# Semiclassical correlation functions in holography

Kostya Zarembo

K.Z.,1008.1059

“Strings, Gauge Theory and the LHC”, Copenhagen, 23.08.11

# AdS/CFT correspondence

Yang-Mills theory with
N=4 supersymmetry

Maldacena'97

String theory on
AdS5xS5 background

$$\lambda = g_{\mathrm{YM}}^2 N$$

't Hooft coupling

$$\frac{\sqrt{\lambda}}{2\pi}$$

string tension

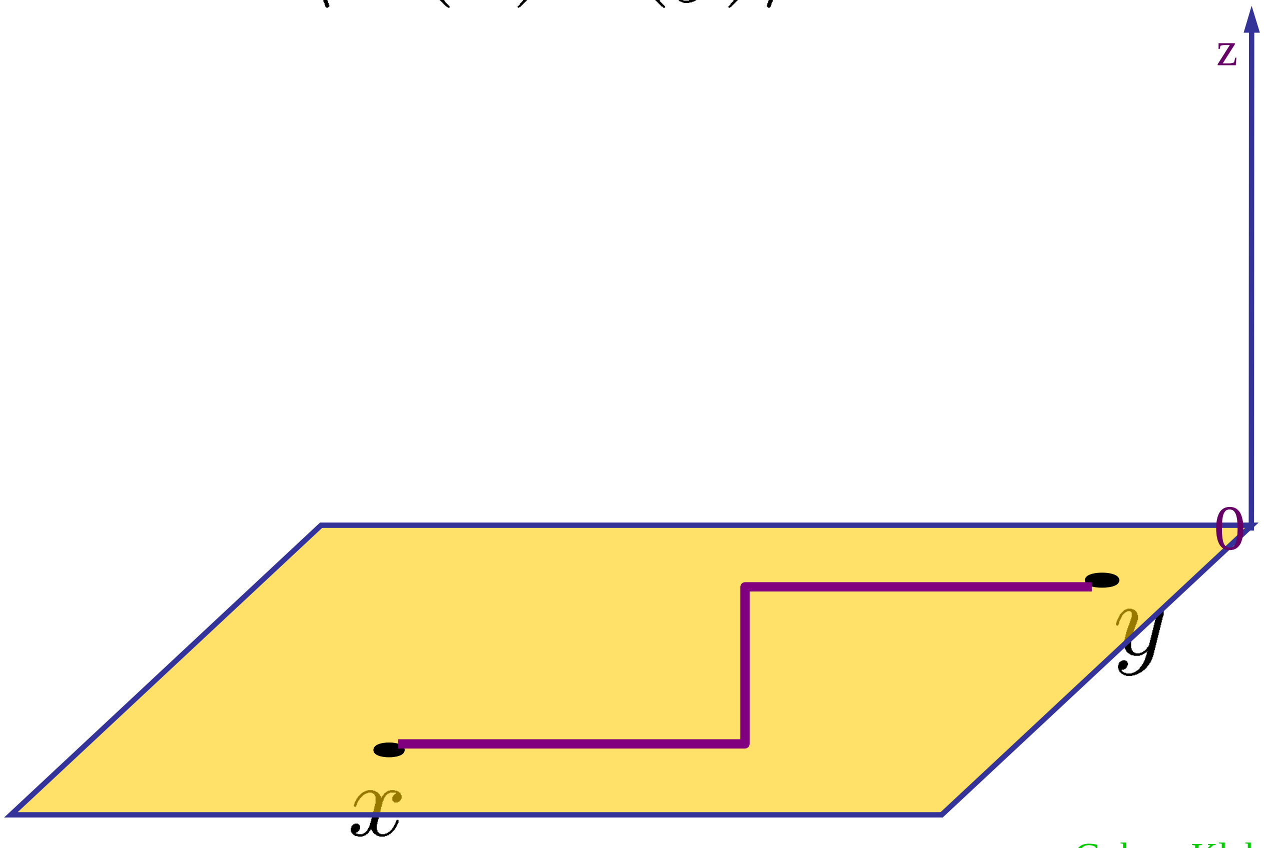
$$N \to \infty$$

planar / no quantum gravity

$$\lambda \gg 1$$

string theory - classical

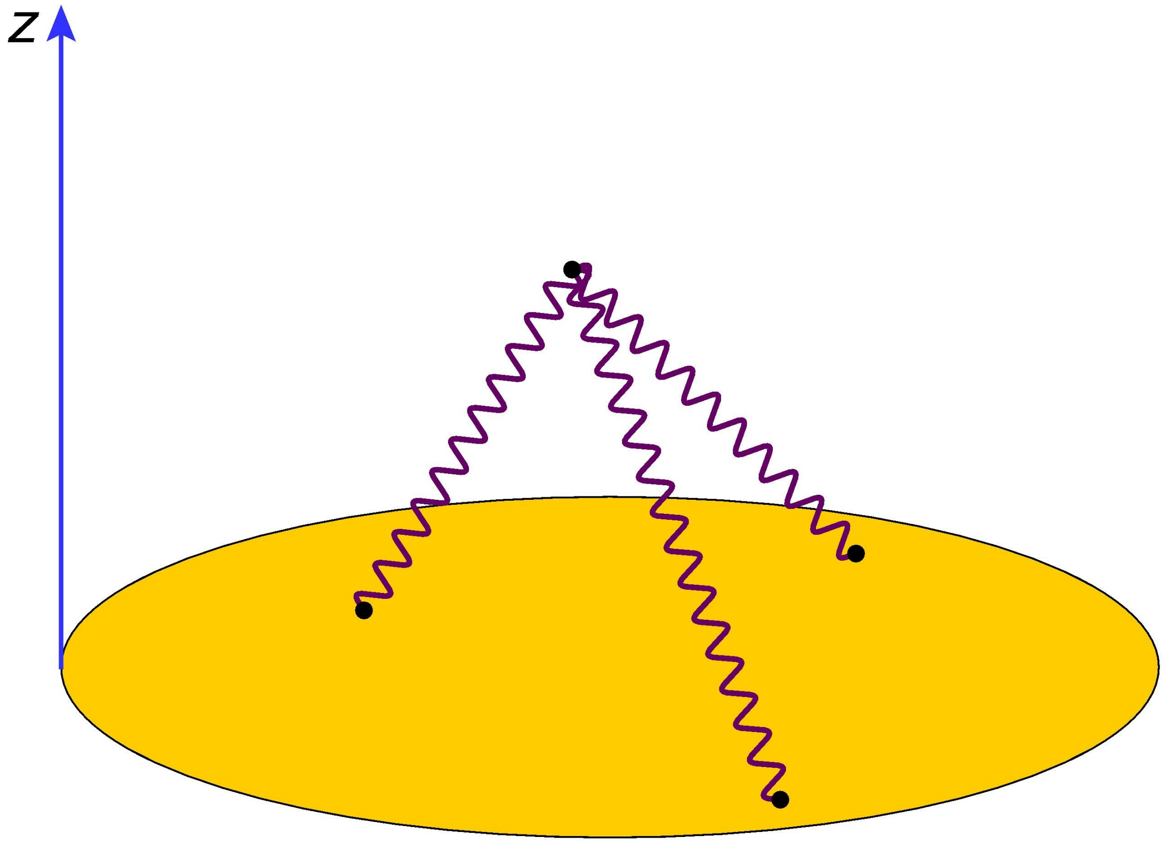
$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle$$



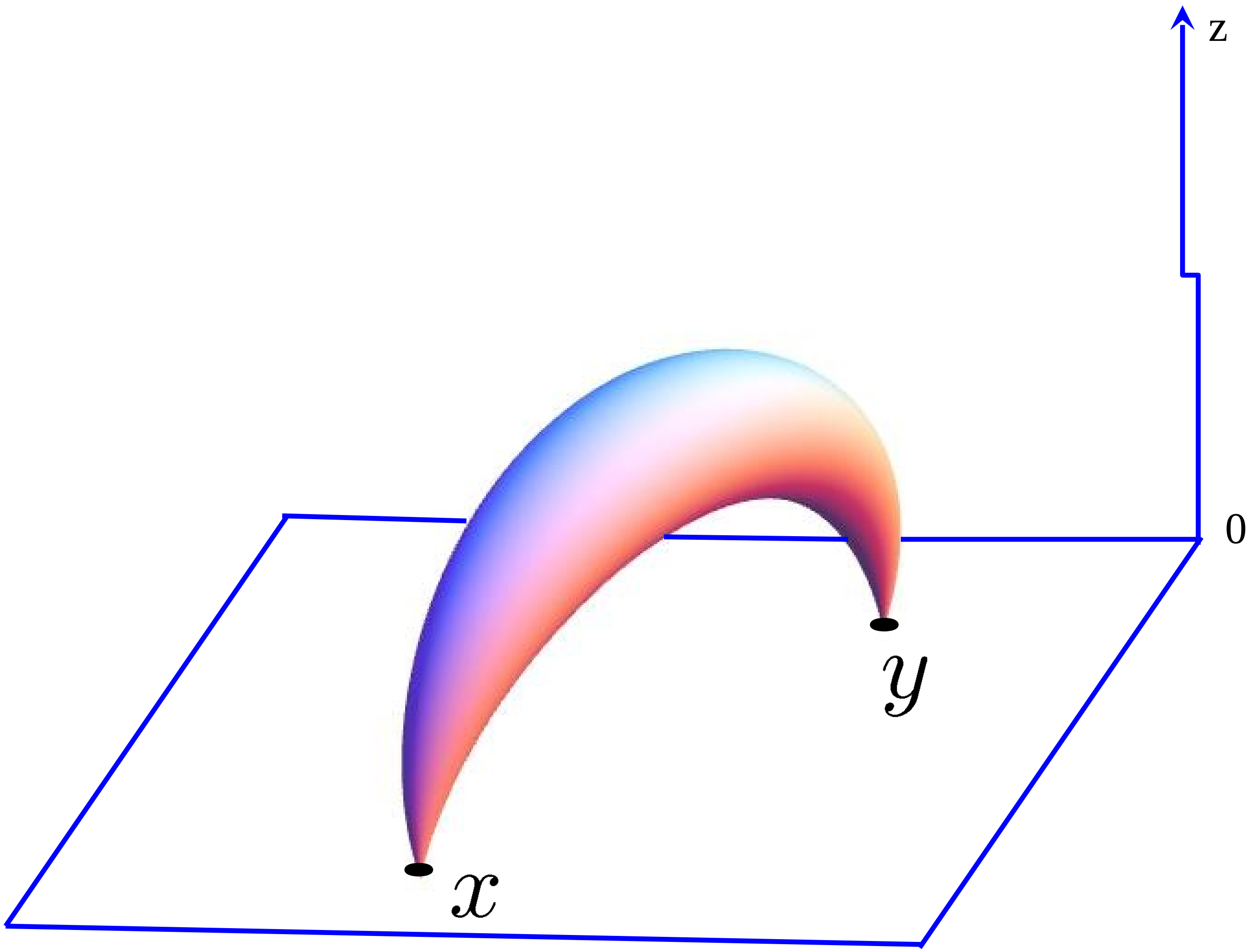

Gubser,Klebanov,Polyakov'98

Witten'98

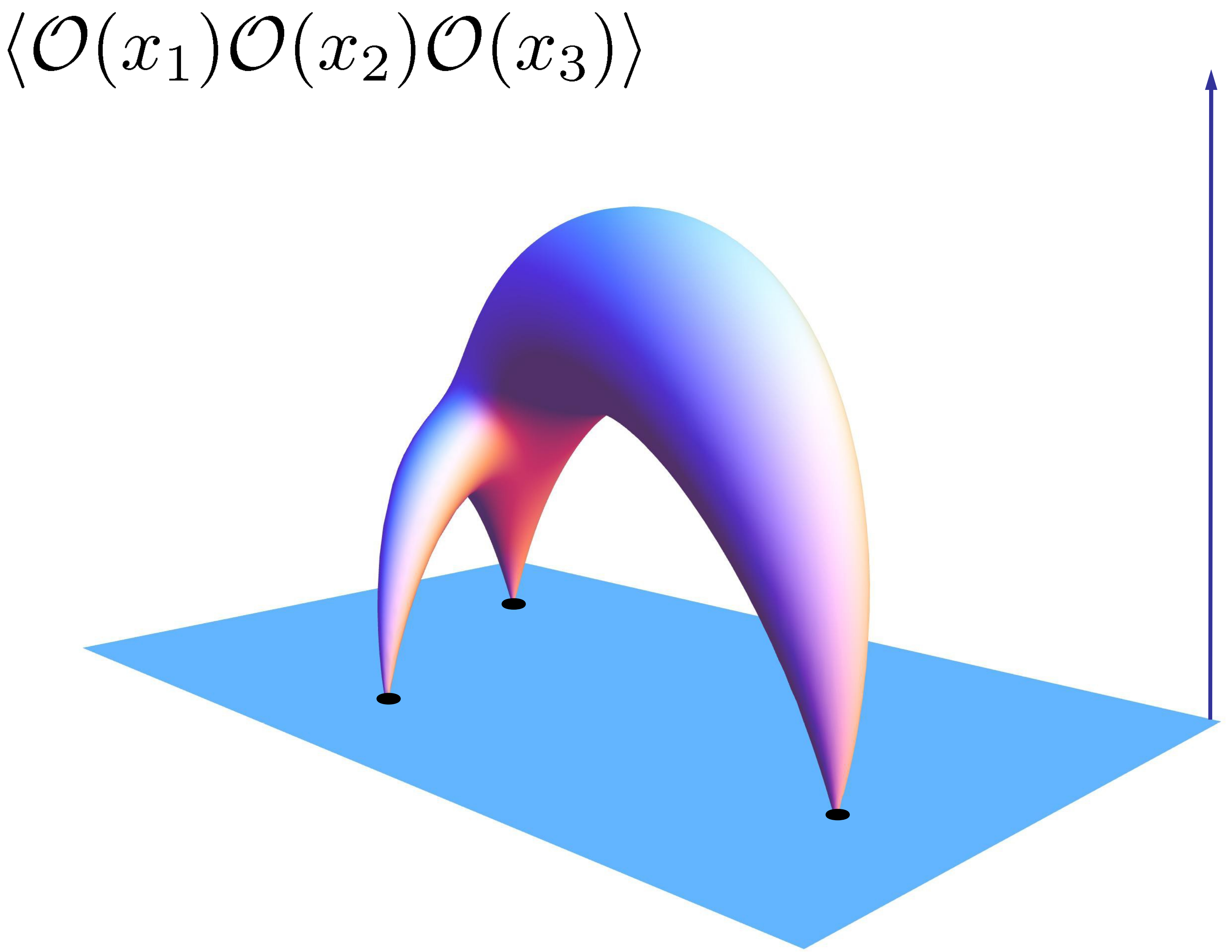
# Witten diagrams




$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\rangle$$




$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle$$


$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\rangle$

# Semiclassical operators:

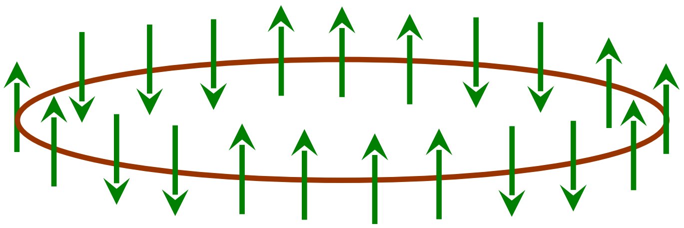
Berenstein,Maldacena,Nastase'02
Gubser,Klebanov,Polyakov'02

$$\Delta \sim \sqrt{\lambda} \gg 1$$

- described by classical strings

Ex: long "spin-chain" operators

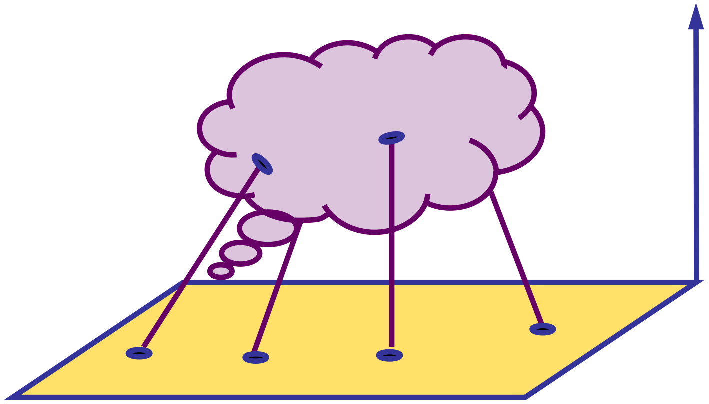
$$\mathrm{tr}\, ZZZZWWZZZWWZZZWWWZZWW$$



$$Z = \Phi_1 + i\Phi_2$$
$$W = \Phi_3 + i\Phi_4$$

# Correlation functions in string theory

$$\langle \mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\rangle_{\rm SYM} = \int d^{2n-6}\sigma_i \; \langle V_1(\sigma_1;x_1)\ldots V_n(\sigma_n;x_n)\rangle_{\rm world-sheet}$$



Vertex operators:

- (1,1) operators in the sigma-model

Semiclassically:

$$\left(\Delta_{AdS^5} + \Delta_{S^5} + M^2\right) V = 0$$

Callan,Gan'86

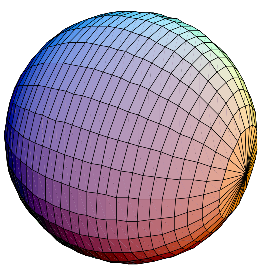
# Vertex operators in AdS5xS5

$$V(Z,X,\mathbf{N};x) \sim Y_I(\mathbf{N}) \times \frac{Z^{\Delta}}{\left[Z^2 + (X-x)^2\right]^{\Delta}}$$

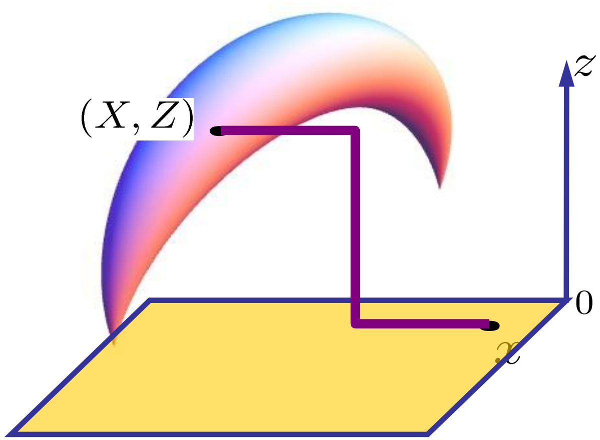
Polyakov'01

Tseytlin'03

Spherical functions



S5




AdS5

# Semiclassical limit

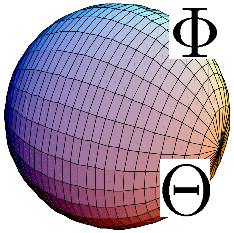
$$S = \frac{\sqrt{\lambda}}{4\pi} \int d^2\sigma \, G_{MN} \partial_a X^M \partial^a X^N - \sum_i \ln V_i(\sigma_i)$$

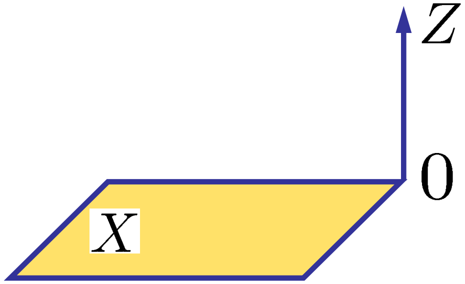
Semiclassical states: $\ln V_i \sim J_i, \Delta_i \sim \sqrt{\lambda}$

Sources in classical equations of motion:

$$\partial_a \partial^a X^M + \Gamma^M_{NL} \partial_a X^N \partial^a X^L = -\frac{2\pi}{\sqrt{\lambda}} G^{MN} \sum_i \frac{\partial \ln V_i}{\partial X^N} \, \delta(\sigma - \sigma_i)$$

Example:




$$V \sim (\sin\Theta)^k \, \mathrm{e}^{\,ik\Phi} \frac{Z^k}{\left[Z^2 + (X-x)^2\right]^k} \xrightarrow{X(\sigma)\to x} (\sin\Theta)^k \, \mathrm{e}^{\,ik\Phi} Z^{-k}$$

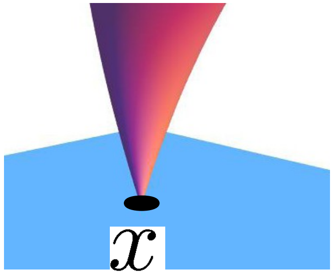
Dual to (same quantum numbers!):

$$\mathcal{O}_{\mathrm{BMN}} = \mathrm{tr}\,(\Phi_1 + i\Phi_2)^k$$

Boundary conditions:

$$
\begin{aligned}
-\partial^2 \ln Z - \frac{(\partial X)^2}{Z^2} &= -\frac{2\pi k}{\sqrt{\lambda}}\delta(\sigma) \\
-\partial^2 \Theta + \sin\Theta\ \cos\Theta\ (\partial\Phi)^2 &= \frac{2\pi k}{\sqrt{\lambda}} \cot\Theta\,\delta(\sigma) \\
-\partial_{\mathbf{a}}\left(\sin^2\Theta\,\partial^{\mathbf{a}}\Phi\right) &= \frac{2\pi k}{\sqrt{\lambda}}\delta(\sigma),
\end{aligned}
$$

$$
\begin{aligned}
\ln Z &\to -\frac{k}{\sqrt{\lambda}}\ \ln|\sigma| \\
\Theta &\to \frac{\pi}{2} \\
\Phi &\to -i\frac{k}{\sqrt{\lambda}}\ \ln|\sigma|
\end{aligned}
$$




de Boer,Ooguri,Robins,Tannenhauser’98

# Two-point functions

$$\left\langle \mathcal{O}_I^\dagger(x)\mathcal{O}_J(y)\right\rangle = \frac{\delta_J^I}{|x-y|^{2\Delta_I}}$$

Spectrum: $\Delta_I$

Known from integrability

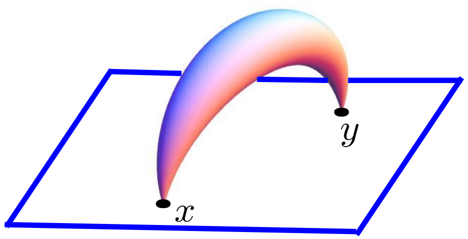
exactly at large-N

Bombardelli,Fioravanti,Tateo'09

Gromov,Kazakov,Vieira'09

Arutyunov,Frolov'09

# Holographic two-point functions




Buchbinder'10
Janik,Surowka,Wereszczynski'10
Buchbinder,Tseytlin'10

Two-point functions ↔ Spectrum ↔ Periodic solutions in global AdS

- start with time-periodic (finite-gap) solution in global AdS
- Wick-rotate
- transform to Poincaré patch

- generates solutions with correct boundary conditions
- classical string action produces the correct - -dependence of the correlator

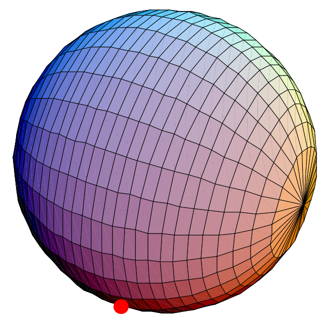
$x$ olutions are in general complex

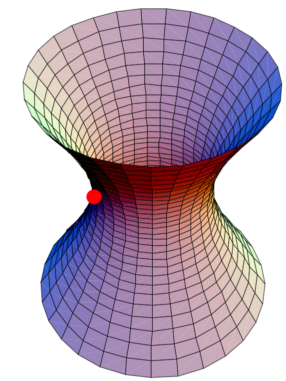
Example: BMN string: $t=\kappa\tau, \rho=0, \varphi=\kappa\tau \iff \mathcal{O} = \mathrm{tr}\, Z^L$

## Standard global-Poincaré map (AdS3):

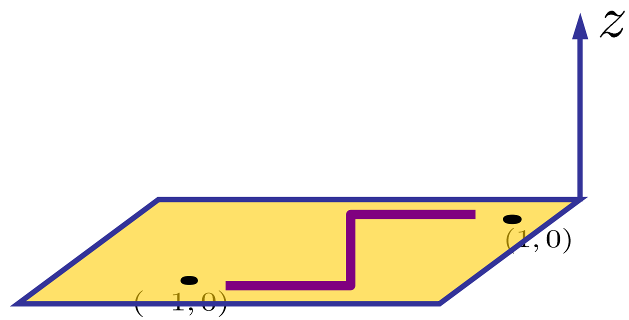
Cartesian coordinates on R3,1

$$
\begin{aligned}
\sinh\rho\cos\varphi &= \frac{x}{z}\\
\sinh\rho\sin\varphi &= \frac{y}{z}\\
\cosh\rho\sinh t &= \frac{z}{2}\left(1+\frac{x^2+y^2-1}{z^2}\right)\\
\cosh\rho\cosh t &= \frac{z}{2}\left(1+\frac{x^2+y^2+1}{z^2}\right)
\end{aligned}
$$





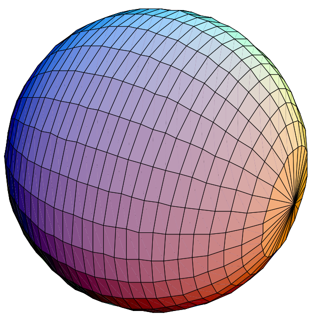
$$
\begin{aligned}
x &= \tanh\kappa\tau\\
z &= \frac{1}{\cosh\kappa\tau}\\
\varphi &= i\kappa\tau
\end{aligned}
$$



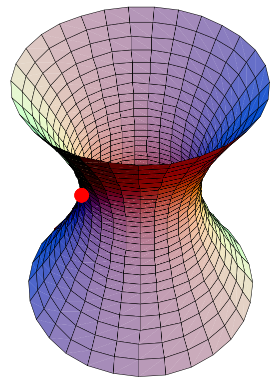

Tsuji'06

Janik,Surowka,Wereszczynski'10

# More general semiclassical states



S5



global AdS5

Gubser,Klebanov,Polyakov'02
Frolov,Tseytlin'03
…

$$\mathcal{O} = \operatorname{tr} Z^{L-M} W^{M} + \text{perm}$$

Periodic solutions in sigma-model $\leftrightarrow$ Long operators in SYM

Energy: $\Delta \sim \sqrt{\lambda}$

Angular momenta: $L, M \sim \sqrt{\lambda}$

…

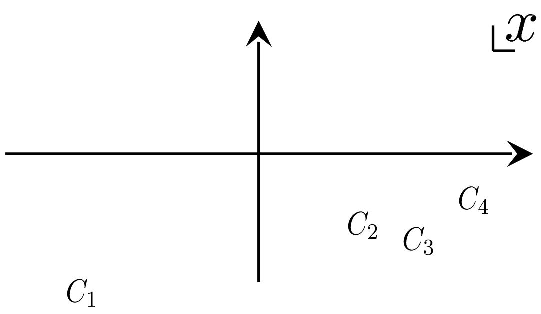
# Finite-gap solutions

$$2\fint \frac{dy\,\rho(y)}{x-y} = \frac{x}{x^2 - \frac{\lambda}{16\pi^2 L^2}}\;\frac{\Delta}{L} + 2\pi n_k, \qquad x \in C_k$$

Kazakov,Marshakov,Minahan,Z.'04

Normalization:

$$\int dx\,\rho(x) = \frac{M}{L} + \frac{\Delta - L}{2L}$$




Level matching:

$$\int \frac{dx\,\rho(x)}{x} = 2\pi m$$

Scaling dimension:

$$\Delta = L + \frac{\lambda}{8\pi^2 L}\int \frac{dx\,\rho(x)}{x^2}$$

vertex operators

$\leftrightarrow$

finite-gap solutions

(?)

# Three-point functions

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{C_{123}}{|x_1-x_2|^{\Delta_1+\Delta_2-\Delta_3}|x_1-x_3|^{\Delta_1+\Delta_3-\Delta_2}|x_2-x_3|^{\Delta_2+\Delta_3-\Delta_1}}$$

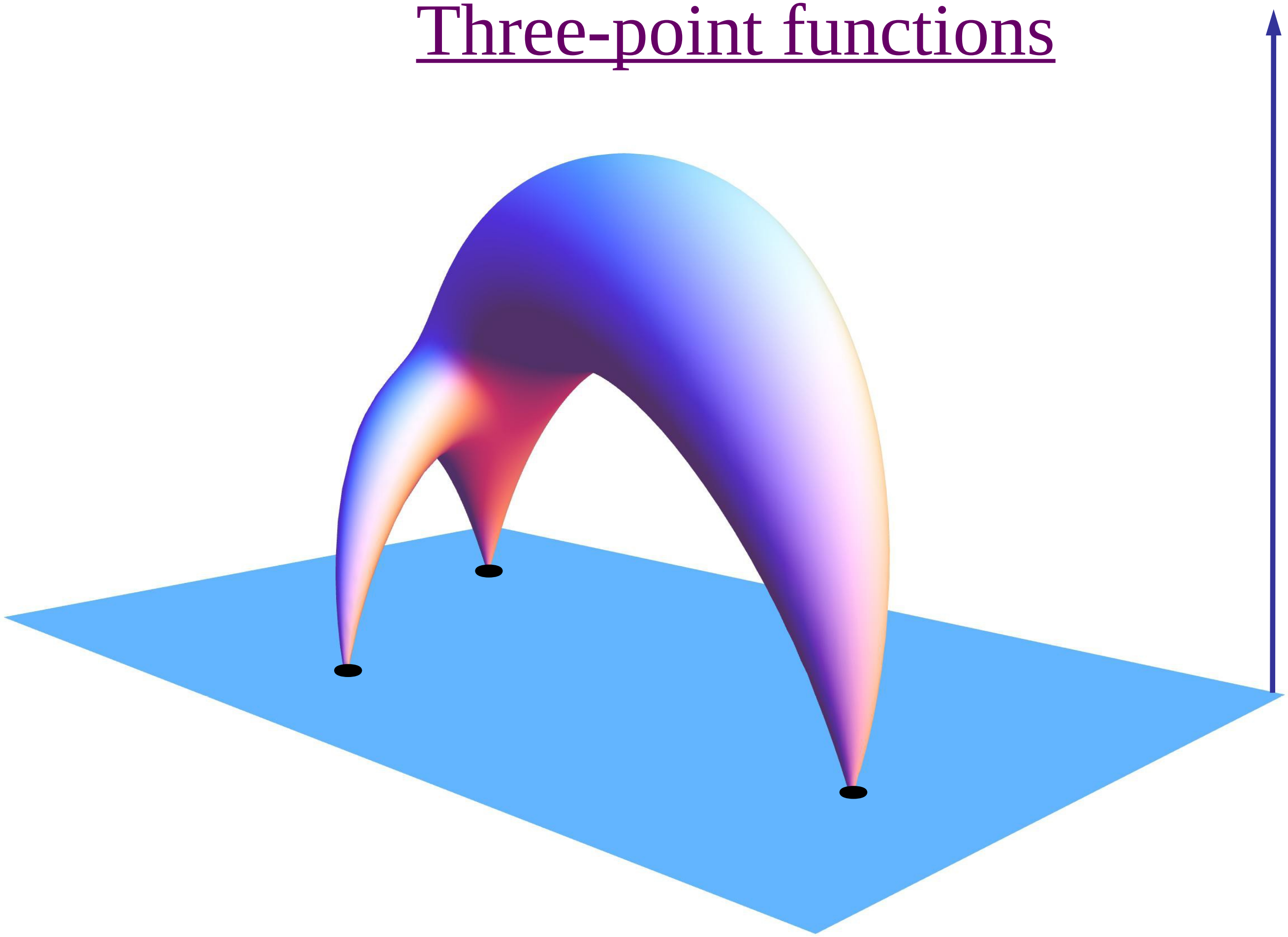
OPE coefficients:

$$\mathcal{O}_J(x)\mathcal{O}_K(0) = \sum_I C^I_{JK}|x|^{\Delta_I-\Delta_J-\Delta_K}\mathcal{O}_I(0) + \text{descendants}$$

Simplest 1/N observables:

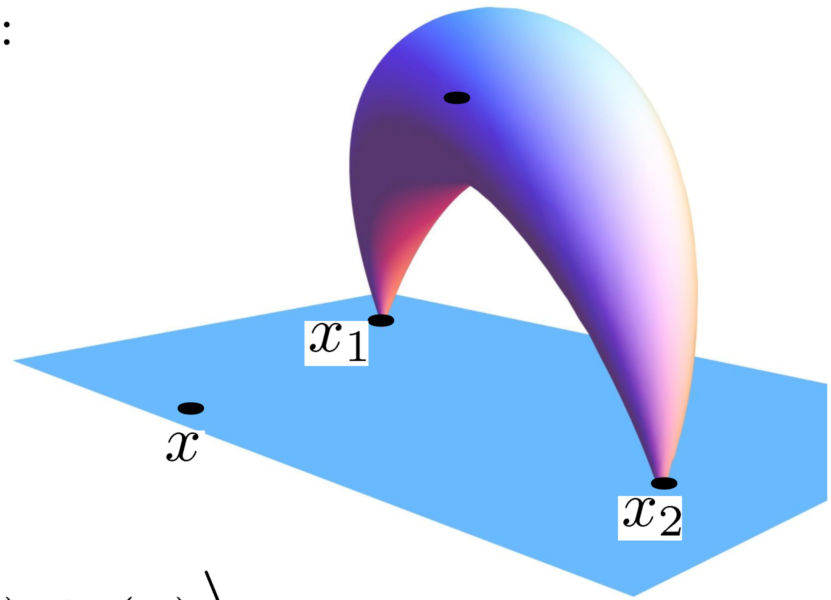
$$C^I_{JK} = O\left(\frac{1}{N}\right)$$

# Three-point functions



- No solutions known

Simpler problem:




Z.'10
Costa,Monteiro,Santos,Zoakos'10
Roiban,Tseytlin'10
Hernandez'10
Arnaudov,Rashkov'10
Georgiou'10
Lee,Park'10,11
Buchbinder,Tseytlin'10
Bak,Chen,Wu'11
Bissi,Kristjansen,Young,Zoubos'11
Arnaudov,Rashkov,Vetsov'11
Bai,Lee,Park'11
Alday,Tseytlin'11
Ahn,Bozhilov'11
Bozholov'11
...

$$\left\langle \mathcal{O}_J^\dagger(x_1)\mathcal{O}_K(x_2)\mathcal{O}_I(x) \right\rangle$$

$$\mathcal{O}_J : \quad \Delta_J \sim \sqrt{\lambda}$$

$$\mathcal{O}_K : \quad \Delta_K \sim \sqrt{\lambda}$$

create fat string

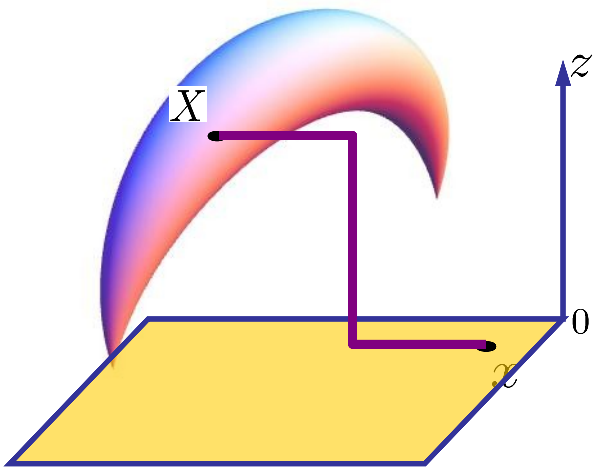
$$\mathcal{O}_I : \quad \Delta_I \sim 1$$

creates slim string

# General formalism

$\mathcal{W}$ big non-local operator that creates classical string

$$\frac{\langle \mathcal{W} \mathcal{O}_I(x) \rangle}{\mathcal{W}} = \lim_{\varepsilon \to 0} \frac{\pi}{\varepsilon^{\Delta_I}} \sqrt{\frac{2}{\Delta_I - 1}} \left\langle \phi_I(x,\varepsilon) \, \frac{1}{Z_{\rm str}} \int \mathcal{D}X \; \mathrm{e}^{-S_{\rm str}[X]} \right\rangle_{\rm bulk} .$$




Berenstein,Corrado,Fischler,Maldacena'98

$$S_{\rm str}=\frac{\sqrt{\lambda}}{4\pi}\int d^2\sigma\,\left[\frac{(\partial X^\mu)^2+(\partial Z)^2}{Z^2}+(\partial\mathbf{N})^2+\gamma_{MN}\partial_a X^M\partial^a X^N\right]$$

metric perturbation

due to operator insertion

$$\gamma_{MN}=V^I_{MN}\left(X,\frac{\partial}{\partial X}\right)\phi_I$$

$$\frac{\langle\mathcal{W}\mathcal{O}_I(x)\rangle}{\langle\mathcal{W}\rangle}=-\frac{\sqrt{2\,(\Delta_I-1)\,\lambda}}{8\pi^2}\int d^2\sigma\,\partial_a X^M\partial^a X^N V^I_{MN}\frac{Z^{\Delta_I}}{\left[Z^2+(X-x)^2\right]^{\Delta_I}}$$

vertex operator

OPE coefficient:

$$C_I[\mathcal{W}]=\lim_{x\to\infty}|x|^{2\Delta_I}\frac{\langle\mathcal{W}\mathcal{O}_I(x)\rangle}{\langle\mathcal{W}\rangle}=-\frac{\sqrt{2\,(\Delta_I-1)\,\lambda}}{8\pi^2}\int d^2\sigma\,\partial_a X^M\partial^a X^N V^I_{MN}Z^{\Delta_I}$$

# Chiral Primary Operators

$$\mathcal{O}_I^{\rm CPO} = \frac{1}{\sqrt{k}} \left(\frac{8\pi^2}{\lambda}\right)^{\frac{k}{2}} K_I^{i_1 \ldots i_k}\, {\rm tr}\, \Phi_{i_1} \ldots \Phi_{i_k}$$

symmetric traceless tensor of SO(6)

Dual to scalar supergravity mode on S5

Wavefunction on S5:

$$Y_I(\mathbf{n}) = K_I^{i_1 \ldots i_k} n_{i_1} \ldots n_{i_k}$$

(spherical function of SO(6))

# Kaluza-Klein reduction

$$h_{mn} = \frac{1}{\mathcal{N}_k} \frac{2}{k+1} Y_I \left[ 2\nabla_m \nabla_n - k(k-1) g_{mn} \right] \phi_I$$

$$h_{\alpha\beta} = \frac{2}{\mathcal{N}_k} k g_{\alpha\beta} Y_I \phi_I$$

Kim,Romans,van Nieuwenhuizen'85

Lee,Minwalla,Rangamani,Seiberg'98

$$\mathcal{N}_k^2 = \frac{N^2 k(k-1)}{2^{k-3}\pi^2(k+1)^2}$$

Vertex operator:

$$C_I^{\mathrm{CPO}}[\mathcal{W}] = \frac{2^{\frac{k}{2}-3}(k+1)\sqrt{k\lambda}}{\pi N} \int d^2\sigma \, Y_I(\mathbf{N}) Z^k \left[ \frac{(\partial X)^2 - (\partial Z)^2}{Z^2} - (\partial \mathbf{N})^2 \right]$$

# Correlator of three chiral primaries

Superconformal highest weight:

$$\mathcal{O}_L = \frac{1}{\sqrt{L}} \left( \frac{4\pi^2}{\lambda} \right)^{\frac{L}{2}} \operatorname{tr} Z^L, \qquad Z = \Phi_1 + i\Phi_2$$

$$\left\langle \mathcal{O}^\dagger_{J+k}(x)\mathcal{O}_J(y)\mathcal{O}_k(\infty) \right\rangle \quad @ \quad \begin{matrix} J & \sim & \sqrt{\lambda} \\ k & \sim & 1 \end{matrix}$$

Spherical function:

$$Y_k = \left( \frac{n_1 + in_2}{\sqrt{2}} \right)^k = 2^{-\frac{k}{2}} \left(\sin\theta\right)^k \, \mathrm{e}^{\,ik\varphi}$$

Classical solution:

$$\begin{aligned} x &= R\tanh\kappa\tau \\ z &= \frac{R}{\cosh\kappa\tau} \\ \varphi &= i\kappa\tau, \qquad \theta = \frac{\pi}{2}\,. \end{aligned}$$

$$\kappa = \frac{J}{\sqrt{\lambda}}$$

$$R = |x - y|$$

OPE coefficient:

$$C_{J,k}^{J+k} = \frac{1}{N}\,2^{-k-1}J(k+1)\sqrt{k}\,\kappa\int_{-\infty}^{+\infty}d\tau\;\frac{\mathrm{e}^{-k\kappa\tau}}{\cosh^{k+2}\kappa\tau} = \frac{1}{N}\,J\sqrt{k}$$

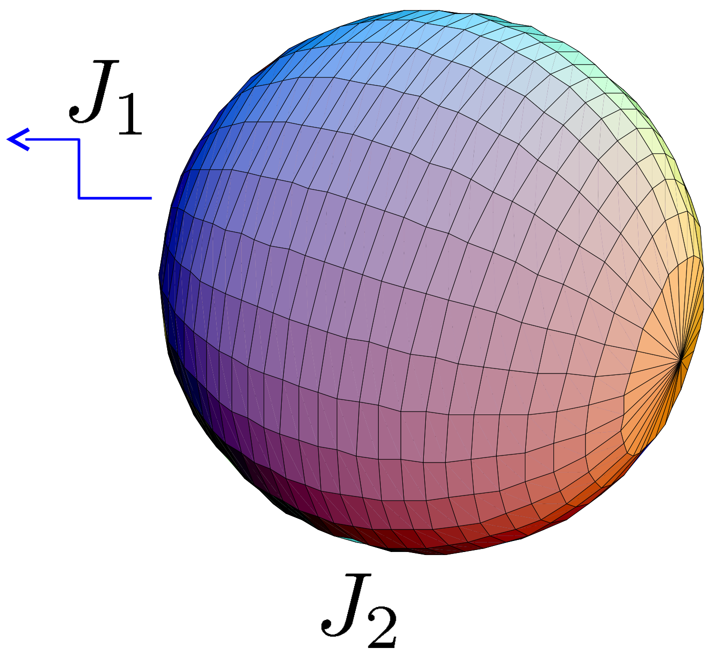
Exact OPE coefficient of three CPO's:

$$C_{Jk}^{J+k} = \frac{1}{N}\,\sqrt{(J+k)Jk}$$

Agree at J>>k

Lee,Minwalla,Rangamani,Seiberg'98

# Spinning string on S5




$$\Phi = i\omega_1 \tau$$

$$\Psi = i\omega_2 \tau$$

Frolov,Tseytlin'03

$$\Theta = \Theta(\sigma)$$

$$\acute{\Theta}^2 = \kappa^2 - \omega_1^2 \sin^2 \Theta - \omega_2^2 \cos^2 \Theta$$

Elliptic modulus:

$$s = \frac{\kappa^2 - \omega_1^2}{\omega_2^2 - \omega_1^2}$$

$$\sqrt{\omega_2^2 - \omega_1^2} = \frac{2K(s)}{\pi}$$

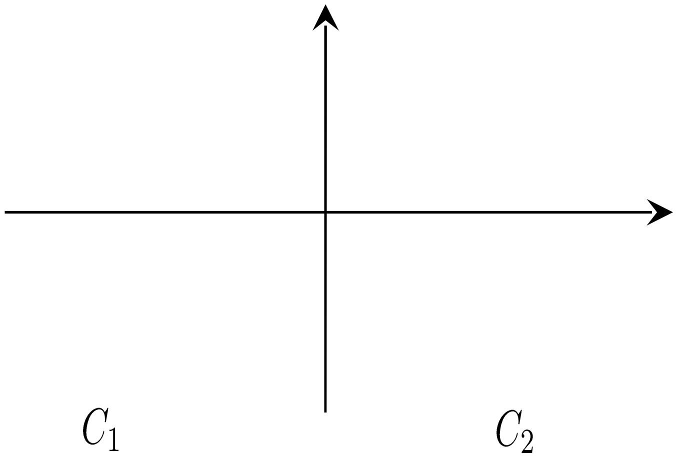
Conserved charges: $J_1 = \frac{\sqrt{\lambda}\omega_1 E(s)}{K(s)}, \qquad J_2 = \frac{\sqrt{\lambda}\omega_2 \left(K(s) - E(s)\right)}{K(s)}, \qquad \Delta = \sqrt{\lambda}\kappa$

Dual to

$$\mathcal{O}_{\text{fold.}} = \text{ tr } \left( Z^{J_1} W^{J_2} + \text{perm.} \right)$$

The concrete operator can be identified by comparing the finite-gap curve to Bethe ansatz

Beisert,Minahan,Staudacher,Z.'03



OPE coefficient:

$$C^{\text{fold.}}_{\text{fold.},k} = \frac{1}{N}\,\frac{\pi\sqrt{k}\Delta\left(1-a^2\right)\Gamma\left(\frac{(1+a)k}{2}\right)\Gamma\left(\frac{(1-a)k}{2}\right)}{8sK(s)(k-1)!}\left[(k+1-s)\,{}_2F_1\left(-\frac{k-1}{2},\frac{1}{2};1;s\right)-(k+1)\,{}_2F_1\left(-\frac{k+1}{2},\frac{1}{2};1;s\right)\right]$$

$$a = \frac{\omega_1}{\kappa} = \frac{J_1K(s)}{\Delta E(s)}$$

What happens when k becomes large?

$$C^{\text{fold.}}_{\text{fold.},k} \simeq \frac{1}{N}\left(\frac{1}{s}-1\right)\sqrt{\frac{\lambda}{k}}\,\mathrm{e}^{\,k\left(\frac{1+a}{2}\,\ln\frac{1+a}{2}+\frac{1-a}{2}\,\ln\frac{1-a}{2}\right)}$$

# Saddle-point approximation

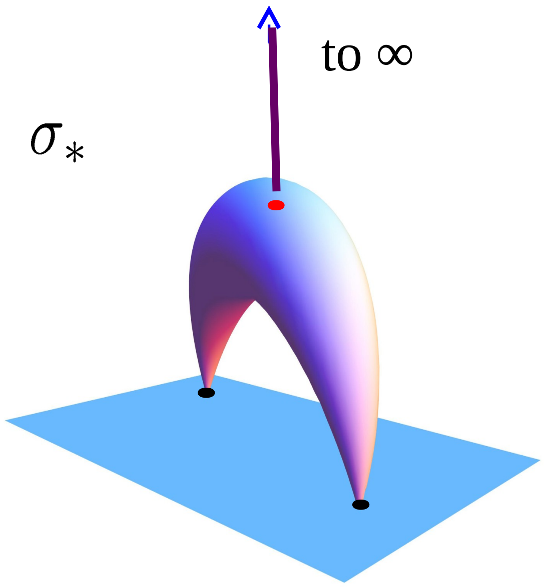
$$C^{\#}_{\#k} = \frac{(k+1)\sqrt{k\lambda}}{8\pi N} \int d^2\sigma \ (\sin\Theta)^k \ \mathrm{e}^{\,ik\Phi} Z^k \left[ \frac{(\partial X)^2 - (\partial Z)^2}{Z^2} - (\partial\Theta)^2 - \sin^2\Theta\,(\partial\Phi)^2 \right]$$

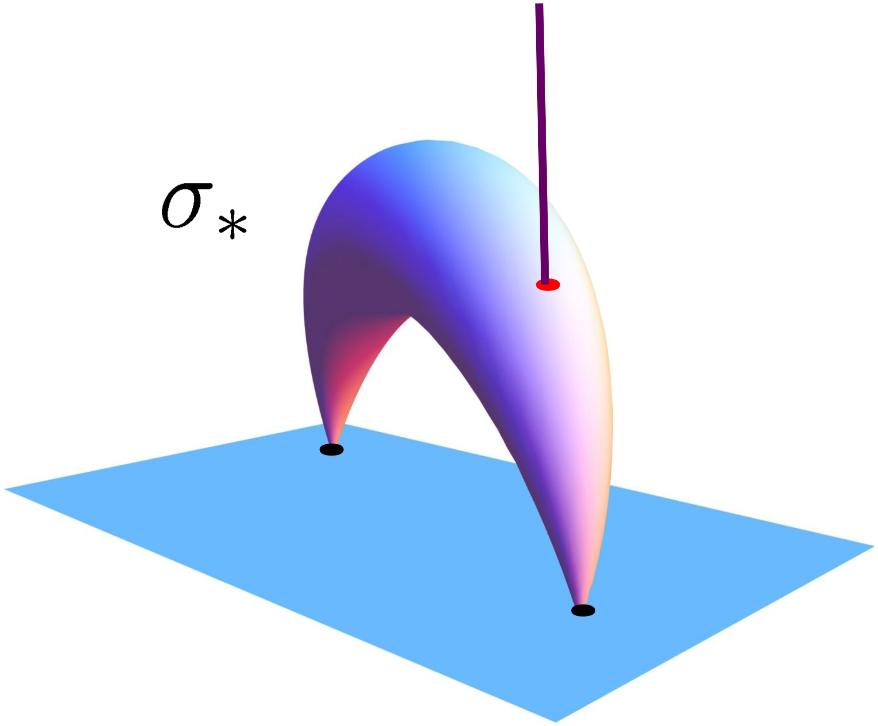
$k \to \infty$

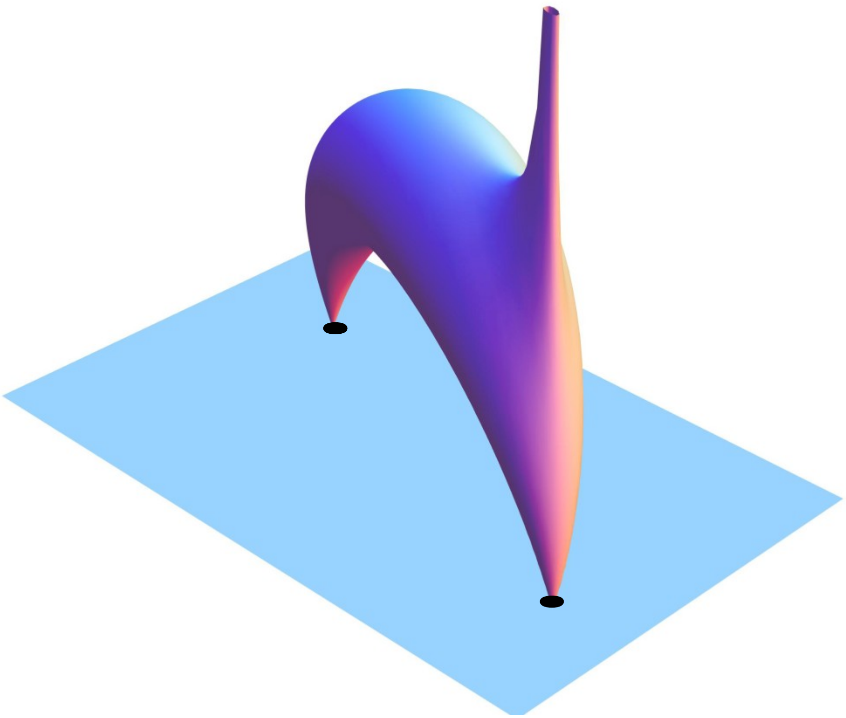
$$C^{\#}_{\#k} \simeq [\ldots]\,\mathrm{det}^{\,-1/2}\,\mathrm{e}^{\,-kS_{\mathrm{eff}}(\sigma_*)}$$

Saddle-point equations:

$$\cot\theta\,\partial_a\theta + i\partial_a\varphi + \frac{\partial_a Z}{Z} = 0$$





$k$ – finite, $\quad k \to \infty$ $\qquad\qquad$ $\frac{k}{\sqrt{\lambda}}$ – finite

Overlapping regime of validity:

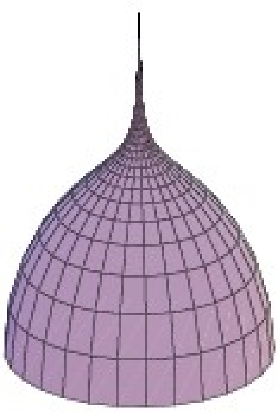
$$\sqrt{\lambda} \gg k \gg 1$$

Exact solution with a spike:

$$\Phi = i\kappa\tau$$
$$Z = \mathrm{e}^{\kappa\tau}\left[\sqrt{\kappa^2+1}\tanh\left(\sqrt{\kappa^2+1}\tau+\xi\right)-\kappa\right]$$
$$X^{1,2} = \left\{\begin{array}{c}\cos\sigma\\ \sin\sigma\end{array}\right\}\frac{\sqrt{\kappa^2+1}\,\mathrm{e}^{\kappa\tau}}{\cosh\left(\sqrt{\kappa^2+1}\tau+\xi\right)}$$

$$\xi = \ln\left(\sqrt{\kappa^2+1}+\kappa\right)$$

Z.'02



Describes

$$\lim_{x\to\infty}|x|^{2\Delta_I}\,\frac{\langle W(C)\mathcal{O}_I(x)\rangle}{\langle W(C)\rangle}$$

for circular Wilson loop

Solution for $\lim_{x\to\infty}|x|^{2\Delta_I}\,\langle\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_I(x)\rangle$ ?

# Boundary conditions at the spike

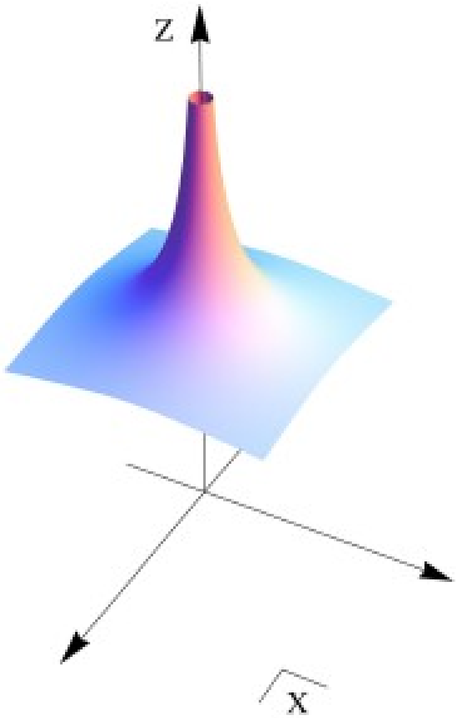
$$\ln Z \quad \to \quad -\chi \ln |w|$$

$$\Theta \quad \to \quad \frac{\pi}{2}$$

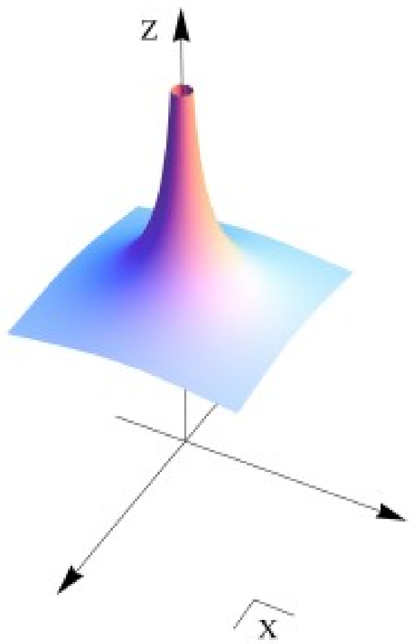
$$\Phi \quad \to \quad -i\chi \ln |w|$$

$$w = \mathrm{e}^{-\tau + i\sigma}$$

$$\chi = \frac{k}{\sqrt{\lambda}}$$



# Fine structure of the spike




Regular solution
without the spike

$$
\begin{aligned}
Z &= |w|^{-\chi}\left(z_0 + \partial z_0 w + \bar{\partial} z_0 \bar{w}\right) + \dots \\
X^\mu &= x_0^\mu + |w|^{\sqrt{1+\chi^2}-1-\chi}\left(\partial x_0^\mu w + \bar{\partial} x_0^\mu \bar{w}\right) + \dots
\end{aligned}
$$

Solution on S5:

$$
\begin{aligned}
e^{i\Phi} &= \frac{|w|^\chi e^{i\varphi_0}\sin\theta_0}{\sqrt{1-|w|^{2\chi}\cos^2\theta_0}}\left[1+i\,\frac{1-|w|^\chi\cos^2\theta_0}{1-|w|^{2\chi}\cos^2\theta_0}\left(\partial\varphi_0 w+\bar\partial\varphi_0\bar w\right)+\frac{(1-|w|^\chi)\cot\theta_0}{1-|w|^{2\chi}\cos^2\theta_0}\left(\partial\theta_0 w+\bar\partial\theta_0\bar w\right)\right]+\ldots \\
\cos\Theta &= |w|^\chi\cos\theta_0-|w|^{\sqrt{1+\chi^2}-1}\left[\frac{1-|w|^\chi\cos^2\theta_0}{\sin\theta_0}\left(\partial\theta_0 w+\bar\partial\theta_0\bar w\right)+i\,(1-|w|^\chi)\cos\theta_0\left(\partial\varphi_0 w+\bar\partial\varphi_0\bar w\right)\right]+\ldots
\end{aligned}
$$

Virasoro constraints:

$$
0=T_{ww}\equiv\frac{(\partial Z)^2+(\partial X)^2}{Z^2}+(\partial\Theta)^2+\sin^2\Theta\,(\partial\Phi)^2
$$

$\chi\longrightarrow 0$ limit:

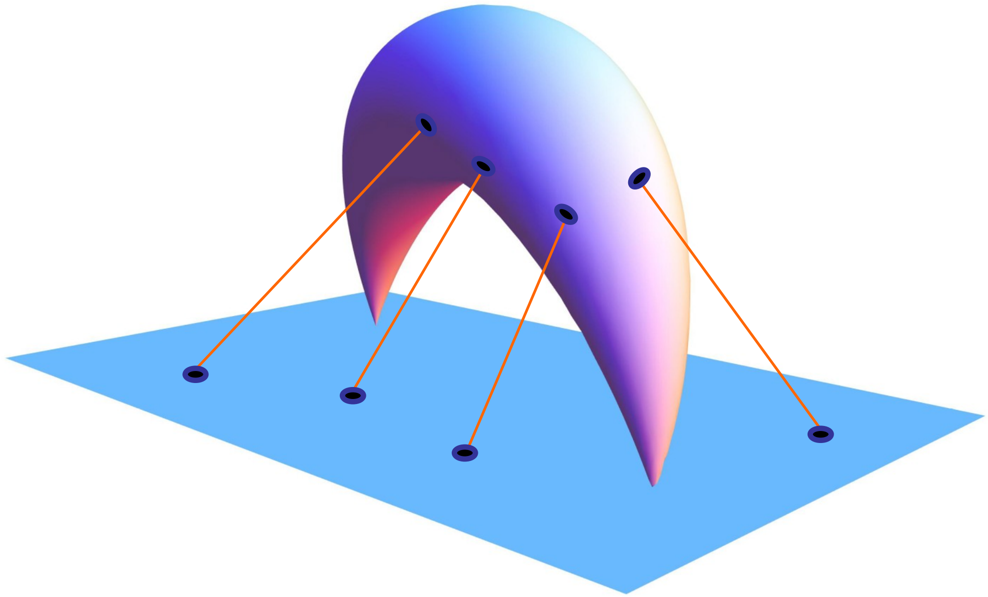
$$
\begin{aligned}
\frac{\partial z_0}{z_0}+\cot\theta_0\,\partial\theta_0+i\partial\varphi_0 &= 0 \\
\frac{\bar\partial z_0}{z_0}+\cot\theta_0\,\bar\partial\theta_0+i\bar\partial\varphi_0 &= 0
\end{aligned}
$$

Determine the position $\sigma_*$ on the worldsheet, where the spike can be attached.

The same as the saddle-point equation for the vertex operator!

# Factorization

Roiban,Tseytlin'10

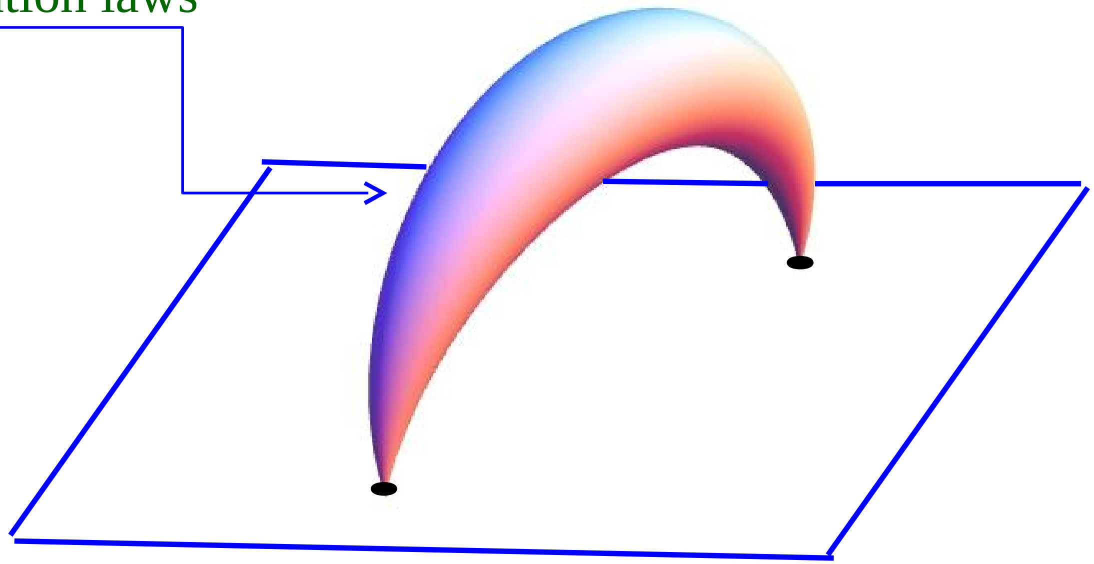


Integration over σi independent:

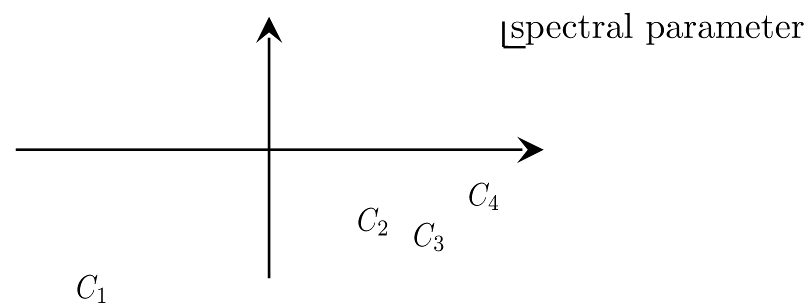
$$\langle HHL_1 \dots L_n \rangle \sim \langle HHL_1 \rangle \dots \langle HHL_n \rangle$$

# Integrability

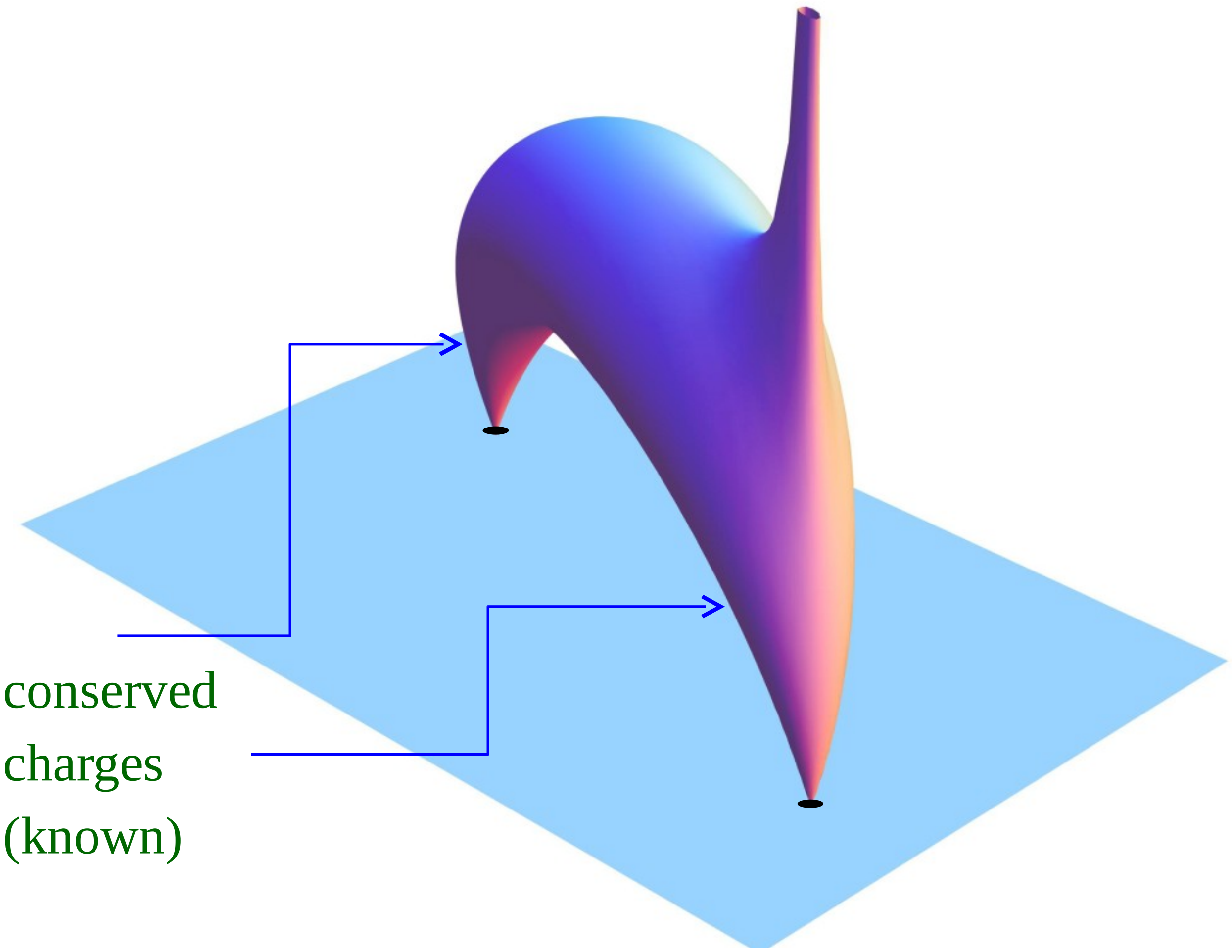
∞ number
of conservation laws



Bookeeping of
conserved charges:



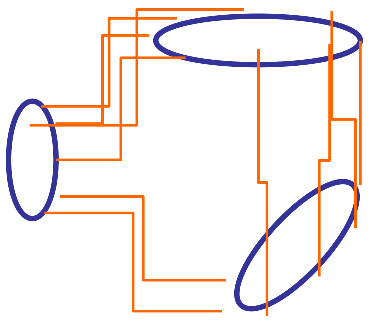
# Integrability in 3-point functions?




Algebraic curves for external states + branching?

# Weak coupling

Escobedo,Gromov,Sever,Vieira'10
Caetano,Escobedo'11



- Overlap of three spin chain states
- Certain resemblance to string field theory vertex Okuyama,Tseng'04
- Can be efficiently computed using ABA Escobedo,Gromov,Sever,Vieira'10
- Still not enough to take the large-charge limit to compare to strong coupling

# Questions

- Possible to compute the <LH…H> correlation functions (H – heavy semiclassical states, L – light supergravity state)

Z.'10
Costa,Monteiro,Santos,Zoakos'10
Roiban,Tseytlin'10
Hernandez'10
Buchbinder,Tseytlin'10

- How to calculate <HHH>?

Can give a clue to exact solution…

- How to use integrability?

- Vertex operators $\overset{?}{\leftrightarrow}$ Classical Solutions $\leftrightarrow$ Bethe ansatz
- Boundary conditions for generic vertex operators