The Momentum Kernel of Gauge and Gravity Theories

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based on works

done in collaboration with

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Explicit amplitude computations display rather unexpectedly simple structures allowing to compute many more processes than expected

- Field theory
  - On-shell recursion methods [cf. R. Britto's talk]
  - twistor geometry, Graßmanian, Symbol,... [cf. N. Arkani-hamed's & S. Carot-Huot talks]
  - Massive amplitudes [cf. M. Kiermaier's talk]
  - new parametrisations and simplified structure [cf. Z. Bern & J.-J. Carrasco & R. Roiban talk]
  - Dual conformal invariance [cf. G. Korchemsky's talk]
- String theory
  - amplitudes relations [cf. O. Schlotterer's talk]
  - constraints from duality [cf. M.B. Green's talk]

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As pointed out Jorge Luis Borges

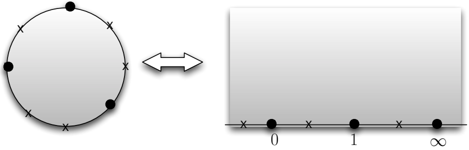
There is no intellectual exercise that is not ultimately pointless

so most of this talk will be focused on string theory

Part I

Tree-level amplitudes

Tree-level amplitudes in Open string

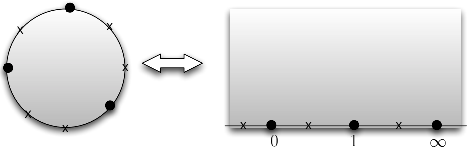


- We first evaluate the open string amplitudes on the disc.
- They can be decomposed into color $(n-1)!/2$ color-ordered sub-amplitudes

$$\mathfrak{A}(1,\ldots,n) \sim \sum_{\sigma\in\mathfrak{S}_{n-1}/\mathbb{Z}_2} \mathrm{Tr}\left(\lambda^{a_1}\lambda^{a_{\sigma(2)}}\cdots\lambda^{a_{\sigma(n)}}\right)\, \mathcal{A}(1,\sigma(2,\ldots,n))$$

- λ^a are generator in the fundamental representation
- $\mathcal{A}(1,\sigma(2,\ldots,n))$ are the color ordered open string amplitudes

Tree-level amplitudes in Open string



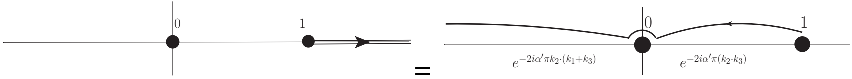
- $PSL(2,\mathbb{R})$ invariance $z_1 = 0$, $z_{n-1} = 1$ and $z_n = +\infty$. (3 marked points)

$$\mathcal{A}(1,\dots,n) = \int_{x_1<\cdots<x_n} d^{n-3}x\, f(x_i - x_j) \prod_{1\leqslant i<j\leqslant n} (x_i - x_j)^{2\alpha' k_i\cdot k_j}$$

- The function $f(x_j)$ does not have branch cut but has poles. Depends on the polarisation of the external states.
- The precise form of $f(x_{ij})$ depends on the type of string we use

Monodromies from contour deformation

Contour deformation [Bjerrum-bohr, Damgaard, Vanhove; Stieberger; Bjerrum-bohr, Damgaard, Søndergaard, Vanhove]



- The real and imaginary part of the monodromy relations lead to a set of linear system of equations

$$\mathcal{A}_n(\beta_1,\ldots,\beta_r,1,\alpha_1,\ldots,\alpha_s,n) = (-1)^r \times$$

$$\times\Re\mathrm{e}\Big[\prod_{1\leqslant i<j\leqslant r} e^{(\beta_i\cdot\beta_j)} \sum_{\sigma\subset \mathrm{OP}\{\alpha\}\cup\{\beta^T\}} \prod_{i=1}^{r}\prod_{j=1}^{s} e^{(\alpha_i,\beta_j)}\mathcal{A}_n(1,\{\sigma\},n)\Big]$$

$$0 = \Im\mathrm{m}\Big[\prod_{1\leqslant i<j\leqslant r} e^{(\beta_i\cdot\beta_j)} \sum_{\sigma\subset \mathrm{OP}\{\alpha\}\cup\{\beta^T\}} \prod_{i=1}^{r}\prod_{j=1}^{s} e^{(\alpha_i,\beta_j)}\mathcal{A}_n(1,\{\sigma\},n)\Big]$$

$\exp(\alpha,\beta) = \exp(2i\pi\alpha' k_\alpha\cdot k_\beta)$ if $\Re\mathrm{e}(z_\beta - z_\alpha) > 0$ or 1 otherwise

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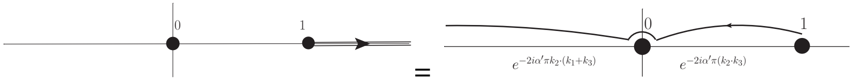


- This leads to an object name momentum kernel $\mathcal{S}$

$$\mathcal{S}_{\alpha'}[i_1,\ldots,i_k|j_1,\ldots,j_k]_p \equiv \prod_{t=1}^{k}\frac{1}{\alpha'}\sin\alpha'(p\cdot k_{i_t}+\sum_{q>t}^{k}\theta(i_t,i_q)\,k_{i_t}\cdot k_{i_q})$$

- This leads to the following set of constraints on the string theory amplitudes for all $\beta\in\mathfrak{S}_{n-2}$

$$\sum_{\sigma\in\mathfrak{S}_{n-2}}\mathcal{S}_{\alpha'}[\sigma(2,\ldots,n-1)|\beta(2,\ldots,n-1)]_{k_1}\mathcal{A}_n(n,\sigma(2,\ldots,n-1),1)=0$$

Minimal basis

- The partial string amplitude satisfy the annihilation relation

$$\sum_{\sigma\in\mathfrak{S}_{n-2}} \mathcal{S}_{\alpha'}[\sigma(2,\dots,n-1)|\beta(2,\dots,n-1)]_{k_1}\mathcal{A}_n(n,\sigma(2,\dots,n-1),1) = 0$$

- The rank of this system is $(n-3)!$ and we can use $(n-3)!$ color ordered string amplitudes as a basis for all tree level color ordered amplitudes

- Starting from the original expression for the amplitude in the fundamental representation

$$\mathfrak{A}(1,\dots,n) \sim \sum_{\sigma\in\mathfrak{S}_{n-1}/\mathbb{Z}_2} \mathrm{Tr}\left(\lambda^{a_1}\lambda^{a_{\sigma(2)}}\cdots\lambda^{a_{\sigma(n)}}\right)\,\mathcal{A}(1,\sigma(2,\dots,n))$$

- What does the expansion on the minimal basis implies?

Tree-level amplitude in closed string I

- For this we consider the gauge amplitudes in the closed (heterotic) string setup. This allows to get both YM and Gravity amplitudes at the same time [Kawai,Lewellen, Tye; Tye, Zhang; Bjerrum-Bohr, Damgaard, Søndergaard, Vanhove]

- The $\alpha' \to 0$ limit reproduces the YM answer, although there are important differences in the α' corrections

- The closed string spectrum is obtained from $|\text{phys}\rangle_{closed} = |\text{phys}\rangle^L_{open} \otimes |\text{phys}\rangle^R_{gauge\ algebra}$ or for gravity amplitudes $|\text{phys}\rangle_{closed} = |\text{phys}\rangle^L_{open} \otimes |\text{phys}\rangle^R_{open}$

- This implies that one can use an holomorphic factorization

$$\mathfrak{M}(1,\dots,n) = \int_{C_x} d^{n-3}x \int_{C_y} d^{n-3}y \prod_{1\leqslant i<j\leqslant n} (x_i - x_j)^{\frac{\alpha' k_i\cdot k_j}{2}} (y_i - y_j)^{\frac{\alpha' k_i\cdot k_j}{2}} f(x_{ij}) g(y_{ij})$$

Tree-level amplitude in closed string II

- One get YM or graviton amplitudes by using vertex operators

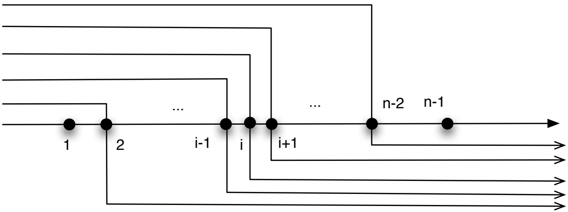
$$V^{YM} = \int d^2z : (A^a_\mu \partial X^\mu J^a + \cdots) e^{ik\cdot X} :$$
$$V^{Grav} = \int d^2z : (g_{\mu\nu} \partial X^\mu \bar\partial X^\nu + \cdots) e^{ik\cdot X} :$$

$$\mathfrak{M}(1,\dots,n) = \int_{C_x} d^{n-3}x \int_{C_y} d^{n-3}y \prod_{1\leqslant i<j\leqslant n} (x_i - x_j)^{\frac{\alpha' k_i\cdot k_j}{2}} (y_i - y_j)^{\frac{\alpha' k_i\cdot k_j}{2}} f(x_{ij}) g(y_{ij})$$
$$\sim \sum \mathcal{A}(\cdots)\, \tilde{\mathcal{A}}(\cdots)$$

- In the field theory limit then $\mathcal{A} \to A^{vector}$ color stripped YM amplitude, $\tilde{\mathcal{A}} \to \tilde{A}^{vector/color}$

Tree-level amplitude in closed string III

- The holomorphic left/right factorization $|z|^{\alpha' k_i\cdot k_j} \to z^{\frac{\alpha'}{2} k_i\cdot k_j} \bar{z}^{\frac{\alpha'}{2} k_i\cdot k_j}$ puts important restriction on the relative x and y integration regions of the previous ordered "open" string amplitudes



- Closing the contour of integration to the right or the left give the most general relations between amplitudes

Tree-level amplitude in closed string IV

$$\mathcal{M}_n \sim \sum_{\sigma\in\mathfrak{S}_{n-3}} \sum_{\gamma\in\mathfrak{S}_j} \sum_{\beta\in\mathfrak{S}_{n-3-j}} \mathcal{S}_{\alpha'}[\gamma\circ\sigma|\sigma]_{k_1} \mathcal{S}_{\alpha'}[\beta\circ\sigma|\sigma]_{k_{n-1}}$$
$$\times \quad \mathcal{A}_n(1,\sigma(\ldots),n-1,n)\,\widetilde{\mathcal{A}}_n(\gamma\circ\sigma,1,n-1,\beta\circ\sigma,n)\,.$$

- The expression is independent of j thanks to the annihilation relation

$$\sum_{\sigma\in\mathfrak{S}_{n-2}} \mathcal{S}_{\alpha'}(\sigma)\mathcal{A}(\sigma) = 0$$

- The expression is a sum over $(n-3)! \times (j-2)! \times (n-1-j)!$ terms.
- The number of terms takes the maximal value $(n-3)! \times (n-3)!$ for $j=2$ or $j=n-1$. This is the most symmetric case
- The choice made by KLT consists in $j=\lceil n/2\rceil$ this leads to the smallest number of terms $(n-3)! \times (\lceil \frac{n}{2}\rceil -2)! \times (\lfloor \frac{n}{2}\rfloor -1)!$

Momentum kernel in field theory I

- Taking the field theory limit $\alpha' \to 0$ we get

$$\begin{aligned}\mathcal{A}_n^{\text{YM}} &= A^{\text{vector}} \otimes \mathcal{S} \otimes A^{\text{scalar}} \\ \mathcal{M}_n^{\text{Grav}} &= A^{\text{vector}} \otimes \mathcal{S} \otimes A^{\text{vector}}\end{aligned}$$

- The form with the maximal number of terms is

$$\mathcal{A}_n^{\text{YM/Grav}} = (-1)^{n-3} \sum_{\sigma,\gamma\in\mathfrak{S}_{n-3}} \mathcal{S}[\gamma(2,\ldots,n-2)|\sigma(2,\ldots,n-2)]_{k_1}$$

$$\times \mathcal{A}_n^{\text{vector}}(1,\sigma(2,\ldots,n-2),n-1,n)\widetilde{\mathcal{A}}_n^{\text{scalar/vector}}(n-1,n,\gamma(2,\ldots,n-2),1)$$

- These formula can as well be derived from unitarity methods in field theory [Bjerrum-Bohr, Damgaard, Feng, Søndergaard; Feng et al.]

Momentum kernel in field theory II

- In the field theory limit $\alpha' \to 0$ the momentum kernel

$$\mathcal{S}[i_1,\dots,i_k|j_1,\dots,j_k]_p \equiv \prod_{t=1}^{k} (p\cdot k_{i_t} + \sum_{q>t}^{k} \theta(i_t,i_q)\, k_{i_t}\cdot k_{i_q})$$

- The color ordered field theory amplitudes satisfy $\forall\beta \in \mathfrak{S}_{n-2}$

$$\sum_{\sigma\in\mathfrak{S}_{n-2}} \mathcal{S}[\sigma(2,\dots,n-1)|\beta(2,\dots,n-1)]_{k_1}\, A_n(n,\sigma(2,\dots,n-1),1) = 0$$

- There is one more amplitude relation in field theory

$$\sum_{\sigma\in\mathfrak{S}_{n-1}} 1\, \mathcal{A}_n(1,\sigma(2,\dots,n)) = 0$$

- This relation is not satisfied in string theory because of the α' corrections

Amplitudes relations

- We have used string theory amplitudes to derive amplitudes relations.
- The relations between color ordered amplitude are valid both in string and field theory, and a independent of any particular parametrisation of the amplitude.
- This can be achieved with any string theory formalism (we only used the generic structure of the amplitudes from the OPE of vertex operators, and Green's function)
- One can have a purely unitarity based derivation in QFT [Bjerrum-Bohr, Damgaard, Feng, Søndergaard; Feng et al.]

Field theory amplitudes

$$\mathcal{A}_n^{\rm YM/Grav} = A^{\rm vector} \otimes \mathcal{S} \otimes \tilde{A}^{\rm color/vector}$$

- Letting the momentum kernel to act on the partial amplitudes
- For $\sigma \in \mathfrak{S}_{n-2}$ with $\sigma = \tilde{\sigma} \cup \{i_o\}$ define

$$n_{1|\tilde{\sigma} i_o|n} = \sum_{\gamma \in \mathfrak{S}_{n-3}} \mathcal{S}[\gamma \circ \tilde{\sigma}|\tilde{\sigma}]_{k_1} \times \widetilde{A}_n(1,\gamma,n,i_o)\,,$$

- Notice since $n = \sum (s_{ij})^{n-3}$ $\tilde{A}$ are of the form $\sum s_{ij}^{n-3}/s_{kl}^{n-3}$
- The interpretation of these quantities will depends on $\tilde{A}$

Field theory amplitudes

$$\mathcal{A}_n^{\text{YM/Grav}} = A^{\text{vector}} \otimes \mathcal{S} \otimes \tilde{A}^{\text{color/vector}}$$

- Letting the momentum kernel acting on the right we get

$$\mathcal{A}_n^{\text{YM}} \sim \sum_{\sigma \in \mathfrak{S}_{n-2}} c_{1|\sigma(2\cdots n-1)|n}\, A_n^{\text{vector}}(1, \sigma(2, \cdots, n-1), n)$$

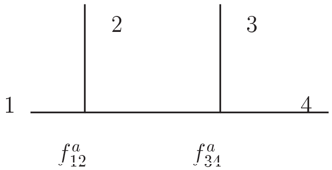
- expanding the vector part we get the dual form

$$\mathcal{A}_n^{\text{YM/Grav}} \sim \sum_{\sigma \in \mathfrak{S}_{n-2}} n_{1|\sigma(2\cdots n-1)|n}\, A_n^{\text{scalar/vector}}(1, \sigma(2), \cdots, \sigma(n-1), n)$$

- This leads to Lorentz $n_{1|\ldots|n}$ that are "dual" to color factor [Bern, Carrasco, Johansson]

Scalar amplitudes

- We can consider scalar φ^3 amplitudes $A_n^{\rm color}$ where vertices are the structure f^{abc}, propagators δ^{ab}/k^2



$$A^{\rm color}(1234) = \frac{f^a_{12} f^a_{34}}{(k_1+k_2)^2} + \frac{f^a_{41} f^a_{23}}{(k_1+k_4)^2}$$

- Applying the momentum kernel leads to

$$n_{1|\sigma|n} \sim \sum_{\gamma \in \mathfrak{S}_{n-3}} \mathcal{S}[\gamma \circ \tilde\sigma | \tilde\sigma]_{k_1} \, \tilde A_n^{\rm color}(1, \sigma, n-1, n)$$

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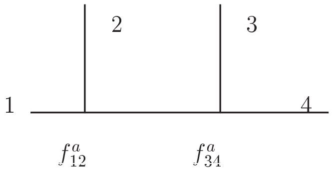


$$n_{1|\sigma|n} = (t_A^{\sigma(i_2)} \cdots t_A^{\sigma(i_{n-1})})_{a_1 a_n} + \delta n$$

$$\mathcal{A}_n^{\rm YM} \sim \sum_{\sigma\in\mathfrak{S}_{n-2}} (t_A^{\sigma(i_2)} \cdots t_A^{\sigma(i_{n-1})})_{1n} A_n^{\rm vector}(1,\sigma,n)$$

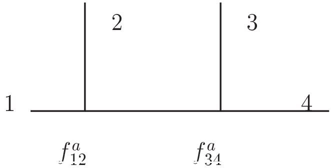
- The extra piece is cancelled by the monodromy relations $\sum S\, A^{\rm vector} = 0$

BCJ parametrisation

- The symmetry between color and lorentz dependence inspired [Bern-Carrasco-Johansson] (BCJ) to introduce the following numerator parametrisation
- The tree level amplitude takes the form

$$\mathcal{A}_n^{\text{YM}} = \sum_i \frac{n_i c^i}{\prod_{r=1}^{n-3} p_r^2}$$

- where c^i are the color factor of the graph in the adjoint basis



- with $c_{1234} = f_{12}^x f_{34}^x$

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$$\mathcal{A}_n^{\rm YM} = \sum_i \frac{n_i c^i}{\prod_{r=1}^{n-3} p_r^2}$$

- Gravity amplitude are constructed as

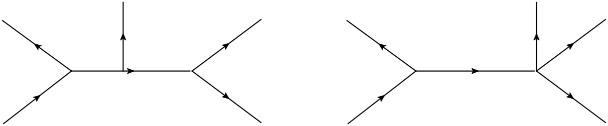
$$\mathcal{A}_n^{\rm Grav} = \sum_i \frac{n_i \tilde{n}^i}{\prod_{r=1}^{n-2} p_r^2}$$

- What are the constraints on the numerator n_i in these expansions?

Lorentz numerators: the five-point case I

- We consider the for color ordered gauge amplitudes

$$A_5^{\text{vector}}(\sigma(1),\dots,\sigma(5)) = \sum_{i=1}^{5} \frac{n_{r_i}}{p_{1,i}^2 p_{2,i}^2}$$



- The numerator factors are *not* gauge invariant but the pairing $n_i c_i$ summed over the graph gives the gauge invariant amplitudes

Lorentz numerators: the five-point case II

- The $\mathcal{S}$-kernel imply the relations on the ordered amplitudes

$$0 = (s_{13}+s_{23})A_5^{\text{vector}}(1,2,3,4,5) - s_{35}A_5^{\text{vector}}(1,2,4,3,5) + s_{13}A_5^{\text{vector}}(1,3,2,4,5)$$

- The system is solved by the generalized dual Jacobi relations

$$X_{ijk} = n_i - n_j + n_k; \qquad 0 = \frac{X_3}{s_{45}} - \frac{X_9}{s_{24}} - \frac{X_2}{s_{12}} - \frac{X_5}{s_{51}}$$

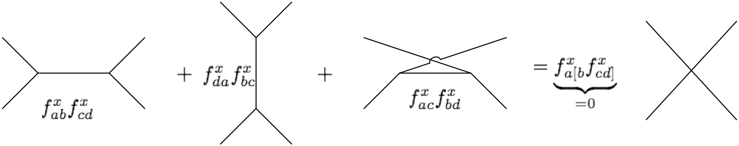
- That can be solved by [Tye, Zhang; Bjerrum-Bohr, Damgaard, Søndergaard, Vanhove]

$$X_{ijk} = f_i\, s_{kl}$$

- BCJ and `[Mafra et al.]` choice is $X_{ijk} = 0$ which is a gauge choice.

Jacobi relations and gauge invariance

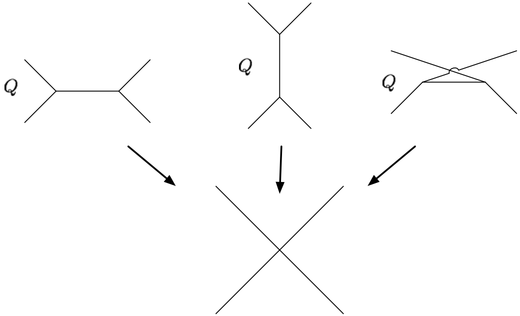
- The right-hand side has to do with reabsorbing the 4-point contact term needed for gauge invariance



- The Jacobi one the f_{abc} implies locally the relation $HIX = 0$ which takes care of the ambiguity

Jacobi relations and gauge invariance

- Very natural in pure spinor formalism where Q such that $Q^2 = 0$ and physical states are in Q-cohomology and $Qb = T \sim p^2$: leads to some multiloop systematic [see Michael Green's talk]



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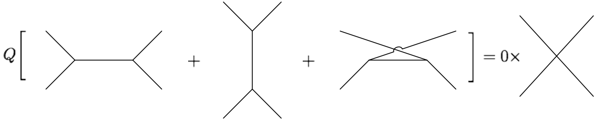

- See as well the pure spinor approach by [Mafra et al.; Schloterrer's talk]

Gravity amplitudes

- The gravity amplitude can be reconstructed from the Lorentz generators n_i and $\tilde{n}_i$ satisfying (generalized) dual Jacobi

$$X_{ijk} = n_i - n_j + n_k = f_{ijk}\, s_*; \qquad \tilde{X}_{ijk} = \tilde{n}_i - \tilde{n}_j + \tilde{n}_k = g_{ijk}\, s_*$$

$$M_5(1,2,3,4,5) \sim \frac{n_1 \tilde{n}_1}{s_{12} s_{45}} + \cdots$$

- With the following consistency constraint [Bjerrum-Bohr et al.; Bern et al.]

$$\sum_i f_i g_j M_{ij} = 0$$

- M_{ij} numerical array
- In general the right-hand side of the extended dual Lorentz-relation $X_{ijk} = P_{n-3}(s_{ij})$ and $\tilde{X}_{ijk} = \tilde{P}_{n-3}(s_{ij})$ will satisfy $P \otimes M \otimes \tilde{P} = 0$
- Strict Jacobi $\tilde{P} = 0$ is enough for a correct amplitude [Bern, Carrasco, Johansson; Bern, Dennen, Huang, Kiermaier]

Comparing string and field theory amplitudes I

- The 3-point vertices the open superstring amplitudes reproduces the field theory result

$$\mathfrak{A}_n^{\text{tree}}(1,2,3) = \text{Tr}(\lambda^{a_1}\lambda^{a_2}\lambda^{a_3})\,\epsilon^{\mu}_{a_1}\epsilon^{\nu}_{a_2}\epsilon^{\rho}_{a_3}\,(k^2_\mu\eta_{\nu\rho} + k^1_\rho\eta_{\mu\nu} + k^3_\nu\eta_{\mu\rho})$$

- The lorentz vertex is totally antisymmetric in the external particle labels

$$V_{a_1a_2a_3} = \epsilon^{\mu}_{a_1}\epsilon^{\nu}_{a_2}\epsilon^{\rho}_{a_3}\,(k^2_\mu\eta_{\nu\rho} + k^1_\rho\eta_{\mu\nu} + k^3_\nu\eta_{\mu\rho})$$

- projects the trace structure on $f^{abc} = \text{Tr}([\lambda^a, \lambda^b]\lambda^c)$ from $\lambda^a\lambda^b = \delta^{ab}/N + (f^{ab}_c + d^{ab}_c)\lambda^c$

Comparing string and field theory amplitudes II

- String and field theory amplitudes take a similar looking form but there is an important difference due to the massive string modes

$$\mathfrak{A}_n^{\rm tree}(\sigma(1,\dots,n)) = \sum_{\gamma\in\mathfrak{S}_{n-3}} \mathcal{S}_{\alpha'}[\sigma|\gamma]\, A_n^{\rm FT}(\gamma(1,\dots,n))\, F_{\alpha'}(\sigma,\gamma)$$

- Where $F_{\alpha'}(\sigma,\gamma) = 1 + O(\alpha')$ is a form factor that goes to 1 when $\alpha' \to 0$. This form is related to the one given by [Mafra et al.; Schloterrer's talk] where there $F^{\sigma,\gamma} = F_{\alpha'} \times \mathcal{S}$

- One dramatic effect of massive string modes is that the $U(1)$ decoupling identities

$$\mathfrak{A}_n^{\rm tree} = \sum_{\sigma\in\mathfrak{S}_{n-2}} c_\sigma A(\sigma) + O(\alpha')$$

Comparing string and field theory amplitudes III

- For instance in the 4-point amplitude the α' correction involve the symmetrized trace that will require the d_{abc}

$$\delta\mathfrak{A} = \sum \mathrm{symTr}(t_8 F^1 \cdots F^4) + \cdots$$

- As well the d_{abc} arises as a coefficient of the chiral anomaly then at loop some generalization may be needed

- In fact one can constraint the effective action of a YM theory with local higher derivative terms from deformation of the field theory KLT relations (see for instance [Bjerrum-Bohr])

- The $\mathcal{S}$ gives constraints on the amplitudes including the higher derivative operators

Part II

Supersymmetry and UV divergences

$\mathcal{N} = 4$ SYM UV divergences [Berkovits, Green, Russo, Vanhove]

$$[\mathfrak{A}^{(D)}_{4;L}] = \Lambda^{(D-4)L-4-2\gamma_L}\, \partial^{2\gamma_L}\, t_8 \mathrm{Tr} F^4 + \Lambda^{(D-4)L-4-2\beta_L}\, \partial^{2\beta_L}\, t_8 (\mathrm{Tr} F^2)^2$$

	L=1	L=2	L=3	L=4	L=5
$\partial^{2\gamma_L}\, t_8 \mathrm{Tr}(F^4)$	$D_c = 8$ $\gamma_1 = 0$	$D_c = 7$ $\gamma_2 = 1$	$D_c = 6$ $\gamma_3 = 1$	$D_c = \frac{11}{2}$ $\gamma_4 = 1$	$D_c = \frac{26}{5}$ $\gamma_5 = 1$
$\partial^{2\beta_L}\, t_8 (\mathrm{Tr} F^2)^2$	$D_c = 8$ $\beta_1 = 0$	$D_c = 7$ $\beta_2 = 1$	$D_c = \frac{20}{3}$ $\beta_3 = 2$	$D_c = 6$ $\beta_4 = 2$	$D_c = \frac{28}{5}$ $\beta_5 = 2$

- Some F-term are in $D < 10$

$$\partial^2 t_8 \mathrm{Tr}(F^4) \sim \int d^8\theta\, \mathrm{Tr}(W_\alpha^4)$$

$$\partial^4 t_8 (\mathrm{Tr}(F^2))^2 \sim \int d^{12}\theta\, (\mathrm{Tr}(W_\alpha^2))^2$$

- Gaugino superfield
  $W_\alpha = \chi_\alpha + \cdots + (\theta\gamma^{mn})_\alpha F_{mn} + (\theta\gamma^p\theta)(\theta\gamma^{mn})_\alpha \partial_p F_{mn} + \cdots$

$\mathcal{N} = 4$ SYM UV divergences [Berkovits, Green, Russo, Vanhove]

$$[\mathfrak{A}^{(D)}_{4;L}] = \Lambda^{(D-4)L-4-2\gamma_L}\,\partial^{2\gamma_L}\, t_8 \mathrm{Tr} F^4 + \Lambda^{(D-4)L-4-2\beta_L}\,\partial^{2\beta_L}\, t_8 (\mathrm{Tr} F^2)^2$$

	L=1	L=2	L=3	L=4	L=5
$\partial^{2\gamma_L}\, t_8 \mathrm{Tr}(F^4)$	$D_c = 8$ $\gamma_1 = 0$	$D_c = 7$ $\gamma_2 = 1$	$D_c = 6$ $\gamma_3 = 1$	$D_c = \frac{11}{2}$ $\gamma_4 = 1$	$D_c = \frac{26}{5}$ $\gamma_5 = 1$
$\partial^{2\beta_L}\, t_8 (\mathrm{Tr} F^2)^2$	$D_c = 8$ $\beta_1 = 0$	$D_c = 7$ $\beta_2 = 1$	$D_c = \frac{20}{3}$ $\beta_3 = 2$	$D_c = 6$ $\beta_4 = 2$	$D_c = \frac{28}{5}$ $\beta_5 = 2$

- Some F-term are descendant of the Konishi operator $\mathrm{tr}(\Phi \cdot \Phi)$ in $D < 10$

$$\partial^2 t_8 \mathrm{Tr}(F^4) \sim \int d^{16}\theta\, \mathrm{Tr}(\Phi \cdot \Phi)$$

$$\partial^4 t_8 (\mathrm{Tr}(F^2))^2 \sim \int d^{16}\theta\, (\mathrm{tr}(\Phi \cdot \Phi))^2$$

- These operators are not protected from quantum corrections

$\mathcal{N}=4$ SYM UV divergences [Berkovits, Green, Russo, Vanhove]

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For $L\geqslant 4$ the UV divergence is dominated by the single trace term

single trace	$\Lambda^{(D-4)L-6}\,\partial^2 t_8\mathrm{tr}(F^4)$	$L\geqslant 2$	$D_c=4+\frac{6}{L}$
double trace	$\Lambda^{(D-4)L-8}\,\partial^4 t_8(\mathrm{tr}F^2)^2$	$L\geqslant 3$	$D_c=4+\frac{8}{L}$

- $\mathcal{N}=3$ superspace explains the leading UV behaviour [Howe, Stelle]
- Confirmed by amplitude computation [Bern, Dixon, Carrasco, Johansson,Roiban]

The ultraviolet behaviour of $\mathcal{N} = 8$ supergravity

- Up to and including 4-loop order the critical UV behaviour is the same in for $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ SUGRA [Bern et al.;Green, Russo, Vanhove]

$$[\mathfrak{M}_{4;L}^{(D)}] \sim \Lambda^{(D-4)L-6}\, \partial^{2L}\, \mathcal{R}^4 \qquad 2 \leqslant L \leqslant 4$$

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$$[\mathfrak{M}^{(D)}_{4;L}] \sim \Lambda^{(D-4)L-6}\, \partial^{2L}\, \mathcal{R}^4 \qquad 2 \leqslant L \leqslant 4$$

- After 4-loop it is expected a worse UV behaviour than for $\mathcal{N}=4$ SYM

$$[\mathfrak{M}^{(D)}_{4;L}] \sim \Lambda^{(D-2)L-14}\, \partial^{8}\, \mathcal{R}^4 \qquad L \geqslant 4$$

- At five-loop order the 4-point amplitude in
  - $\mathcal{N}=4$ SYM divergences for $5 < 26/5 \leqslant D$
  - $\mathcal{N}=8$ SUGRA divergences for $24/5 \leqslant D$

Would imply a *seven-loop* divergence in $D=4$ with counter-term $\partial^8\mathcal{R}^4$

[Green, Russo, Vanhove;Vanhove;Green, Bjornsson]

Linearized $D = 4$ $\mathcal{N} = 8$ supergravity

- At the linearized level one can construct the invariants

$$\int d^4x \int d^{8+2L}\theta d^{8+2L}\bar\theta\, (W\bar W)^2 \sim \partial^{2L}\,\mathcal{R}^4, \qquad L = 0, 2, 3, 4$$

where

$$W_{ijkl} = \phi_{ijkl} + \cdots + (\theta_i\gamma^{mn}\theta_j)(\theta_k\gamma^{pq}\theta_k)R_{mnpq} + \cdots$$

- ϕ_{ijkl} and $\bar\phi^{ijkl} = \frac{1}{24}\,\epsilon^{ijklmnpq}\phi_{mnpq}$ are the **70** scalar fields parametrizing the coset space $E_7/(SU(8)/\mathbb{Z}_2)$
- The structure of these F-terms is determined by the $SU(2,2|8)$ superconformal representations [Petkova, Dobrev;Drummond, Heslop, Howe, Kerstan]
- Classification can be obtained as well from soft limit properties of scattering amplitudes [Elvang et al.; Beisert et al.]
- expression $SU(8)$ invariant but not E_7 invariant

Harmonic superspace

Harmonic superspace is an extension of the usual superspace $\mathbb{R}^{4|4\mathcal{N}}$ with the addition of extra bosonic coordinates in the flag manifold [Rosly; Galperin, Ivanov, Ogievetsky, Sokatchev]

$$\mathbb{F}_{q,p} = (U(p) \times U(\mathcal{N}-p-q) \times U(q)) \backslash U(\mathcal{N})$$

- For $\mathcal{N}=8$ at the linearized level we can consider the 1/2, 1/4, and 1/8 BPS measure [Drummond, Heslop, Howe, Kerstan]

$$\int d^4x d\tilde{\mu}_{(8,4-L,4-L)} \, (W\bar{W})^4 \sim \int d^4x \, \partial^{2L} \, \mathcal{R}^4 \qquad L = 0,2,3,4$$

$$d\tilde{\mu}_{(8,p,p)} = d^{16-2p}\theta d^{16-2p}\bar{\theta} \, du$$

- There are F-terms satisfying the non-renormalisation theorems where L is the maximal loop order for their quantum corrections [Green, Russo, Vanhove]

What about non-protected operators?

- So far we have discussed protected $1/2$, $1/4$ and $1/8$-BPS operators
- What about non-protected operators?

For $\mathcal{N} = 8$ supergravity that with 32 supercharges dimension analysis indicates that the dimensions 16 operator $\nabla^8\mathcal{R}^4$ could be an D-term given by the volume of superspace

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Доверяй, но проверяй!

Harmonic superspace

Only the harmonic measures with $d\mu_{(\mathcal{N},1,1)}$ can be extended to full superspace

[Bossard, Howe, Stelle, Vanhove]

- We can make special the coordinates $\zeta^\alpha := \theta^\alpha_1$ and $\bar\zeta^{\dot\alpha} := \bar\theta^{\mathcal{N}\,\dot\alpha}$ because of the obstruction from the dimension 1/2 torsion $T^{ij\dot\gamma\,k}_{\alpha\beta} = \epsilon_{\alpha\beta}\bar\chi^{\dot\gamma\,ijk}$

$$\begin{aligned} \hat{E}_{\hat{A}} &:= \{\tilde{E}^1_\alpha, \tilde{E}_{\dot\alpha\,\mathcal{N}}, d^1{}_r, d^r{}_{\mathcal{N}}, d^1{}_{\mathcal{N}}\}\,, \qquad 2\leqslant r\leqslant \mathcal{N}-1 \\ \{\hat{E}_{\hat{A}}, \hat{E}_{\hat{B}}\} &= C_{\hat{A}\hat{B}}{}^{\hat{C}}\,\hat{E}_{\hat{C}}\,, \end{aligned}$$

preserved by the structure group $SL(2,\mathbb{C})\times U(1)\times U(\mathcal{N}-2)\times U(1)$

- Normal coordinates

$$\zeta^{\hat{A}} := \{\zeta^\alpha, \bar\zeta^{\dot\alpha}, z^r{}_1, z^{\mathcal{N}}{}_r, z^{\mathcal{N}}{}_1\},$$

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- Extending to $\mathcal{N} \geqslant 4$ the flot equation in [Kuzenko et al.] one shows that

$$\zeta^{\hat\alpha}\partial_{\hat\alpha}\ln E = -\frac{1}{3}B_{\alpha\dot\beta}\zeta^\alpha\bar\zeta^{\dot\beta} + \frac{1}{18}B_{\alpha\dot\beta}B_{\alpha\dot\alpha}\zeta^\alpha\zeta^\beta\bar\zeta^{\dot\alpha}\bar\zeta^{\dot\beta}\,.$$

- The supervielbein takes the very simple form without a quadratic term in $\zeta^2\bar\zeta^2$

$$E(\hat x,\zeta,\bar\zeta) = E|_{\zeta=0}\left(1 - \frac{1}{6}B_{\alpha\dot\beta}\zeta^\alpha\zeta^{\dot\beta} + 0\zeta^2\bar\zeta^2\right)$$

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- G-analytic field $D_{\alpha\,1}B_{\alpha\dot\beta} = \bar D_{\dot\alpha\,\mathcal{N}}B_{\alpha\dot\beta} = 0$

$$B_{\alpha\dot\beta} = \begin{cases} \bar\chi_{\dot\beta}^{1ij}\chi_{\alpha\,\mathcal{N}ij} & \text{for } \mathcal{N}=4,5,8 \\ \bar\chi_{\dot\beta}^{1ij}\chi_{\alpha\,6ij} + \frac{1}{3}\chi_\alpha^{1ijkl}\bar\chi_{\dot\beta\,6ijkkl} & \text{for } \mathcal{N}=6 \end{cases}$$

- The superfield χ_α^{ijk} has dimension 1/2

$$\bar\chi_{\dot\alpha}^{ijk} = \bar D_{\dot\alpha\,l}\bar W^{ijkl} = \cdots + \bar\theta^3 R + \theta^5\nabla R + \cdots$$

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[Bossard, Howe, Stelle, Vanhove]

- One then defines the $1/\mathcal{N}$ harmonic measure (over $4(\mathcal{N}-1)$ θs)

$$\int d^4x d^{4\mathcal{N}}\theta E(x,\theta)\,\Phi(x,\theta) =: \int d^4x\, d\mu_{(\mathcal{N},1,1)}\,(D^1)^2(\bar{D}_{\mathcal{N}})^2\Phi|_{\zeta=0}$$

- With $\Phi = 1$ one shows that the duality invariant volume is vanishing

$$\int d^4x d^{4\mathcal{N}}\theta E(x,\theta) = 0, \qquad 4 \leqslant \mathcal{N} \leqslant 8$$

- The $\mathcal{N}-1$-loop candidate counter-term $\nabla^{2(\mathcal{N}-4)}\mathcal{R}^4$ term

$$\int d\mu_{(\mathcal{N},1,1)}\,\bar{\chi}^{1mn}\chi_{8mn}\bar{\chi}^{1pq}\chi_{8pq} \sim \int d^4x e\,(\nabla^{2(\mathcal{N}-4)}\,\mathcal{R}^4 + \cdots)$$

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$$\int d^4x d^{4\mathcal{N}}\theta E(x,\theta) = 0, \qquad 4 \leqslant \mathcal{N} \leqslant 8$$

- For $\mathcal{N}=8$ this is explicitly E_7-invariant (dim 1/2 torsion $T\sim\chi$)

$$\int d\mu_{(8,1,1)}\,\bar{\chi}^{1mn}\chi_{8mn}\bar{\chi}^{1pq}\chi_{8pq} \sim \int d^4xe\,(\nabla^8\,\mathcal{R}^4+\cdots)$$

- Supersymmetric E_7 invariant candidate counter-term.

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- For $\mathcal{N}=8$ the 1/8 BPS coupling $\nabla^6\,\mathcal{R}^4$

$$\int d^4x d\mu_{(8,1,1)}\,E(x,\theta,u)\,F(\mathcal{V}) = \int d^4x\,e\,(f_{(0,1)}(\phi)\,\nabla^6\,\mathcal{R}^4 + \text{susy completion})$$

- Fully supersymmetric, $SU(8)$ invariant but not E_7 invariant expression $\mathcal{V}\in E_7/(SU(8)/\mathbb{Z}_2)$

Full superspace integrals for $\mathcal{N} = 8$

- In case there is no 7-loop divergence in $D = 4$, we can construct a host of E_7 invariant full superspace integral, like the 8-loop candidate counter-term [Kallosh; Howe, Lindstrom]

$$\int d^4x d^{32}\theta E(x,\theta)\,(\chi\bar{\chi})^4 \sim \int d^4x e(x)\,\nabla^{10}R^4 + \cdots$$

- Green: explicitly checked by field theory or string theory computation
- Black 'allowed': Allowed by not explicitly checked
- Red: First possible ultraviolet divergence. Coefficient has not been evaluated.

	R^4	$\partial^4 R^4$	$\partial^6 R^4$	$\partial^8 R^4$	$\partial^{10} R^4$	$\partial^{12} R^4$
D=11	– –	– –	– –	– –	– –	L=2 yes
D=10	– –	– –	– –	– –	L=2 yes	– –
D=9	– –	– –	– –	L=2 yes	– –	– –
D=8	L=1 yes	– –	L=2 yes	– –	– –	L=3 yes
D=7	– –	L=2 yes	– –	– –	– –	– –
D=6	– –	– –	L=3 yes	– –	L=4 yes	– –
D=5	L=2 no	– –	L=4 no	– –	– –	L=6
D=4	L=3 no	L=5 no	L=6 no	L=7 !	L=8	L=9

An allowed E_7 invariant 7-loop counterterm in $D = 4$ constructed in [Bossard, Howe, Stelle, Vanhove]

Outlook

The S kernel provides that best possible reorganization of the tree-level amplitude both in gravity and gauge theory and in string theory

- At tree-level this allows to derive the BCJ parametrisation
- What is the extension at loop orders?
  - Need to relate different topologies related by the $HIX = 0$ relation [See Carrasco's talk].
  - The Pure spinor formalism seems to be the most promising
- UV divergences of $\mathcal{N} = 8$ if no 7-loop divergences, the divergences are pushes to 8-loop where bona fida D-term counter-terms exists

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Lucky soon we can call us expert in $\mathcal{N} = 8$ supergravity since

An expert is a person who has made
all the mistakes that can be made in a very narrow field.
Niels Bohr