Renormalization in the Unitarity Method

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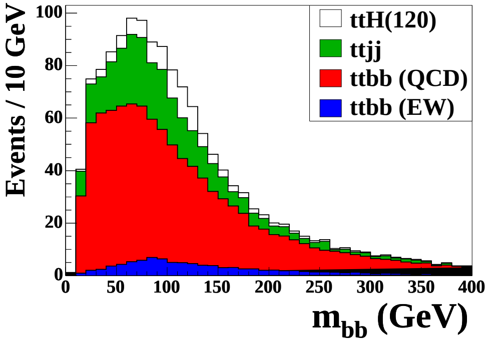
One-loop amplitudes



Higgs Signal With Background (top quark associated production)

NLO (Next-to-Leading-Order) Calculations

- provide some essential precision
- reduce scale dependence

Prioritized Wish List, Next-to-Leading Order

[Les Houches Physics at TeV colliders 2007, NLO multileg working group: Summary report; updated 2009]

Done [a]	$p\,p \to t\,\bar{t}\,b\,\bar{b}$	background for $t\bar{t}H$
Done [b]	$p\,p \to t\,\bar{t} + 2$ jets	relevant for $t\bar{t}H$
✓	$p\,p \to V\,V\,b\,\bar{b}$	relevant for benchmark processes
Done [c]	$p\,p \to V\,V + 2$ jets	VBF $\to H \to VV$
Done [d]	$p\,p \to V + 3$ jets	new physics
Done [e]	$p\,p \to b\,\bar{b}\,b\,\bar{b}$	Higgs and new physics
	$p\,p \to t\,\bar{t}\,t\,\bar{t}$	new physics
	$p\,p \to W\,b\,\bar{b}\,j$	new physics

[a]: Bredenstein, Denner, Dittmaier, Pozzorini; Bevilacqua, Czakon, Papadopoulos, Pittau, Worek [b]: Bevilacqua, Czakon, Papadopoulos, Worek [c]: Jager, Melia, Melnikov, Nason, Rontsch, Zanderighi [d]: Berger, Bern, Dixon, Febres Cordero, Forde, Gleisberg, Ita, Kosower, Maître; Ellis, Melnikov, Zanderighi [e]: Greiner, Guffanti, Reiter, Reuter

2001 Experimenter's Wish List [Knuteson, Campbell]

single boson	diboson	triboson	heavy flavor
$W+\leq 5j$	$WW+\leq 5j$	$WWW+\leq 3j$	$t\bar{t}+\leq 3j$
$Wb\bar{b}+\leq 3j$	$WWb\bar{b}+\leq 3j$	$WWWb\bar{b}+\leq 3j$	$t\bar{t}\gamma+\leq 2j$
$Wc\bar{c}+\leq 3j$	$WWc\bar{c}+\leq 3j$	$WWW\gamma\gamma+\leq 3j$	$t\bar{t}W+\leq 2j$
$Z+\leq 5j$	$ZZ+\leq 5j$	$Z\gamma\gamma+\leq 3j$	$t\bar{t}Z+\leq 2j$
$Zb\bar{b}+\leq 3j$	$ZZb\bar{b}+\leq 3j$	$WZZ+\leq 3j$	$t\bar{t}H+\leq 2j$
$Zc\bar{c}+\leq 3j$	$ZZc\bar{c}+\leq 3j$	$ZZZ+\leq 3j$	$t\bar{b}+\leq 2j$
$\gamma+\leq 5j$	$\gamma\gamma+\leq 5j$		$b\bar{b}+\leq 3j$
$\gamma b\bar{b}+\leq 3j$	$\gamma\gamma b\bar{b}+\leq 3j$		$b\bar{b}t\bar{t}$
$\gamma c\bar{c}+\leq 3j$	$\gamma\gamma c\bar{c}+\leq 3j$		
	$WZ+\leq 5j$		
	$WZb\bar{b}+\leq 3j$		
	$WZc\bar{c}+\leq 3j$		
	$W\gamma+\leq 3j$		
	$Z\gamma+\leq 3j$		

Numerical Programs for Virtual Corrections

- `BlackHat` [Berger, Bern, Diana, Dixon, Febres Cordero, Forde, Gleisberg, Höche, Ita, Kosower, Maître, Ozeren]
- `CutTools/HELAC-NLO` [Ossola, Papadopoulos, Pittau; Actis, Bevilacqua, Czakon, Draggiotis, Garzelli, van Hameren, Worek, ...]
- `GOLEM` [Binoth, Guillet, Heinrich, Pilon, Reiter, Schubert]
- `MadLoop` [Hirschi, Frederix, Frixione, Garzelli, Maltonin, Pittau]
- `NGluon` [Badger, Biedermann, Uwer]
- `Rocket` [Ellis, Giele, Kunszt, Melnikov, Zanderighi]
- `SAMURAI` [Mastrolia, Ossola, Reiter, Tramontano]

Analytic one-loop amplitudes

Completed recently:

pp $\longrightarrow$ Higgs + 2 jets. [Badger, Berger, Campbell, Del Duca, Dixon, Ellis, Giele, Glover, Mastrolia, Risager, Sofianatos, Williams, Zanderighi]

pp $\longrightarrow$ $t\bar{t}$ [Badger, Sattler, Yudin]

$0 \longrightarrow d\bar{u}Q\bar{Q}\bar{\ell}\ell$, W-mediated. [Badger, Campbell, Ellis]

$gg \longrightarrow WW \longrightarrow \nu ll\nu$ including SM Higgs [Campbell, Ellis, Williams]

Analytic techniques

- may have advantages in stability and speed,
- extend readily to larger numbers of external particles,
- and have numerical counterparts.

Outline

1. Structure of one-loop amplitudes, and the unitarity method

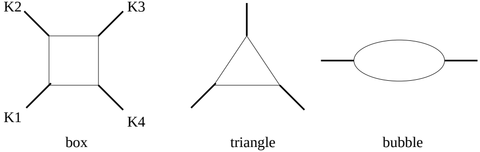
2. When cuts diverge: a prescription for treating the wavefunction renormalization of external massive fermions
   [work with E. Mirabella, to appear soon]

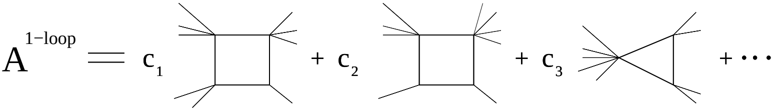
One-Loop Amplitudes

In 4-dimensional massless theories, reduction of Feynman integrals brings the one-loop amplitude to the form

$$A = \sum_i d_i \text{ (box)} + \sum_i c_i \text{ (triangle)} + \sum_i b_i \text{ (bubble)} + \text{rational}$$

where the master integrals have scalar structure and are known explicitly. [in dim. reg.: Bern, Dixon, Kosower]

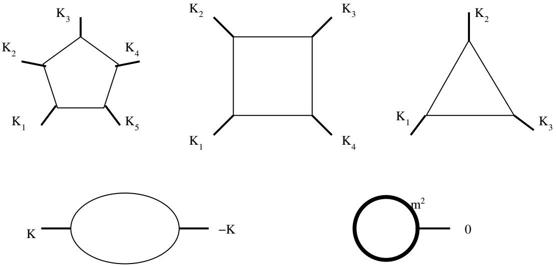




One-Loop Amplitudes

In $D = 4 - 2\epsilon$ dimensions, and allowing for internal masses, the result of reduction is

$$A = \sum_i e_i \text{ (pentagon)} + \sum_i d_i \text{ (box)} + \sum_i c_i \text{ (triangle)} + \sum_i b_i \text{ (bubble)} + \sum_i a_i \text{ (tadpole)}$$



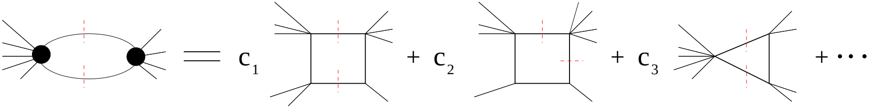
Singularities in One-Loop Amplitudes

- As kinematic functions, one-loop amplitudes have poles, branch cuts, and divergences.
- These singularities constrain the amplitude – enough to bypass a Feynman diagram expansion.
- This is the principle of the "unitarity method" and its numerical cousins.

Amplitudes from unitarity cuts

$$\Delta A^{1-\mathrm{loop}} = \sum c_i \Delta I_i$$

Tree level input.



Matching 4-dimensional cuts can suffice to determine reduction coefficients! Logarithms with unique arguments.
"Cut-constructibility"

[Bern, Dixon, Dunbar, Kosower]

But: we get several coefficients together in the same equation.

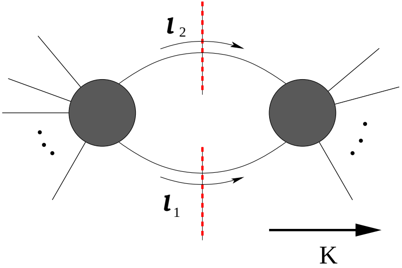
How do we evaluate a unitarity cut?

Unitarity Cuts: Loops from Trees

$$\Delta A^{1-\mathrm{loop}} = \int d\mu \;\; A^{\mathrm{tree}}_{\mathrm{Left}} \;\times\; A^{\mathrm{tree}}_{\mathrm{Right}}$$

where

$$d\mu = d^4\ell_1 \; d^4\ell_2 \; \delta^{(4)}(\ell_1 + \ell_2 - K) \; \delta(\ell_1^2) \; \delta(\ell_2^2)$$



By unitarity, this is the discontinuity of the amplitude across a branch cut. [Cutkosky]

Cut integrals

The cut integrand is a rational function of the loop momentum.

$$\Delta A^{1-\mathrm{loop}} = \int d\mu \; A_{\mathrm{Left}}(\ell) \;\times\; A_{\mathrm{Right}}(\ell)$$

$$d\mu = d^4\ell \;\; \delta^+(\ell^2)\; \delta^+((\ell - K)^2)$$

Change to homogeneous (CP^1) spinor variables with

$$\ell_{a\dot{a}} = t \; \lambda_a \tilde{\lambda}_{\dot{a}}.$$

Integration measure:

$$\int d^4\ell \; \delta^+(\ell^2) \; (\bullet) = \int_0^\infty dt \;\; t \int_{\tilde{\lambda}=\bar{\lambda}} \langle \lambda \; d\lambda \rangle \; [\tilde{\lambda} \; d\tilde{\lambda}] \; (\bullet)$$

[Cachazo, Svrček, Witten]

Spinor variables

From Lorentz 4-vector to spinor indices with Pauli matrices:

$$p_{a\dot{a}} = \sigma^{\mu}_{a\dot{a}}\ p_{\mu} \qquad\qquad a, \dot{a} = 1, 2$$

For a null vector (massless particle):

$$0 = p^2 = \det(p_{a\dot{a}}) \implies p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}.$$

Lorentz-invariant spinor products:

$$\begin{aligned} \langle \lambda\ \lambda' \rangle &\equiv \epsilon_{ab}\ \lambda^a\ \lambda'^b \\ [\tilde{\lambda}\ \tilde{\lambda}'] &\equiv \epsilon_{\dot{a}\dot{b}}\ \tilde{\lambda}^{\dot{a}}\ \tilde{\lambda}'^{\dot{b}} \end{aligned}$$

Spinor integration

[Anastasiou, RB, Buchbinder, Cachazo, Feng, Kunszt, Mastrolia]

- Change variables, $\ell = t\lambda\tilde{\lambda}$, and use the spinor measure,

$$\int d^4\ell \; \delta(\ell^2)\delta((\ell - K)^2) = \int dt \; t \int \langle\lambda \; d\lambda\rangle[\tilde{\lambda} \; d\tilde{\lambda}]\delta((t\lambda\tilde{\lambda} - K)^2)$$

- Use 2nd delta function to perform t-integral.
- Each term of integrand takes the form:

$$\frac{(K^2)^{n+1}}{\langle\lambda|K|\tilde{\lambda}]^{n+2}} \frac{\prod_{i=1}^{n+k}\langle\lambda|R_i|\tilde{\lambda}]}{\prod_{j=1}^{k}\langle\lambda|Q_j|\tilde{\lambda}]}$$

- Evaluate with residue theorem.
- Identify cuts of basis integrals and read off coefficients.
- We have given formulas for the resulting coefficients.

Cuts of Master Integrals

$$\Delta I_2 = \int \langle\lambda\, d\lambda\rangle[\tilde\lambda\, d\tilde\lambda] \frac{K^2}{\langle\lambda|K|\tilde\lambda]^2}$$

$$\Delta I_3 = \int \langle\lambda\, d\lambda\rangle[\tilde\lambda\, d\tilde\lambda] \frac{1}{\langle\lambda|K|\tilde\lambda]\langle\lambda|Q_1|\tilde\lambda]}$$

$$\Delta I_4 = \int \langle\lambda\, d\lambda\rangle[\tilde\lambda\, d\tilde\lambda] \frac{1}{K^2}\frac{1}{\langle\lambda|Q_1|\tilde\lambda]\langle\lambda|Q_2|\tilde\lambda]}$$

$$Q_j \equiv -K_j + \frac{K_j^2}{K^2}K$$

Formulas for coefficients

From a general cut integrand $\mathcal{T}^{(N)}(\ell)$. N is the degree.

Original integral:

$$i(4\pi)^{\frac{D}{2}} \int \frac{d^D\ell}{(2\pi)^D} \delta^{(+)}(\ell^2)\ \delta^{(+)}((K-\ell)^2)\mathcal{T}^{(N)}(\ell),$$

To define the target master integrals:

$$D_i(\ell) \ = \ (\ell - K_i^2)$$

Box coefficients

$$C[K_r, K_s, K] = \frac{1}{2} \left(\mathcal{T}^{(N)}(\ell) D_r(\ell) D_s(\ell) \right)\Big|_{\lambda \to P_{sr,1}, \tilde{\lambda} \to P_{sr,2}} + \{P_{sr,1} \leftrightarrow P_{sr,2}\}$$

$$\begin{aligned}
P_{sr,1} &= Q_s + \left(\frac{-Q_s \cdot Q_r + \sqrt{\Delta_{sr}}}{Q_r^2} \right) Q_r, \\
P_{sr,2} &= Q_s + \left(\frac{-Q_s \cdot Q_r - \sqrt{\Delta_{sr}}}{Q_r^2} \right) Q_r, \\
\Delta_{sr} &= (Q_s \cdot Q_r)^2 - Q_s^2 Q_r^2 .
\end{aligned}$$

Triangle coefficients

$$
\begin{aligned}
C[K_s, K] \;=\;& \frac{1}{2(N+1)!\sqrt{\Delta_s}^{N+1}\langle P_{s,1}\; P_{s,2}\rangle^{N+1}} \\
&\times \frac{d^{N+1}}{d\tau^{N+1}} \left(\mathcal{T}^{(N)}(\ell) D_s(\ell) \langle\lambda|K|\tilde{\lambda}]^{N+1} \Big|_{\tilde{\lambda}\to Q_s\lambda,\, \lambda\to P_{s,1}-\tau P_{s,2}} \right) \Bigg|_{\tau\to 0} \\
&+\{P_{s,1} \leftrightarrow P_{s,2}\}
\end{aligned}
$$

$$
\begin{aligned}
P_{s,1} &= Q_s + \left(\frac{-Q_s\cdot K + \sqrt{\Delta_s}}{K^2} \right) K, \\
P_{s,2} &= Q_s + \left(\frac{-Q_s\cdot K - \sqrt{\Delta_s}}{K^2} \right) K, \\
\Delta_s &= (Q_s\cdot K)^2 - Q_s^2 K^2.
\end{aligned}
$$

Bubble coefficient

$$C[K] = K^2 \sum_{q=0}^{N} \frac{(-1)^q}{q!} \frac{d^q}{ds^q} \left(\mathcal{B}^{(0)}_{N,N-q}(s) + \sum_{r=1}^{k} \sum_{a=q}^{N} \left(\mathcal{B}^{(r;a-q;1)}_{N,N-a}(s) - \mathcal{B}^{(r;a-q;2)}_{N,N-a}(s) \right) \right) \Bigg|_{s=0} ,$$

$$\mathcal{B}^{(0)}_{N,m}(s) \equiv \frac{d^N}{d\tau^N} \left(\frac{(2\eta \cdot K)^{m+1} \langle \lambda | K | \tilde{\lambda}]^N}{N! [\eta|\eta' K|\eta]^N (m+1)(K^2)^{m+1} \langle \lambda\ \eta \rangle^{N+1}} \mathcal{T}^{(N)}(\ell) \Bigg|_{\substack{\tilde{\lambda} \to (K+s\eta)\cdot\lambda \\ \lambda \to (K-\tau\eta')\cdot\eta}} \right) \Bigg|_{\tau \to 0} ,$$

$$\mathcal{B}^{(r;b;1)}_{n,m}(s) \equiv \frac{(-1)^{b+1}}{b!(m+1)\sqrt{\Delta_r}^{b+1} \langle P_{r,1}\ P_{r,2} \rangle^b} \times \frac{d^b}{d\tau^b} \left(\frac{\langle \lambda|\eta|P_{r,1}]^{m+1} \langle \lambda|Q_r\eta|\lambda\rangle^b \langle \lambda|K|\tilde{\lambda}]^{N+1}}{\langle \lambda|K|P_{r,1}]^{m+1} \langle \lambda|\eta K|\lambda\rangle^{n+1}} \mathcal{T}^{(N)}(\ell) D_r(\ell) \right) \Bigg|_{\substack{\tilde{\lambda} \to (K+s\eta)\lambda,\ \lambda \to P_{r,1} - \tau P_{r,2} \\ \tau = 0}}$$

$$\mathcal{B}^{(r;b;2)}_{n,m}(s) \equiv \frac{(-1)^{b+1}}{b!(m+1)\sqrt{\Delta_r}^{b+1} \langle P_{r,1}\ P_{r,2} \rangle^b} \times \frac{d^b}{d\tau^b} \left(\frac{\langle \lambda|\eta|P_{r,2}]^{m+1} \langle \lambda|Q_r\eta|\lambda\rangle^b \langle \lambda|K|\tilde{\lambda}]^{N+1}}{\langle \lambda|K|P_{r,2}]^{m+1} \langle \lambda|\eta K|\lambda\rangle^{n+1}} \mathcal{T}^{(N)}(\ell) D_r(\ell) \right) \Bigg|_{\substack{\tilde{\lambda} \to (K+s\eta)\lambda,\ \lambda \to P_{r,2} - \tau P_{r,1} \\ \tau = 0}}$$

Unitarity in $D = 4 - 2\epsilon$ dimensions

Orthogonal decomposition, keeping external momenta in 4 dimensions. [Bern, Chalmers, Mahlon, Morgan]

$$\int d^{4-2\epsilon}\ell_{4-2\epsilon} \quad = \quad \frac{(4\pi)^\epsilon}{\Gamma(-\epsilon)} \int_0^1 du\ u^{-1-\epsilon} \int d^4\ell_4.$$

where $\ell^2_{-2\epsilon} = \frac{K^2}{4} u$.

The integral over u will remain. The u-dependence is controlled:

$$\Delta A \quad = \quad \int_0^1 du\ u^{-1-\epsilon} \int d^4\ell\ \ \delta(\ell^2)\ \delta(\sqrt{1-u}\ K^2 - 2K\cdot\ell)$$

Recognize and perform the 4-d integral as before.

(Cf. methods by Ossola, Papadopoulos, Pittau; Forde; Ellis, Giele, Kunszt; Kilgore; Giele, Kunszt, Melnikov; Badger)

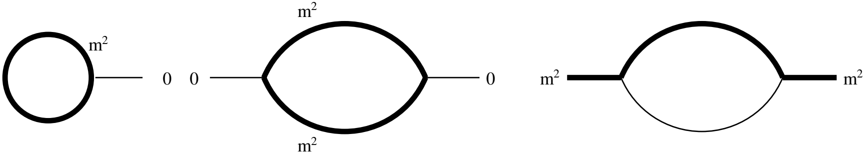
Massive particles

Cut amplitude:

$$\int_0^1 du\ u^{-1-\epsilon} \int \langle \lambda\ d\lambda \rangle [\tilde{\lambda}\ d\tilde{\lambda}] \left(\frac{\sqrt{\Delta[K^2, M_1^2, M_2^2]}}{K^2} \right) \frac{(K^2)^{n+1}}{\langle \lambda | K | \tilde{\lambda}]^{n+2}} \frac{\prod_{j=1}^{n+k} \langle \lambda | R_j | \tilde{\lambda}]}{\prod_{i=1}^{k} \langle \lambda | Q_i | \tilde{\lambda}]}$$

- The formalism/formulas for integral coefficients have the same form..
- Integral coefficients are polynomials in u.
- New master integrals.

The special "massive" master integrals



These integrals have cuts only in the m^2 channel.

$$\begin{aligned} I_1 &= m^{2-2\epsilon}\frac{\Gamma(1+\epsilon)}{\epsilon(\epsilon-1)} \\ I_2(0;m^2,m^2) &= m^{-2\epsilon}\frac{\Gamma(1+\epsilon)}{\epsilon} \\ I_2(m^2;0,m^2) &= m^{-2\epsilon}\frac{\Gamma(1+\epsilon)}{\epsilon(1-2\epsilon)} \end{aligned}$$

Tadpole approaches

1. Universal divergent behavior

   Examples:
   4-gluon amplitude with a massive fermion loop [Bern, Morgan].
   $t\bar{t}gg$, using Mitov-Moch small-mass factorization result [Badger].

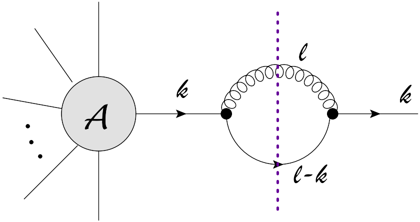
2. Add an "auxiliary propagator"
   [RB, Feng]

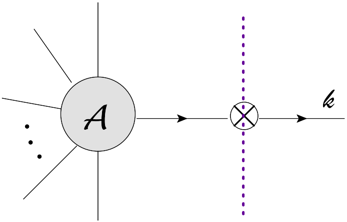
3. Generalized unitarity: single cuts
   [RB, Mirabella]

   Other single cut studies: [Catani, Gleisberg, Krauss, Rodrigo, Winter; Glover, Williams; Caron-Huot]

Divergent cuts

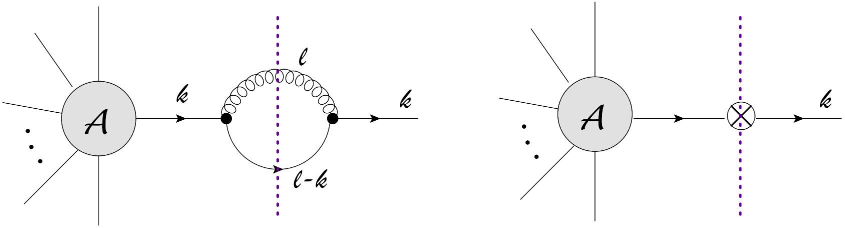
- Try to apply (generalized) unitary cuts to the special massive master integrals
- Single cut of tadpole, and cut of massless on-shell bubble diverge, due to internal on-shell propagators
- Single cut can be regularized specially for the tadpole. Double cut of bubble involves adding the counterterms explicitly.





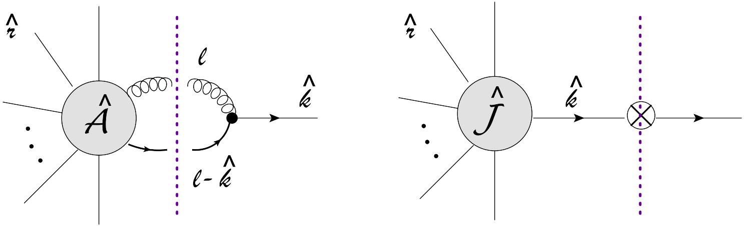
Masses, fermions and unitarity [EGKM]

[Ellis, Giele, Kunszt, Melnikov (2008)]



- Isolate and remove the divergent diagrams
- Implicit use of counterterm
- Feynman-diagram decomposition is gauge dependent
- Embedded in a numerical algorithm

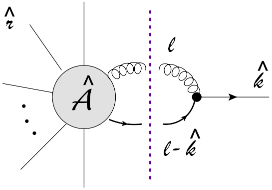
Our proposal [RB, Mirabella (to appear)]



Use an off-shell continuation of the fermion mass. The cut is finite until we take the on-shell limit.

- Power series expansion in the off-shell parameter
- In the on-shell limit, divergences are guaranteed to cancel: keep only finite terms
- Explicit use of counterterms. Gauge dependence enters only in tree level currents.
- Clean analytic results

Off-shell continuation



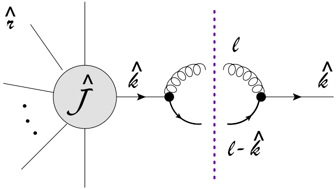
Momentum-conserving shift:
$k \longrightarrow \hat{k} = k + \xi \bar{k}, \qquad r \longrightarrow \hat{r} = r - \xi \bar{k}.$

Propagator of interest: $k^2 - m^2 \longrightarrow \xi(2k \cdot \bar{k})$

Can choose $\bar{k} = r$ for some null external momentum r, so it stays on shell: $\hat{r}^2 = r^2 = 0$.

Cut diverges as $1/\xi$.

Reduction of the shifted divergent diagram



$$\mathcal{A}_L = \frac{1}{\hat{k}^2 - m^2} \left(\bar{u}_{\hat{k}-\ell}\, \not{\epsilon}^*_\ell \,(m + \hat{\not{k}})\, \hat{\mathcal{J}} \right),$$

$$\mathcal{A}_3 = \bar{u}_k \,\not{\epsilon}_\ell\, u_{\hat{k}-\ell}$$

For internal helicity sum, use Feynman gauge as in EGKM:

$$\sum_{\lambda=\pm} \varepsilon^\lambda_\mu \left(\varepsilon^\lambda_\nu\right)^\star = -g_{\mu\nu}$$

Reduction of the shifted divergent diagram

Simple reduction of linear bubble gives

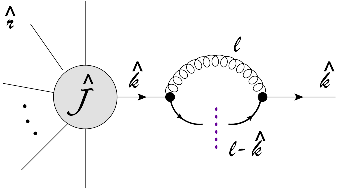
$$\int \mathcal{A}_3 \mathcal{A}_L \to \frac{1}{\xi(2k\cdot\bar{k})}(4m^2\,\bar{u}_k\hat{\mathcal{J}} + 2\xi m\,\bar{u}_k\bar{\not{k}}\hat{\mathcal{J}})\ B_0(\hat{k}^2).$$

Also expand off-shell current and scalar bubble to 1st order:

$$\begin{aligned}\hat{\mathcal{J}} &= \mathcal{J} + \xi\mathcal{J}' \\ B_0(\hat{k}^2) &= B_0(m^2) + \xi(2k\cdot\bar{k})B_0'(m^2)\end{aligned}$$

Result:

$$\frac{1}{\xi}\frac{4m^2\bar{u}_k\mathcal{J}\ B_0}{(2k\cdot\bar{k})} + \frac{4m^2(2k\cdot\bar{k})\,\bar{u}_k\mathcal{J}\ B_0' + 4m^2\,\bar{u}_k\mathcal{J}'\ B_0 + 2m\,\bar{u}_k\bar{\not{k}}\mathcal{J}\ B_0}{(2k\cdot\bar{k})}.$$

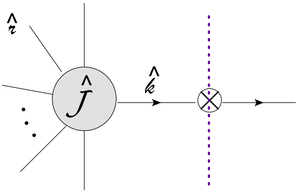


Similar, except gluon polarization sum → propagator, $-ig_{\mu\nu}/\ell^2$.

Result:

$$\left[\left(\frac{2}{\xi(2k\cdot\bar{k})}-\frac{1}{m^2}\right)\bar{u}_k\,\mathcal{J}+\frac{1}{(2k\cdot\bar{k})m}\bar{u}_k\,\not{\bar{k}}\,\mathcal{J}+\frac{2}{(2k\cdot\bar{k})}\bar{u}_k\,\mathcal{J}'\right]\,A_0.$$

Counterterm



$$-\frac{1}{\xi(2k\cdot\bar{k})}\bar{u}_k\left(\not{k}\delta Z_\psi + \xi\bar{\not{k}}\delta Z_\psi - m\delta Z_\psi - m\delta Z_m\right)(\not{k} + \xi\bar{\not{k}} + m)\,\hat{\mathcal{J}}$$

Renormalization constants in on-shell scheme:

$$\begin{aligned}\delta Z_m &= \frac{A_0}{m^2} + 2B_0,\\ \delta Z_\psi &= \frac{A_0}{m^2} - 4m^2 B_0'.\end{aligned}$$

Verify total cancellation of divergent diagram.

Small examples with Feynman Diagrams

- $H \to b\bar{b}$
  3 loop diagrams + 2 counterterm diagrams
- $q\bar{q} \to t\bar{t}$
  12 loop diagrams + 2 counterterm diagrams

1. Implemented momentum shift
2. Computed bubble and tadpole coefficients from unitarity cut
3. Checked cancellation of divergences against counterterm and agreement of finite result with Passarino-Veltman reduction.

The fermion-channel cut in the spinor-helicity formalism

The spinor-helicity convention for the polarization vectors requires axial gauge:

$$\varepsilon^{-}(p) = -\sqrt{2}\,\frac{|p\rangle\,[q| + |q]\,\langle p|}{[qp]}, \qquad \varepsilon^{+}(p) = -\sqrt{2}\,\frac{|p]\,\langle q| + |q\rangle\,[p|}{\langle qp\rangle}.$$

The completeness relation is

$$\sum_{\lambda=\pm} \varepsilon^{\lambda}_{\mu}(p)\left(\varepsilon^{\lambda}_{\nu}(p)\right)^{\star} = -g_{\mu\nu} + \frac{p_\mu q_\nu + q_\mu p_\nu}{p\cdot q}.$$

Cf. Feynman gauge, where the 2nd term is absent.

Specific gauge choice = choice of q for each p.

Additional counterterm in axial gauge, for spinors

The double cut gets an extra $\mathcal{O}(\xi^0)$ contribution:

$$\frac{1}{\xi(2k\cdot\bar{k})}\int d\mu_{2,k}\left[\frac{\left(\bar{u}_k\,\not{\ell}\,u_{\hat{k}-\ell}\right)\left(\bar{u}_{\hat{k}-\ell}\,\not{q}\,(m+\hat{\not{k}})\,\hat{\mathcal{J}}\right)}{q\cdot\ell}\right.$$
$$\left.+\frac{\left(\bar{u}_k\,\not{q}\,u_{\hat{k}-\ell}\right)\left(\bar{u}_{\hat{k}-\ell}\,\not{\ell}\,(m+\hat{\not{k}})\,\hat{\mathcal{J}}\right)}{q\cdot\ell}.\right]$$

Second term vanishes by Ward identity with cut gluon.
First term is cancelled by a new (non-divergent) counterterm:

$$\mathcal{M}^k = -\frac{1}{2k\cdot\bar{k}}\bar{u}_k\left[(\bar{\not{k}}-m)\,\hat{\not{k}}\not{q}\,\delta Z'_k\right](\hat{\not{k}}+m)\hat{\mathcal{J}},$$
$$\delta Z'_k = \frac{B_0}{q\cdot k}$$

Example from $t\bar{t} \to gg$ amplitude

Full analytic result computed previously by other methods. [Körner, Merabashvili; Badger, Sattler, Yundin]

- Color decomposition
- 3-point tree $\times$ 5-point tree for m^2 on-shell bubbles
- Checked cancellation of divergence and evaluated on-shell bubble coefficient, for equal-helicity gluons

On-shell recursion for off-shell currents

Finite parts of the counterterm need off-shell $\widehat{\mathcal{J}}$ explicitly, not the gauge-invariant $\mathcal{A}_{\mathcal{L}}\mathcal{A}_{\mathcal{R}}$.

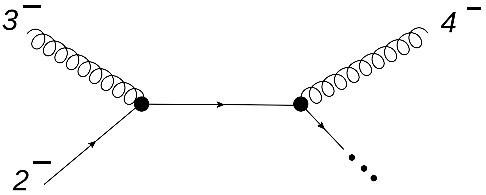
The current $\widehat{\mathcal{J}}$ depends on gauge choices of external gluons—these cancel among counterterms.

Generate $\widehat{\mathcal{J}}$ by BCFW-type relations, starting purely from 3-point vertices with the polarizations

$$\varepsilon^-(p) = -\sqrt{2}\,\frac{|p\rangle\,[q| + |q]\,\langle p|}{[qp]}, \qquad \varepsilon^+(p) = -\sqrt{2}\,\frac{|p]\,\langle q| + |q\rangle\,[p|}{\langle qp\rangle}.$$

BCFW shifts available for any pair of massless quarks/gluons.

On-shell recursion for off-shell currents



Example:

$$\mathcal{J} = \frac{|\hat{4}\rangle [q_4| + |q_4] \langle \hat{4}|}{[q_4 4]} \frac{(p_2 - p_3 + m)}{-2p_2 \cdot p_3} \frac{|3\rangle [4| + |4] \langle 3|}{[4\hat{3}]} u_2$$

$$= \frac{p_{34}\, |2|3\rangle\, [q_4 2] + m\, |q_4]\, \langle 43\rangle\, [42] + (m^2 - p_{234}^2)\, |q_4]\, \langle 32]}{2p_2 \cdot p_3\, [34]\, [q_4 4]}$$

Summary

- Unitarity method at one loop constructs amplitudes from branch cuts, using the fact of the expansion in known master integrals.
- Residue theorem gives analytic formulas for coefficients of most master integrals.
- Massive particles require an expanded set of master integrals. Challenge for the unitarity method.
  - Tadpole techniques still merit further study;
  - On-shell bubbles via unitarity cuts are available from an off-shell continuation and 1-term Taylor expansion.
  - Must check efficiency.