \nabla_a T^{ai} = \frac{1}{c} \mathcal{H}^i + \mathcal{G}^i + \mathcal{O}(c^2) \end{cases}$$

c^2

→ Doubling of equations?

→ in the perfect fluid case $F = H = 0$

and

$$\mathcal{E} = -\partial_t \epsilon - (\epsilon + p)\,\theta$$

$$G_J = (\epsilon + p)\left[\varphi_J + \sigma^c_J + \beta_J \theta^c\right] + \beta_J\, \partial_t(\epsilon + p) + \hat{\partial}_J p$$

Some qns.

1. How does a C. fluid equilibrate if the cones are closed

2. Carrollian symmetries

2. Conservation laws, where else?

3. What is a C. eqn of state?

4. What is such a fluid?

Why do we care??

- I care about gravity! Fluids are a different class of beasts!

- linearly deg. vs truly non linear!
  - $u_{,t} = u_{,x}$ vs $u_{,t} = u\, u_{,x}$.
    smooth soln $\Big\}$ shocks!

- smooth solns from smooth data

- shocks from smooth data

- prop. speeds

c or gauge indep. of state of field

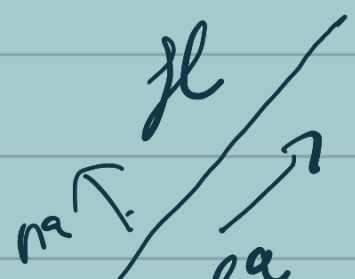
- dependent of state of field $v = c_s + v_0$

- Why would we learn anything useful?!

- "only" non linear classical field that treats non-linearities seriously...

Fluid - gravity correspondence...

- $\boxed{\text{Damour}}$ What do we get if one projects EEs onto $\mathcal{H}$?



- define $\pi_A = - n_a \ell^a_{;A}$

$$\theta = \frac{1}{2}\gamma^{AB}\frac{D\gamma_{AB}}{dt}$$

$n^a l_a = 1$

γ_{AB} int. metric of "cut"

$$\sigma_{AB} = \frac{1}{2}\frac{D\gamma_{AB}}{dt} - \frac{1}{2}\theta\gamma_{AB}$$

- Raychaudhuri eqn.

$$\dot{\theta} - \kappa\theta + \frac{1}{2}\theta^2 + \sigma_{AB}\sigma^{AB} = 0$$

- Damour (-NS) eqn

$$\dot{\pi}_A + \theta\pi - \nabla_A\kappa - \frac{1}{2}\nabla_A\theta + \nabla^B\sigma_{AB} = 0$$

With the identifications:

$$\epsilon = \theta \; ; \; p = -\kappa \quad \rightarrow \text{EoS!}$$

$$\Pi_{AB} = 2\eta\sigma_{AB} + \zeta\theta\gamma_{AB} \; ; \; \eta = \frac{1}{2} \; ; \; \zeta = -\frac{1}{2}$$

$$\pi_A = -\frac{1}{\kappa}\left(\nabla_A\kappa - H^B\dot{\gamma}_{AB} - \frac{1}{\kappa}H^A\dot{\kappa}\right)$$

$$\Sigma^{AB} = J^A = 0$$

$\rightsquigarrow$ $\boxed{\text{Carrollian Hydrodynamics!}}$

$\rightarrow$ for a particular fluid.

From gravity $\longrightarrow$ fluids

- How does a fluid with "closed" cones thermalizes?

From fluid $\longrightarrow$ gravity

- How about non linear behavior?

- For Carrollian fluids dual to horizons we can link these questions!

[Redondo-Yuste-LL]

- But, how to close the system of eqs?
  - k?, σ_{AB}?
    - $\rightarrow$ diss: pertoon/heating?
    - $\rightarrow$ EOS?

$\rightarrow$ k, σ_{AB} are "sensitive" to the <u>bulk</u>! null surface is C $d+1$ s.t.!

$\rightsquigarrow$ Involve rest of EEs!

$\boxed{R_{AB}}$

- $\theta^{(1)} + k\theta^{(1)} + \theta^{(1)}\theta^{(1)} - [D_A H_A + H_A H^A] + \frac{R}{2} = 0$

- $2\sigma^{c\,(1)}_{AB} - 4\,\sigma^{(1)}_{c(A}\,\sigma_{B)c} + \theta^{1}\sigma_{AB} + (2k-\delta)\sigma^{(1)}_{AB} + R^{TF}_{AB} - 2(H_A H_B)^{TF} - 2\left[D_{(A}H_{B)}\right]^{TF} = 0$

- Horizon is teleological!

$\theta \to 0$

$u \to \infty$

$\rightsquigarrow$ In Carrollian picture

$\theta \to$ energy <u>and</u> "cup"

- fluid thermalizes by "deforming" the cup!

- Horizon "uniformizes" by adjusting behavior of generators!

Back to gravity!

[Back to gravity]

- perturbatively & resummation

$$-\dot{H}_A = \nabla^2 H_A + \nabla_A \nabla_B H^B + \frac{1}{m}\left[H_A \nabla^A H_B + H_B \nabla^B H_A\right] + \dots$$

Reynolds number.

$$\nabla^2 H_A + \nabla_A \nabla_B H^B \ll H \nabla H \quad ?$$

$\rightsquigarrow$ condition

+ non-linear mode generation, etc...

Let's go to $r \to \infty$ [AdS bdry]

- consider a perturbed BH

$\rightsquigarrow$ typically $g = g_B + h_1 + h_2 + \dots$

<u>Instead</u> consider

$$g = g_B(M, a) \to g(M(x), a(x))$$

and assume a gradient expansion

i.e. $|\partial^n F| > |\partial^{n+1} F|$

- Plug this into EEs in characteristic form ...

$R_{ra} \to$ "constraints on r"

<u>Rest</u> Radial eqns inwards

$\to D_a(T^{ab}) = 0 \to$ for a symmetric tensor

$\to \nabla^a{}_a = 0$

Say the BH was a "boosted" black brane with boost u^a, radius $r_H = \frac{1}{b}$

$$T^{ab} = \frac{3}{b^4} u^a u^b + \frac{1}{b^4}(u^a u^b + \eta^{ab})$$

$P = \frac{\epsilon}{3}$ (trace condition)

1st order in gradients?

$$T_{ab} = \frac{3(1+\chi)}{b^4} + \frac{(1+\chi)}{b^4}\left(u^a u^b + \eta^{ab}\right)$$

$$- \frac{2}{b^3}\sigma_{ab} - \frac{8}{b^4}\left(u_{(a}\xi_{b)}\right)$$

with $\sigma^{ab} = P^{c(a}\partial_c u^{b)} - \frac{1}{3}P^{ab}\partial_c u^c$

$P^{ab} = u^a u^b + \eta^{ab}$

χ, ξ^a = "corrections" which are gauge dependent.

- Notice
  - transport coeff for shear visc. defined
  - Can choose $\chi = \xi^a = 0$ [Bhattacharyya '2006].

- Can go to even 2nd order & 1-to-1 correspondence with Landau-Lifschitz!

- Pert's of shear & sound modes $\longleftrightarrow$ Lowest QNMs of BH.

What else?

- Long-wavelengths pert
  $\rightarrow$ turbulent regime!
  - turbulence in gravity???

- Freedom in X, g^a?

$\boxed{\text{in AdS}}$

6:16 Sat, Jan 17 · *Beyond astro: AdS/CFT...* 92%

- The AdS/CFT correspondence relates a d-dimensional QFT with a (d+1)–dimensional gravity.

[Image: J. Santos]

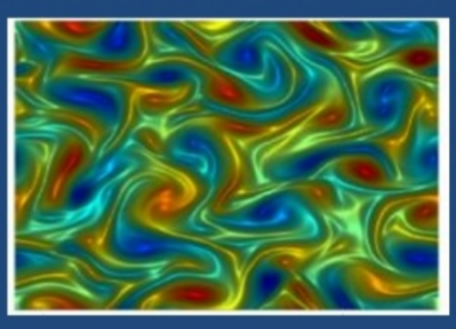
  - Any gravitational phenomena should have an equivalent CFT analog, *and vice versa*.
  - A natural arena to study field theory open questions: transport properties in strongly coupled field theories, quantum quenches, thermalization, etc.
  - Plenty of applications. Most of which in equilibrium situations and in the probe limit (phase space analysis) (e.g. CMT applications)
  - More recently, *dynamical applications*: e.g. examining quark-gluon plasma behavior, thermalization,...

  - But also... new physics in gravity?

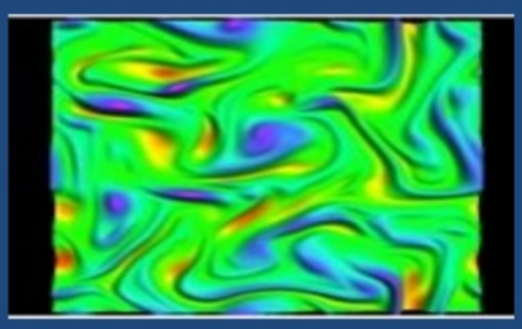
Gravitational turbulence?

- Einstein eqns do not develop shocks (propagation speeds = c, regardless of the state of the solution -> no crossing of characteristics). *VERY* different from fluids!

- But, AdS/CFT motivates (& makes 'rigorous') a fluid/gravity correspondence, previously identified through a 'membrane paradigm programme' [Damour, Price-Thorne-McDonald] -> [Hubeny-Rangamani-Minwalla-Bhattacharya]
  - Relies on a 'non-standard' [in GR] perturbative approach



[Adams,Chesler,Liu. PRL 2014]



[Green,Carrasco,LL, PRX 2013]

(in AF) Perturbative analysis

- The problem is non-linear….what can we draw from turbulence in AdS?
- *Back to AF spacetimes*, beyond linear perturbation? Take h1 (with l,m,n); a further mode on the 'time-dependent' background, obeys:

$$[\ \mathrm{Box}_{kerr} + O(h_1)\ ]\ \Phi = 0.$$

- With the solution having the form: $e^{t(\alpha - \omega_l)}$ with

$$\alpha = \pm\sqrt{|Hh_0(t)/Qm'|^2 - (\omega'_R - \omega_R/2)^2}\,,$$

- So exponentially growing solution if:

$$h_0(t)/(m'\omega'_I) - |Q/H|\sqrt{(\omega'_R - \omega_R/2)^2/\omega'^2_I + 1} > 0\,.$$

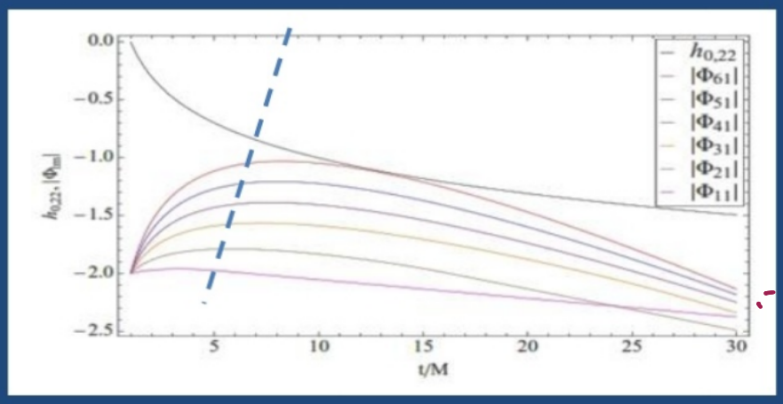
- if Φ has l, m/2 -> a parametric instability can turn on; i.e. inverse cascade.
- Further, one can find 'critical values' for growth onset.
- And can define a max value as:

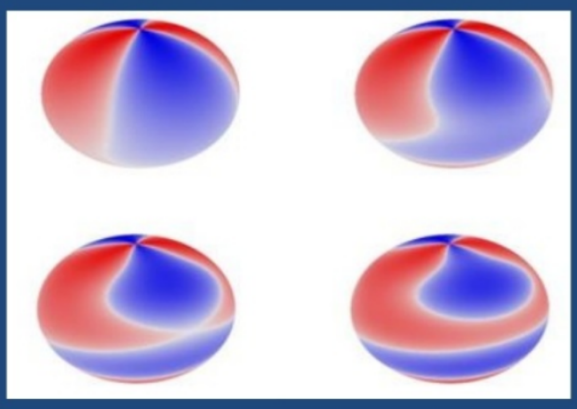
$$Re_g = h_o/(m\ \omega_\nu)$$

- identify λ<-> 1/m ; v <-> h_o ; η/ρ <-> ω_ν

$$\rightarrow\ Re_g = Re$$

‘Turbulence’ , AF spacetimes and ‘beyond linear’



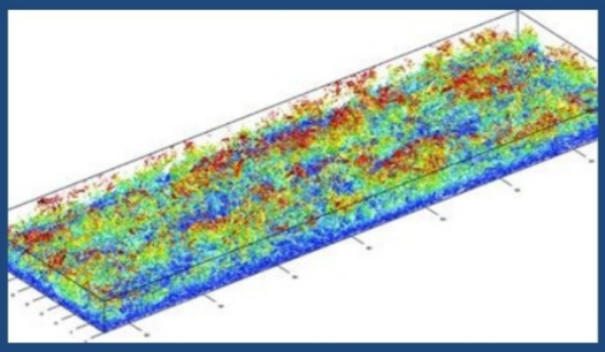


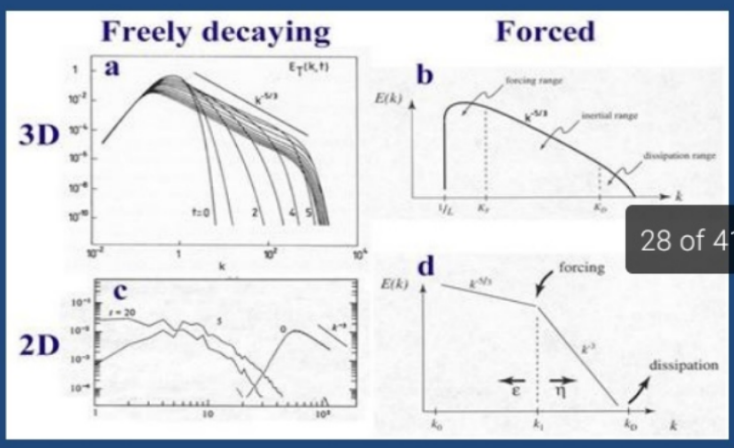
a = 0.998, perturbation ~ 0.02%, initial mode l=2,m2

Could ‘potentially’ have observational consequences. Signal is different from what might be expected at the linear level

[Yang,Zimmemarn,LL ‘14]

Exploring non-linearities in FD





Full non-linear

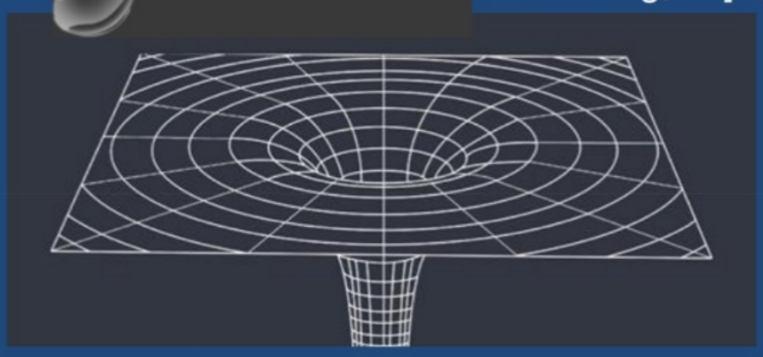
quasi) 'Steady state' analysis

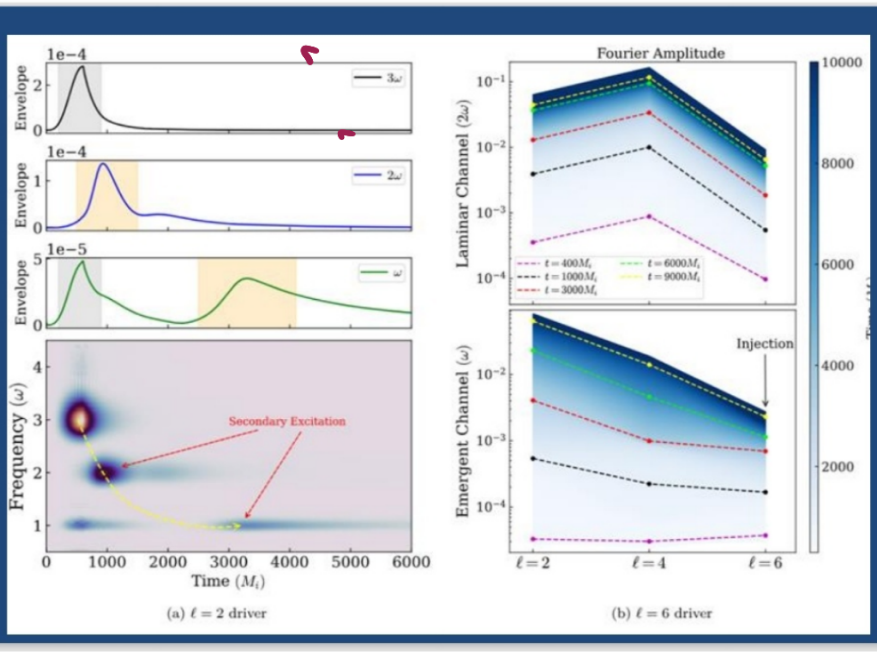
- 'Stirr' a (region of)spacetime with incoming GWs: F(A,ω,ℓ, m=0)

- Choices :
  - non-spinning BH
  - ω = 1/2m ; 1/m ; 2/m
  - ℓ = 2, 6
  - A = 'small', 'large' [R_{eg}]
  - Drive with linear growth unil t_*, steady afterwards

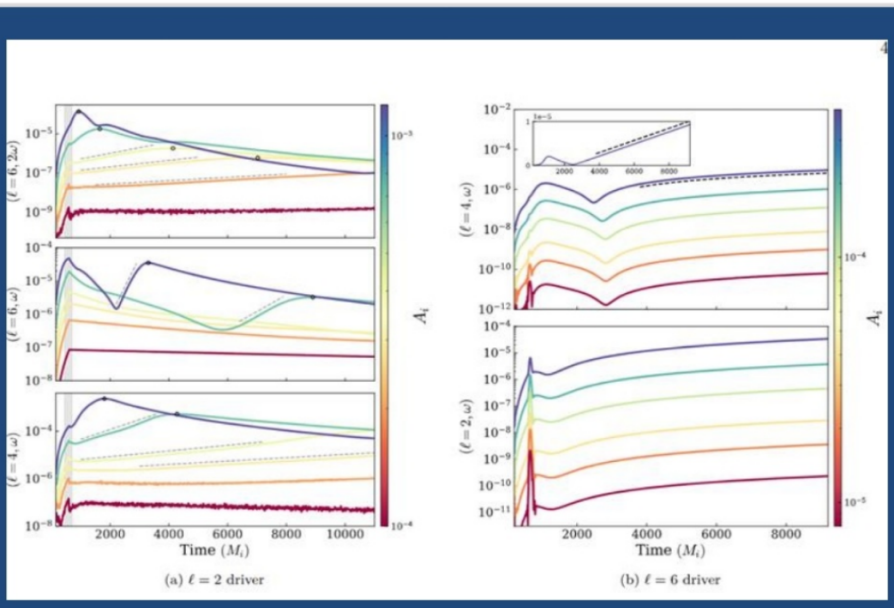
[Sizheng Ma

H. Yang, LL]





(a) ℓ = 2 driver (b) ℓ = 6 driver



(a) ℓ = 2 driver (b) ℓ = 6 driver

Gauge choice? $\{q^a, \chi\}$

- Unperturbed BH. $\begin{cases} K = \frac{2}{b}; & \Theta = 0 \\ \pi_a = 0; & \varphi_a = 0 \end{cases}$

- 1st order in gradient...

  - $K = \frac{2}{b} - \frac{2}{3}\theta + \frac{\chi}{2b} + \frac{u^a \partial_a b}{b}$

  - $\Theta = \theta - 3\frac{u^a \partial_a b}{b}$

  - $\pi_a = \frac{2 q_a}{b} - \frac{P_a{}^b \partial_b b}{b}$

<u>Options</u>? : $\pi_a = 0 \longleftrightarrow \ell^a \parallel u^a$

$\hookrightarrow q^a = -b\, a^a + P^{ab}\partial_b b$

$: r_H = \frac{1}{b} + \frac{\Theta}{6} + \frac{\chi}{4b}$

at 1st order $\Theta = 0 \rightarrow \chi \propto \Theta$ keeps r_H

$\hookrightarrow \chi = \alpha\, b$ ⊕

<u>Implications</u>?

$$\Rightarrow \mathcal{E}^{(1)} = \frac{3}{b^3}\alpha\left(\partial_a u^a + \frac{u^a \partial_a e}{e+p}\right)$$

$$Q^a = \frac{4}{b^3}\left(a^a + \frac{P^{ab}\partial_b e}{3(e+p)}\right)$$

$\Rightarrow$ Of the type argued by <u>BDNK</u>!

- 1st order in gradient T_{ab}
- contains dissipation
- is strongly hyperbolic $\rightarrow$ well posedness!
- bypasses thm! (entropy "issue")

$\boxed{\text{QN}}$ Why does a gauge "natural" [horizon description

for "non-GR description"

"leads" to BDNK?

Deeper analysis of GWs

↳ for beyond GR

↳ for beyond "ingredients"

a.k.a. how would one tell?

① Null tests → Model A vs B

GR vs Beyond GR

⇝ requires understanding well A

and/or "smoking guns" for B.

⇝ $\boxed{\text{Options}}$

* Identify a theory

- predict deviations

* Agnostic approach

Theory? what theory? (where from)

- what changes?
- how to study it?
  (where and how?)

Derive theories in an EFT way...

<u>Ex</u> Euler vs Navier Stokes

incompressible fluid example

$$\underbrace{\partial_t v + v\,\partial v}_{\text{"base theory"}} = \underbrace{\left(\frac{\eta}{\rho}\right)\partial^2 v}_{\text{"correction"}}$$

- <u>Notice:</u> • correction changes character
  → hyperbolic to parabolic!

<u>For GR</u>... $\mathcal{L} \Rightarrow R + \Lambda R^p f(\phi)$

- R^p: curvature comb of order p

- $f(\phi)$: possible extra d.o.f (beyond those in g_{ab})

<u>Eg</u> : ESGB : $\phi\,[R^2, R_{ab}R^{ab}, R_{abcd}R^{abcd}]$

dCS : $\phi\; R_{abcd}{}^{*}R^{abcd}$

cubic : $R_{ab}{}^{cd}R_{cd}{}^{lm}R_{lm}{}^{ab}$

<u>Predictions?</u>

$\boxed{\text{Merger}}$ (fully non-linear regime of ppk)

<u>EoB</u>

- ESGB ... $\rightsquigarrow \Box g = \Lambda\, \partial^a g\, \partial^b g\, D_{ab} g$
  $\rightsquigarrow$ character can/does change!

- Cubic $\rightsquigarrow \Box g = \Lambda\, \partial^3 g$!

$\rightarrow$ unclear if relevant problems are well posed!
$\rightarrow$ though behaviour in GR helps!

→ if it breaks, EFT can't work
needs something else
(e.g. collapse scenarios!)

How to asses?

Option 0 : $\Box g = \lambda S(g^p)$

Red. Ord. → secular behavior kills it!

Option 1 :

Fix eqs

$$\Box g = \lambda \pi$$
$$\Box \pi = k \partial_t \pi + \gamma(-\pi + S)$$

Option 2 :

Redef. with higher order op.

$$\int (\partial\phi)^2 + \lambda (\partial\phi)^4$$
$$\phi \to \phi + \gamma \Box\phi$$
$$\rightsquigarrow \int (\partial\phi)^2 + \lambda(\partial\phi)^4 + \gamma(\partial\phi)\partial\Box\phi$$
$$\hookrightarrow \Box\phi = \lambda S + \gamma \Box\Box\phi$$

Inspiral: use pert approaches
(PN, NRGR, amplitudes)

and compute precession

Ring down:

- Find solns for BHs
- Compute QNM behaviour.

But in what theory!?

$\boxed{\text{Alternatives?}}$

- parameterize deviations

PPN $\varphi(f) = \sum_i (C_i^{GR} + \delta_i) f^i$

→ Search for δ_i in data

→ Can one do better?

$\boxed{\text{Note}}$: evidence so far points to...

i) GR does quite well!

ii) If new scale L → $L \lesssim$ "km"

iii) in GR no significant transfer of energy towards UV

→ EFT!

Recall h is dimensionless!

$\rightsquigarrow h \simeq h_{GR} + \delta$

what can one say about δ?

① $\delta \sim \left(\frac{L}{M}\right)^q \delta_b$

② $\delta_b \sim$ lead order PN on inspiral.

$\boxed{\text{Strategy}}$

① $R + \Lambda \text{"}R\text{"}^p \rightarrow \Lambda = L^{2(p-1)}$

$\rightarrow q = 2(p-1)$

② $R + \Lambda \phi \text{"}R\text{"}^p \rightarrow \Lambda = L^{2(p-1)}$

$\rightsquigarrow$ but $q = 2(2[p-1])$

EOM

$$G_{ab} \approx \nabla_a\phi\, \nabla_b\phi \quad (\lambda^2)$$

$$\Box\phi = \lambda\, G$$

Can follow this in general for any case!

$\rightsquigarrow$ $q \to$ integer: in several events look for $\delta \sim \frac{1}{M^q}$

$$\boxed{q \geq 4 \text{ from EFT in vacuum}}$$

$q = 0$ will imply an ECO!

Example: $q = 4 \to$ ESGB, dCS, cubic ✓

- will eliminate $R_{\_}R^{-}R_{\_}R^{-}$ and others but one can do better.

EFT "rules": Consider all operators consistent with symmetries

and assumptions so...

it can't be ESGB or dCS

→ it must be both!

⇝ one can then ask which one to compute?

→ quasi circular case $\left(\frac{v}{c}\right)$ is small

⇒ an inspiral ESGB leads

⇒ if ϕ is present → dipolar radn

cubic → only starts at 5PN

$\boxed{\text{SMOKING GUN}}$ $q=4$ & PN < 5

→ ESGB + dCS

(and one can go even deeper)

New strategy: multiple events at a time

· search for q & PN

→ $q \geq 4$ & integer Beyond GR

→ $q = 0$ → ECO!

→ $q = 1 \leftrightarrow 2$ environmental effect

→ q other astrophysical

Final thoughts

- Useful to think of problems anew
- In particular to gain new inspiration & tools from other fields